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# COMPLEX WEIGHT FUNCTIONS FOR CLASSICAL ORTHOGONAL POLYNOMIALS

*Dedicated to our friend P. G. (Tim) Rooney on his 65th birthday*

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ABSTRACT. We give complex weight functions with respect to which the Jacobi, Laguerre, little $q$-Jacobi and Askey-Wilson polynomials are orthogonal. The complex functions obtained are weight functions in a wider range of parameters than the real weight functions. They also provide an alternative to the recent distributional weight functions of Morton and Krall, and the more recent hyperfunction weight functions of Kim.

1. **Introduction.** The classical Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}$ are orthogonal on $[-1,1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$, when $\alpha > -1$ and $\beta > -1$. When $\alpha > -1$, the Laguerre polynomials $\{L_n^{(\alpha)}(x)\}$ are orthogonal on $[0,\infty)$ with respect to the weight function $x^\alpha e^{-x}$. The Jacobi and Laguerre polynomials have the explicit representations

$$(1.1)\qquad P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2\mathbf{F}_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}\,\middle|\,\frac{1-x}{2}\right)$$

$$(1.2)\qquad L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1\mathbf{F}_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix}\,\middle|\,x\right)$$

respectively, Rainville [15, Chapters 12 and 16], or [7, § 10.8 and § 10.12].

Every family of orthogonal polynomials $\{p_n(x)\}$ satisfies a three term recurrence relation

$$(1.3)\qquad xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \quad n = 1, 2, \ldots .$$

With the normalization

$$(1.4)\qquad p_0(x) = 1, \quad p_1(x) = (x-\beta_0)/\alpha_0 \quad \alpha_0 \neq 0,$$

a sequence of polynomials $\{p_n(x)\}$ such that $p_n(x)$ is of precise degree $n$ is orthogonal with respect to a positive measure with finite moments and infinite support if and only if

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it satisfies a three term recurrence relation of the type (1.3) such that $\beta_n$'s are real and a positivity condition

$$\alpha_n \gamma_{n+1} > 0, \quad n = 0, 1, \ldots, \tag{1.5}$$

holds.

From the three term recurrence relation of the Jacobi and Laguerre polynomials [15, Chapters 12 and 16], or [7, § 10.8 and § 10.12] it follows that the positivity condition (1.5) is satisfied if and only if $\alpha > -1$ and $\beta > -1$ in the case of Jacobi polynomials and $\alpha > -1$ in the case of Laguerre polynomials. It is then natural to ask what will happen if $\alpha \leq -1$ or $\beta \leq -1$. Morton and Krall [13] attempted to answer this question. They gave distributional weight functions for Jacobi and Laguerre polynomials and proved that their distributional weight functions reduce to the classical weight functions when the positivity condition is satisfied. Morton and Krall then attempted to find a distributional weight function for the Bessel polynomials but pointed out that the series representation their approach gave did not converge in the space of distributions. They then raised the question of finding a distributional weight function for the Bessel polynomials. Recently Kim [11] further extended the work of Morton and Krall and obtained hyperfunctions as weight functions for Jacobi and Laguerre polynomials and Bessel polynomials.

This work grew out of an attempt to replace the use of distributional or hyperfunction weight functions by a more classical approach using complex weight functions. The idea is to observe that if $\{p_n(x)\}$ is orthogonal with respect to a positive measure $d\mu$ supported on a finite interval $[-A, A]$ then the function $X(z)$

$$X(z) := \frac{1}{2\pi i} \int_{-A}^{A} \frac{d\mu(t)}{z - t}, \quad z \notin [-A, A], \tag{1.6}$$

is a complex weight function since

$$\begin{aligned} \frac{1}{2\pi i} \int_{|z|=R>A} p_n(z) p_m(z) X(z)\, dz &= \frac{1}{2\pi i} \int_{|z|=R>A} p_n(z) p_m(z) \int_{-A}^{A} \frac{d\mu(t)}{z-t}\, dz \\ &= \int_{-A}^{A} p_n(t) p_m(t)\, d\mu(t), \end{aligned}$$

by Cauchy's theorem. It turns out that in the case of Jacobi polynomials $X(z)$ continues to be a complex weight function when $\alpha + \beta + 2 \neq 0, -1, \ldots$, but $\alpha < -1$ and $\beta > -1$ or $\alpha > -1$ and $\beta < -1$. Similar results hold for the little $q$-Jacobi polynomials [1], [3] the continuous $q$-ultraspherical polynomials [2], [3] and the Askey-Wilson polynomials [3]. We believe that this is a general analytic continuation property shared by all polynomials orthogonal on a finite interval. The complex weight function for Jacobi polynomial has also appeared in Rusev [16]. Rusev [16] pointed out that the complex orthogonality follows from the fact Jacobi polynomials satisfy a linear second order differential equation of Sturm-Liouville type. In general this technique is not applicable. Furthermore the complex weight function for Laguerre polynomials in Rusev [16] is different from the one we use.

In Section 2 we discuss Jacobi polynomials. In Section 3 we treat the Laguerre polynomials for real $\alpha$. This is particularly interesting because the Laguerre polynomials are orthogonal on the unbounded interval $[0, \infty)$ when $\alpha > -1$. In Sections 4 and 5 we treat $q$-analogues of the classical orthogonal polynomials. The little $q$-Jacobi polynomials [1]

$$p_n(x; \alpha, \beta : q) := {}_2\Phi_1\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1} \\ q^{\alpha+1} \end{matrix}; q, q^{x+1}\right), \tag{1.7}$$

are studied in Section 4 where a complex weight function is found. In Section 5 we give a complex weight function with respect to which the Askey-Wilson polynomials of [3] are orthogonal. A special case of the Askey-Wilson polynomials are the continuous $q$-ultraspherical polynomials [2], so by specializing the parameters of the Askey-Wilson polynomials we obtain a complex weight function of the continuous $q$-ultraspherical polynomials. Part of the proof of the complex orthogonality relation for the Askey-Wilson polynomials is given in Section 5 and the proof is continued in Section 6.

Proofs of the real orthogonality relations for Jacobi and Laguerre polynomials are available in most books on the subject, [7, Chapter 8], [15, Chapters 12 and 16], [17, Chapters 4 and 5]. Proofs of the real orthogonality relations for little $q$-Jacobi and Askey-Wilson polynomials are in the recent book [9]. Reference [9] also contains an excellent bibliography on the subject. After [9] appeared, Atakishiyev and Suslov [4] found a new and elegant proof of the real orthogonality relations for the Askey-Wilson polynomials.

2. **Jacobi polynomials.** Recall that, Szegö [17, §4.61]

$$\int_{-1}^{1} \frac{(1-t)^\alpha(1+t)^\beta}{z-t}\,dt = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(z-1)\Gamma(\alpha+\beta+2)}\,{}_2\mathbf{F}_1\left(\begin{matrix} 1, \alpha+1 \\ \alpha+\beta+2 \end{matrix}\,\middle|\, 2/(1-z)\right). \tag{2.1}$$

Since the Jacobi polynomials are orthogonal on $[-1, 1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ we consider the complex weight function

$$X^{\alpha,\beta}(z) = [\Gamma(\alpha+\beta+2)(z-1)]^{-1}\,{}_2\mathbf{F}_1\left(\begin{matrix} 1, \alpha+1 \\ \alpha+\beta+2 \end{matrix}\,\middle|\, 2/(1-z)\right), \quad |z-1| > 2. \tag{2.2}$$

Consider the integral

$$I_{m,n} := \frac{1}{2\pi i}\int_C (1-z)^m P_n^{(\alpha,\beta)}(z) X^{\alpha,\beta}(z)\,dz, \tag{2.3}$$

where $C$ is a closed circle containing the circle $|z-1| = 2$ in its interior. Using the representations (1.1) and (2.1) we obtain

$$I_{m,n} =$$
$$-\frac{(\alpha+1)_n}{2\pi i(n!)}\int_C \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k(1-z)^{m+k-1}}{k!(\alpha+1)_k 2^k} \sum_{j=0}^{\infty} \frac{(\alpha+1)_j}{\Gamma(\alpha+\beta+j+2)}\left(\frac{2}{1-z}\right)^j dz.$$

Termwise integration is justified and the integrals of various terms will vanish except when $j = m+k$. This identifies $I_{m,n}$ as a multiple of a ${}_3\mathbf{F}_2$ in the form

$$I_{m,n} = \frac{(\alpha+1)_m(\alpha+1)_n}{n!\,\Gamma(\alpha+\beta+m+2)} 2^m\,{}_3\mathbf{F}_2\left(\begin{matrix} -n, n+\alpha+\beta+1,\ \alpha+m+1 \\ \alpha+1,\ \alpha+\beta+m+2 \end{matrix}\,\middle|\, 1\right).$$

The above ${}_3\mathbf{F}_2$ can be summed by the Pfaff-Saalschütz theorem, [15], and we get

$$I_{m,n} = \frac{2^m(\alpha+1)_m}{n!\,\Gamma(\alpha+\beta+m+2)}\,\frac{(-n-\beta)_n(-m)_n}{(-m-n-\alpha-\beta-1)_n}.$$

When $m \leq n$ the above relationship reduces to

$$I_{m,n} = \frac{2^m(\alpha+1)_m}{n!\,\Gamma(\alpha+\beta+m+2)}\,\frac{(-1)^n n!\,(\beta+1)_n}{(\alpha+\beta+m+2)_n}\delta_{m,n}. \tag{2.4}$$

This shows that when $m \leq n$

$$\begin{aligned}\frac{1}{2\pi i}\int_C P_n^{(\alpha,\beta)}(z)P_m^{(\alpha,\beta)}(z)X^{\alpha,\beta}(z)\,dz &= \frac{(-m)_m(\alpha+\beta+m+1)_m}{m!\,m!}2^{-m}I_{m,n}\\ &= \frac{(\alpha+1)_n(\beta+1)_n}{n!\,(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}\delta_{m,n}.\end{aligned} \tag{2.5}$$

Using the notation

$$Y^{\alpha,\beta}(z) := 2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)X^{\alpha,\beta}(z) \tag{2.6}$$

the orthogonality relation (2.5) takes the form

$$\frac{1}{2\pi i}\int_C P_n^{(\alpha,\beta)}(z)P_m^{(\alpha,\beta)}(z)Y^{\alpha,\beta}(z)\,dz = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\,(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}\delta_{m,n}. \tag{2.7}$$

Of course when $\alpha > -1$, $\beta > -1$ the orthogonality relation (2.7) and the integral representation (2.1) imply the familiar orthogonality relation, Szegö [17]

$$\int_{-1}^{1}(1-t)^\alpha(1-t)^\beta P_n^{(\alpha,\beta)}(t)P_m^{(\alpha,\beta)}(t)\,dt = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\,(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}\delta_{m,n}. \tag{2.8}$$

It is evident that (2.7) holds as long as neither $\alpha$, $\beta$ nor $\alpha+\beta+1$ is a negative integer. It is interesting to note that (2.7) implies orthogonality with respect to a distributional weight function if $\alpha$ or $\beta$ is less than $-1$. To see this first consider the case

$$k+\alpha > -1 > \alpha,\ \beta > -1, \text{ for some integer } k,\ k > 0. \tag{2.9}$$

Let $f(x)$ be a polynomial of degree at least $k$ and let $g(x)$ be the first $k$ terms in its Taylor series about $x = 1$, that is

$$g(x) = \sum_{j=0}^{k-1}\frac{f^{(j)}(1)}{n!}(x-1)^j. \tag{2.10}$$

For $j = 0, 1, \ldots,$ and $\alpha+\beta \neq -2, -3, \ldots,$ Cauchy's theorem yields

$$\int_C (z-1)^j Y^{\alpha,\beta}(z)\,dz = 2^{j+\alpha+\beta+1}(-1)^j\Gamma(\beta+1)\Gamma(\alpha+j+1)/\,\Gamma(j+\alpha+\beta+2). \tag{2.11}$$

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Thus we have

$$\begin{aligned}\int_C f(z)Y^{\alpha,\beta}(z)\,dz &= \int_C g(z)Y^{\alpha,\beta}(z)\,dz + \int_C [f(z)-g(z)]Y^{\alpha,\beta}(z)\,dz \\ &= \sum_{j=0}^{k-1}(-1)^j 2^{j+\alpha+\beta+1} f^{(j)}(1)\frac{\Gamma(\beta+1)\Gamma(\alpha+j+1)}{j!\,\Gamma(j+\alpha+\beta+2)} \\ &\quad + \int_C [f(z)-g(z)]Y^{\alpha,\beta}(z)\,dz.\end{aligned}$$

On the other hand Cauchy's theorem, (2.1), (2.6) and the fact that $[f(z)-g(z)]/(z-1)^k$ is a polynomial show that

$$\begin{aligned}\int_C [f(z)-g(z)]Y^{\alpha,\beta}(z)\,dz &= \int_C \frac{f(z)-g(z)}{(1-z)^k}Y^{\alpha+k,\beta}(z)\,dz \\ &= \int_{-1}^{1}\frac{f(t)-g(t)}{(1-t)^k}(1-t)^{\alpha+k}(1+t)^\beta\,dt.\end{aligned}$$

Thus we proved

$$\begin{aligned}\int_C f(z)Y^{\alpha,\beta}(z)\,dz &= \sum_{j=0}^{k-1}(-1)^j 2^{j+\alpha+\beta+1} f^{(j)}(1)\frac{\Gamma(\beta+1)\Gamma(\alpha+j+1)}{j!\,\Gamma(j+\alpha+\beta+2)} \\ &\quad + \int_{-1}^{1}(1-t)^{-k}\Big[f(t)-\sum_{j=0}^{k-1}\frac{f^{(j)}(1)}{j!}(t-1)^j\Big](1-t)^{\alpha+k}(1+t)^\beta\,dt, \qquad (2.12)\end{aligned}$$

where $f$ is a polynomial, provided that (2.9) is fulfilled. The restriction $\alpha+\beta+2\neq 0,-1,\ldots$, which was used implicitly in the derivation of (2.12) can be removed by analytic continuation.

Based on the above calculations one can define an indefinite inner product with respect to a distributional weight function by

(2.13)

$$\begin{aligned}(f,g) &= \sum_{j=0}^{k-1}(-1)^j 2^{j+\alpha+\beta+1}(fg)^{(j)}(1)\frac{\Gamma(\beta+1)\Gamma(\alpha+j+1)}{j!\,\Gamma(j+\alpha+\beta+2)} \\ &\quad + \int_{-1}^{1}(1-t)^{-k}\Big[f(t)g(t)-\sum_{j=0}^{k-1}\frac{(fg)^{(j)}(1)}{j!}(t-1)^j\Big](1-t)^{\alpha+k}(1+t)^\beta\,dt.\end{aligned}$$

The symmetry relation

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x), \qquad (2.14)$$

Szegö [17, (4.1.4)], reduces the case $\beta<-1$, $\alpha>-1$ to the case $\alpha<-1$, $\beta>-1$ where (2.12),(2.13) and (2.14) are applicable.

3. **The Laguerre polynomials.** We have been unable to find a direct proof of the complex orthogonality of the Laguerre polynomials without using the real orthogonality, Rainville [15, Chapter 12]

$$\int_0^\infty L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)\frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}\,dx = \frac{(\alpha+1)_n}{n!}\delta_{m,n}, \quad \alpha>-1. \qquad (3.1)$$

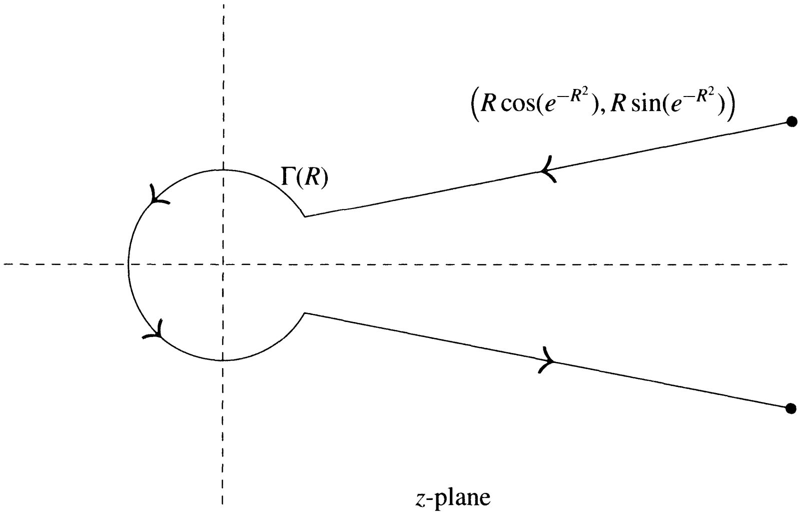


FIGURE 1

If one assumes the real orthogonality of a family of polynomials then the method of this section shows how one may extend the real orthogonality to orthogonality with respect to a complex weight function. We now proceed to show how to extend (3.1) to the cases when $\alpha < -1$ and $\alpha$ is not an integer. Note that in (3.1) we normalized the weight function to have a total mass equal to unity.

The complex weight function is

$$w(z;\alpha) = -\Psi(1, 1-\alpha, -z), \tag{3.2}$$

where $\Psi$ is the Tricomi $\Psi$ function, [6, Chapter 6]. When $\alpha > -1$, the moments of the real weight function are given by

$$\mu_n = \int_0^\infty \frac{x^n x^\alpha e^{-x}}{\Gamma(\alpha+1)}\,dx = (\alpha+1)_n. \tag{3.3}$$

Furthermore, [6],

$$w(z,\alpha) = \sum_{n=0}^{N} \mu_n z^{-n-1} + O(|z|^{-N-2}), \quad z \to \infty, \quad N = 0, 1, \ldots .$$

The explicit representation (1.2) is

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} d_{n,k} x^k, \quad d_{n,k} := \frac{(\alpha+1)_n(-n)_k}{n!\,k!\,(\alpha+1)_k}. \tag{3.4}$$

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LEMMA 3.5. *The identity*

$$\sum_{j=0}^{m}\sum_{k=0}^{n} d_{m,j}d_{n,k}\mu_{k+j} = \frac{(\alpha+1)_n}{n!}\delta_{m,n}, \tag{3.6}$$

*holds for all $\alpha$ (real or complex).*

PROOF. The identity (3.6) is just the orthogonality relation (3.1) when $\alpha > -1$. Since $(\alpha+1)_n/(\alpha+1)_k = (\alpha+k)\cdots(\alpha+n-1)$ the $d_{j,n}$'s and the $\mu_n$'s are polynomials in $\alpha$ then (3.6) holds for all $\alpha$.

THEOREM 3.7. *If $\alpha \neq -1,-2,\ldots$, then*

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\Gamma(R)} w(z;\alpha)L_n^{(\alpha)}(z)L_m^{(\alpha)}(z)\,dz = \frac{(\alpha+1)_n}{n!}\delta_{m,n}, \tag{3.8}$$

*where $\Gamma(R)$ is as in Figure 1.*

PROOF. Let $\Gamma_2(R) = \{z : |z| = R, \exp(-R^2) \le \arg z \le 2\pi - \exp(-R^2)\}$, in the clockwise sense. Then

$$\int_{\Gamma(R)} w(z;\alpha)L_n^{(\alpha)}(z)L_m^{(\alpha)}(z)\,dz = -\int_{\Gamma_2(R)} w(z;\alpha)L_n^{(\alpha)}(z)L_m^{(\alpha)}(z)\,dz,$$

since $w(z;\alpha)$ has its singularities in $[0,\infty)$. Now choose $N > m+n$ and observe that

$$\begin{aligned}\int_{\Gamma_2(R)} &w(z;\alpha)L_n^{(\alpha)}(z)L_m^{(\alpha)}(z)\,dz\\ &= \int_{\Gamma_2(R)}\Big[\sum_{j=0}^{m}\sum_{k=0}^{n} d_{m,j}d_{n,k}z^{j+k}\Big]\Big[\sum_{s=0}^{N}\mu_s z^{-s-1} + O(|z|^{-N-2})\Big]\,dz.\end{aligned}$$

Therefore

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\Gamma_2(R)} w(z;\alpha)L_m^{(\alpha)}L_n^{(\alpha)}(z)\,dz = -\sum_{j=0}^{m}\sum_{k=0}^{n} d_{m,j}d_{n,k}\mu_{k+j} = -\frac{(\alpha+1)_n}{n!}\delta_{m,n}.$$

This establishes Theorem 3.7.

Theorem 3.7 continues to hold also when $\alpha = -k$, $k = 1,2,\ldots$. However the orthogonality is then for only a finite number of polynomials $\{L_n^{(-k)}(x)\}_{n=0}^{k-1}$. The measure may be reexpressed as a real distribution with support at $x = 0$.

COROLLARY 3.9. *If $\alpha = -k$, $k = 1,2,\ldots$ then*

$$\int_{-\infty}^{\infty} L_n^{(-k)}(x)L_m^{(-k)}(x)\,d\mu(x;k) = \frac{(1-k)_n}{n!}\delta_{m,n}, \tag{3.10}$$

*where*

$$d\mu(x;k) := \Big(1+\frac{d}{dx}\Big)^{k-1}\delta(x) = \sum_{j=0}^{k-1}\binom{k-1}{j}\delta^{(j)}(x)\,dx. \tag{3.11}$$

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PROOF. From [6, (6.7.13)]

$$w(z,-k) = \sum_{j=0}^{k-1} \frac{(-1)^j (k-1)!}{(k-j-1)!} z^{-j-1}. \tag{3.12}$$

Thus, when $\alpha = -k$, the left side of (3.8) can be replaced by

$$\begin{aligned} &\frac{1}{2\pi i} \sum_{j=0}^{k-1} \int_{|z|=1} \frac{(-1)^j (k-1)!}{(k-j-1)!} z^{-j-1} L_n^{(-k)}(z) L_m^{(-k)}(z)\, dz \\ &\qquad = \left(1 - \frac{d}{dz}\right)^{k-1} L_n^{(-k)}(z) L_m^{(-k)}(z) \Big|_{z=0}, \end{aligned} \tag{3.13}$$

and the proof is complete.

4. **The little $q$-Jacobi polynomials.** We shall assume throughout this section that

$$0 < q < 1.$$

Recall the definition of the $q$-integral [9]

$$\int_0^a f(t)\, d_q(t) := (1-q)a \sum_{n=0}^{\infty} q^n f(aq^n), \tag{4.1}$$

the $q$-shifted factorial
(4.2)

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=1}^{n} (1 - aq^{k-1}) \quad , n = 1, 2, \ldots, \text{ or } \infty, \quad (a;q)_b = \frac{(a;q)_\infty}{(aq^b;q)_\infty},$$

and the $q$-gamma function

$$\Gamma_q(x) := (q;q)_\infty (1-q)^{1-x} / (q^x;q)_\infty, \quad 0 < q < 1. \tag{4.3}$$

When $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\beta > -1$ note the evaluation of the $q$-beta integral [9]

$$\int_0^1 t^\alpha (qt;q)_\beta\, d_q t = [\Gamma_q(\alpha+1)\Gamma_q(\beta+1)/\Gamma_q(\alpha+\beta+2)]. \tag{4.4}$$

The analog of (2.2) is

$$\begin{aligned} \int_0^1 \frac{t^\alpha (tq;q)_\beta}{z-t}\, d_q t &= \sum_{n=0}^{\infty} z^{-n-1} \int_0^1 t^{\alpha+n} (qt;q)_\beta\, d_q t \\ &= \frac{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+2)} z^{-1}\, {}_2\Phi_1\left( \begin{matrix} q, q^{\alpha+1} \\ q^{\alpha+\beta+2} \end{matrix} ; q, z^{-1} \right), \quad |z| > 1. \end{aligned} \tag{4.5}$$

The ${}_2\Phi_1$ in (4.5) is the familiar function

$${}_2\Phi_1\left( \begin{matrix} a, b \\ c \end{matrix} ; q, z \right) := \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} z^n,$$

[8],[9]. We are then led to consider the complex weight function

$$X^{\alpha,\beta}(z;q) := [z]^{-1}{}_2\Phi_1\left(\begin{matrix} q, q^{\alpha+1} \\ q^{\alpha+\beta+2} \end{matrix}; q, z^{-1}\right). \tag{4.6}$$

THEOREM 4.7. *The little q-Jacobi polynomials satisfy the orthogonality relation*

(4.8)

$$\frac{1}{2\pi i}\int_{|z|=r} X^{\alpha,\beta}(z;q)p_n(z)p_m(z)\,dz = \frac{(1-q^{\alpha+\beta+1})(q, q^{\beta+1};q)_n}{(1-q^{\alpha+\beta+1+2n})(q^{\alpha+\beta+1}, q^{\alpha+1};q)_n}q^{(\alpha+1)n}\delta_{m,n},$$

*where*

$$p_n(x) = {}_2\Phi_1\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1} \\ q^{\alpha+1} \end{matrix}; q, qx\right), \tag{4.9}$$

*and*

$$(a_1, a_2, \ldots, a_n; q)_m = \prod_{j=1}^{n}(a_j;q)_m.$$

PROOF. There is no loss of generality in assuming that $m \geq n$.

The left-hand side in (4.8) =

$$\sum_{k=0}^{m}\sum_{j=0}^{n}\frac{(q^{-m}, q^{m+\alpha+\beta+1};q)_k(q^{-n}, q^{n+\alpha+\beta+1};q)_j(q^{\alpha+1};q)_{j+k}}{(q, q^{\alpha+1};q)_k(q, q^{\alpha+1};q)_j(q^{\alpha+\beta+2};q)_{j+k}}q^{j+k}.$$

The $j$ sum is

$${}_3\Phi_2\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{k+\alpha+1} \\ q^{\alpha+1}, q^{\alpha+\beta+2+k} \end{matrix}; q, q\right) = \frac{(q^{\beta+1}, q^{k-n+1};q)_n}{(q^{k+\alpha+\beta+2}, q^{-n-\alpha};q)_n},$$

by the $q$ analog of the Pfaff-Saalschütz formula, [9]. Therefore the $j$ sum vanishes if $k < n$. Thus in the left-hand side of (4.8) the summation index $k$ must satisfy $n \leq k \leq m$. Replacing $k$ by $k+n$, we find after elementary manipulations that the left-hand side in (4.8)

$$\begin{aligned}
&= \frac{(q^{\beta+1}, q^{-m}, q^{m+\alpha+\beta+1};q)_n}{(q^{-n-\alpha};q)_n(q^{\alpha+\beta+2};q)_{2n}}q^n{}_2\Phi_1\left(\begin{matrix} q^{n-m}, q^{n+m+1+\alpha+\beta} \\ q^{2n+\alpha+\beta+2} \end{matrix}; q, q\right) \\
&= \frac{(q^{\beta+1}, q^{-m}, q^{m+\alpha+\beta+1};q)_n}{(q^{-n-\alpha};q)_n(q^{\alpha+\beta+2};q)_{2n}}q^n\frac{(q^{1+n-m};q)_{m-n}}{(q^{2n+2+\alpha+\beta};q)_{m-n}}
\end{aligned}$$

which, after some simplification, reduces to the right-hand side in (4.8).

## 5. **The Askey-Wilson polynomials.**

Let $-1 < q < 1$ and

(5.1)

$$w(t) := \frac{(q, ab, ac, ad, bc, bd, cd, e^{2i\theta}, e^{-2i\theta};q)_\infty}{2\pi(abcd;q)_\infty|(ae^{i\theta}, be^{i\theta}, ce^{i\theta}, de^{i\theta};q)_\infty|^2}(1-t^2)^{-1/2}, \quad t = \cos\theta \in [-1,1].$$

Askey and Wilson proved that the polynomials

(5.2)

$$p_n(\zeta;a,b,c,d) := {}_4\Phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, az, a/z \\ ab, ac, ad \end{matrix}; q, q\right), \quad \zeta = \frac{1}{2}(z+1/z), \quad |z| \leq 1,$$

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later called the Askey-Wilson polynomials, are orthogonal with respect to the weight function $w(t)$ on $[-1,1]$, when $a,b,c,d \in (-1,1)$. For two exceptional cases when $abcd = q$ or $q^2$, see Gupta and Masson [10]. Masson in earlier work [12] treated the case $q = 1$. We shall not discuss any exceptional cases in this work. The weight function $w(t)$ has been normalized to have a unit total mass. Rahman [14] proved that the function $F(\zeta)$ defined by

$$F(\zeta) := \int_{-1}^{1} \frac{w(t)}{\zeta - t}\, dt, \tag{5.3}$$

has the explicit representation

$$F(\zeta) = \frac{2z(1-abcz)}{(1-az)(1-bz)(1-cz)}\, {}_8\mathbf{W}_7(abcz; ab, ac, bc, q, zq/d; q, dz), \tag{5.4}$$

where the ${}_8\mathbf{W}_7$ is a very well poised ${}_8\Phi_7$,

(5.5)

$${}_8\mathbf{W}_7(a; b, c, d, f, g; q, z) = {}_8\Phi_7\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, f, g \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/f, aq/g \end{matrix}; q, z\right).$$

In the above formulas we assume

$$\zeta = (z + 1/z)/2, \tag{5.6}$$

and $|z| < 1 < |1/z|$. The series defining the ${}_8\mathbf{W}_7$ in (5.4) converges if none of the denominator parameters is of the form $q^{-n}$, $n = 0, 1, \ldots$, and $|dz| < 1$. The symmetry of $F(\zeta)$ in $a, b, c, d$, which is a consequence of the Bailey transformation formulas (§ 2.10 in [9]) implies that $|dz| < 1$ can be replaced by "one of $|az|, |bz|, |cz|, |dz|$ is less than 1." If at least one of the parameters $a, b, c, d$ is in $(-1, 1)$ then this latter condition is automatically satisfied because of (5.6). If not, we can always take a smaller circle in the $z$-plane to ensure it.

It is obvious that $F(\zeta)$ is analytic in any closed disc of radius $r$ and center $\zeta = 0$ in the $\zeta$-plane when $r < \min(|a|^{-1}, |b|^{-1}, |c|^{-1}, |d|^{-1})$. We shall use ${}_8\mathbf{W}_7$ to mean the series defining the ${}_8\mathbf{W}_7$ as well as its analytic continuations.

Our main objective is to evaluate the complex integral

$$\frac{1}{2\pi i} \int_K p_n(\zeta; a, b, c, d) p_m(\zeta; a, b, c, d) F(\zeta)\, d\zeta,$$

where $K$ is a simple closed positively oriented contour enclosing the origin in the $\zeta$-plane and $p_n$ is as in (5.2). Furthermore we assume that $K$ contains the unit circle in its interior. Since

$$(az, a/z; q)_k = \prod_{j=0}^{k-1} (1 - 2\zeta a q^j + a^2 q^{2j}),$$

the $p_n$'s have a relatively simple explicit expression in terms of $\zeta$ but the weight function does not share this property. So, it is convenient to transform the above complex integral to the $z$-plane. From (5.6) we see that

$$z = -\sum_{n=1}^{\infty} \frac{(-1/2)_n}{n!} \zeta^{-2n-1}, \quad |\zeta| > 1,$$

hence $2d\zeta = (1 - z^{-2})\,dz$, and $|z| < 1$. We choose $r$ such that

$$r < \min\{1, |a|^{-1}, |b|^{-1}, |c|^{-1}, |d|^{-1}\}. \tag{5.7}$$

Thus $K$ can be chosen such that the integral over $K$ is transformed to the integral

$$J_{m,n}(a,b,c,d) = \frac{1}{4\pi i}\int_C \frac{1-z^2}{z^2} p_m(\zeta\,; a,b,c,d)p_n(\zeta\,; a,b,c,d)F(\zeta\,)\,dz, \tag{5.8}$$

where $C$ is the circle $|z| = r$ with $r$ satisfying (5.7).

The orthogonality relation to be proved is

$$\frac{1}{4\pi i}\int_C \frac{1-z^2}{z^2} p_m(\zeta\,; a,b,c,d)p_n(\zeta\,; a,b,c,d)F(\zeta\,)\,dz = \delta_{m,n}/\,h_n, \tag{5.9}$$

where

$$h_n := \frac{(1-abcdq^{2n-1})(ab,ac,ad,abcd;q)_n}{(1-abcdq^{n-1})(q,bc,bd,cd;q)_n}a^{-2n},$$

It is clear from (5.4) and (5.5) that $F(\zeta\,)$ is an analytic function of $z$ in and on $C$ and that neither $p_m(\zeta\,; a,b,c,d)$ nor $p_n(\zeta\,; a,b,c,d)$ has any singularity inside $C$ other than the origin $z = 0$, where $p_m(\zeta\,)$ has a pole of order $m$. So we need only to consider the residue of the integrand in (5.8) at $z = 0$.

Although $F(\zeta\,)$ is analytic in $C$ as a function of $z$, it is not easy to express it as a power series in $z$, so we shall take an indirect route to compute the residue at $z = 0$.

Bailey's formula [9, (2.10.10)] yields the representation

$$\begin{aligned} F(\zeta\,) = F(a,b,c,d;\zeta\,) &= \frac{-2a}{(1-az)(1-a/\,z)}\,{}_4\Phi_3\left(\begin{matrix} ab, ac, ad, q \\ abcd, aqz, aq/\,z \end{matrix}; q, q\right) \\ &+ \frac{2z(q,ab,ac,ad,bcdz,qz^2;q)_\infty}{(bz,cz,dz,abcd,az,a/\,z;q)_\infty}\,{}_3\Phi_2\left(\begin{matrix} bz, cz, dz \\ bcdz, qz^2 \end{matrix}; q, q\right). \end{aligned} \tag{5.10}$$

Since $|q| < 1$, both series on the right-hand side of (5.10) are, of course, convergent. If $N$ is any positive integer then it follows that

(5.11)

$$\begin{aligned} F(aq^N, b, c, d;\zeta\,) &= \frac{-2aq^N}{(1-azq^N)(1-aq^N/\,z)}\,{}_4\Phi_3\left(\begin{matrix} abq^N, acq^N, adq^N, q \\ abcdq^N, azq^{N+1}, aq^{N+1}/\,z \end{matrix}; q, q\right) \\ &+ \frac{2z(abq^N, acq^N, adq^N;q)_\infty}{(abcdq^N, azq^N, aq^N/\,z;q)_\infty}\frac{(q,bcdz,qz^2;q)_\infty}{(bz,cz,dz;q)_\infty}\,{}_3\Phi_2\left(\begin{matrix} bz, cz, dz \\ bcdz, qz^2 \end{matrix}; q, q\right). \end{aligned}$$

Combining (5.10) and (5.11) with the observation

$$\begin{aligned} &\frac{1}{(1-az)(1-a/\,z)}\,{}_4\Phi_3\left(\begin{matrix} ab, ac, ad, q \\ abcd, aqz, aq/\,z \end{matrix}; q, q\right) \\ &= \sum_{r=0}^{\infty} \frac{(ab,ac,ad;q)_r q^r}{(abcd;q)_r(az,a/\,z;q)_{r+1}} \\ &= \sum_{r=0}^{N-1} \frac{(ab,ac,ad;q)_r q^r}{(abcd;q)_r(az,a/\,z;q)_{r+1}} \\ &+ \frac{(ab,ac,ad;q)_N q^N}{(abcd;q)_N(az,a/\,z;q)_{N+1}}\,{}_4\Phi_3\left(\begin{matrix} abq^N, acq^N, adq^N, q \\ abcdq^N, azq^{N+1}, aq^{N+1}/\,z \end{matrix}; q, q\right), \end{aligned}$$

we establish the functional equation

$$(5.12)\quad \begin{aligned} F(a,b,c,d;\zeta) &= \frac{(ab,ac,ad;q)_N}{(abcd,az,a/z;q)_N}F(aq^N,b,c,d;\zeta) \\ &\quad - 2a\sum_{r=0}^{N-1}\frac{(ab,ac,ad;q)_r q^r}{(abcd;q)_r(az,a/z;q)_{r+1}}. \end{aligned}$$

Now for the sake of definiteness we shall *assume that* $n \geq m$, then apply Sears' ${}_4\Phi_3$ transformation, namely [9, (2.10.4)]

$$(5.13)\quad {}_4\Phi_3\left(\begin{matrix} q^{-n},a,b,c \\ d,e,f \end{matrix};q,q\right) = \frac{(e/a,f/a;q)_n}{(e,f;q)_n}a^n{}_4\Phi_3\left(\begin{matrix} q^{-n},a,d/b,d/c \\ d,aq^{1-n}/e,aq^{1-n}/f \end{matrix};q,q\right),$$

provided that $abc = defq^{n-1}$, to (5.2) in order to establish the representation

$$(5.14)\quad p_m(x;a,b,c,d) = \frac{(bc,bd;q)_m}{(ac,ad;q)_m}(a/b)^m{}_4\Phi_3\left(\begin{matrix} q^{-m},abcdq^{m-1},bz,b/z \\ ba,bc,bd \end{matrix};q,q\right).$$

The $q$-analog of the Pfaff-Saalschütz summation theorem [9, (1.7.2)] implies the identity

$$(bz,b/z;q)_s = (abq^k,bq^{-k}/a;q)_s\,{}_3\Phi_2\left(\begin{matrix} q^{-s},azq^k,aq^k/z \\ abq^k,aq^{k+1-s}/b \end{matrix};q,q\right),\quad k=0,1,\ldots,$$

which enables us to derive the representation

$$(5.15)\quad \begin{aligned} p_m(\zeta;a,b,c,d)p_n(\zeta;a,b,c,d) &= \frac{a^m(bc,bd;q)_m}{b^m(ac,ad;q)_m}\sum_{s=0}^{m}\frac{(q^{-m},abcdq^{m-1};q)_s}{(q,ba,bc,bd;q)_s}q^s \\ &\cdot\sum_{k=0}^{n}\frac{(q^{-n},abcdq^{n-1};q)_k}{(q,ab,ac,ad;q)_k}q^k(abq^k,bq^{-k}/a;q)_s\sum_{j=0}^{s}\frac{(q^{-s};q)_j(az,a/z;q)_{j+k}}{(q,abq^k,aq^{k+1-s}/b;q)_j}q^j, \end{aligned}$$

for the product $p_n(\zeta)p_m(\zeta)$ in a straightforward manner. Thus in order to prove the orthogonality of the $p_n$'s with respect to the complex measure $w(\zeta)$ we need only to compute the integrals

$$(5.16)\quad M_k := \frac{1}{2\pi i}\int_C \frac{1-z^2}{2z^2}(az,a/z;q)_k\,F\Big(\frac{1}{2}(z+z^{-1})\Big)\,dz,\quad k=0,1,\ldots.$$

We set $N = j+k+1$ in (5.12) and obtain

$$(5.17)\quad \begin{aligned} M_{j+k} &= \frac{(ab,ac,ad;q)_{j+k+1}}{2\pi i(abcd;q)_{j+k+1}}\int_C \frac{1-z^2}{2z^2}\frac{F\big(aq^{j+k+1},b,c,d;\frac{1}{2}(z+1/z)\big)}{(1-azq^{j+k})(1-aq^{j+k}/z)}\,dz \\ &\quad - a\sum_{r=0}^{j+k}\frac{(ab,ac,ad;q)_r q^r}{2\pi i(abcd;q)_r}\int_C \frac{1-z^2}{z^2}\frac{(az,a/z;q)_{j+k}}{(az,a/z;q)_{r+1}}\,dz. \end{aligned}$$

The representation (5.17) is all that we need to prove the orthogonality relation of the Askey-Wilson polynomials (5.9). The integrals on the right-hand side of (5.17) will be evaluated in the next section where we will complete the proof of the orthogonality relation of the Askey-Wilson polynomials.

6. **Orthogonality of the Askey-Wilson polynomials.** The integrals on the right-hand side of (5.17) will be computed using the calculus of residues. First observe that, in view of (5.4), the integrand in the first integral is analytic in a neighborhood of $z = 0$ so its only singularity in the unit circle is at $z = aq^{j+k}$. On the other hand the only singularities of the integrand in the second integral which are within $C$ are $z = 0$ and $z = aq^{j+k}$.

The residue at $z = 0$ of the second integrand in (5.17) is

$$a\,\mathrm{Res}\Big[(1-z^{-2})(az, a/z; q)_{j+k}\sum_{r=0}^{j+k}\frac{(ab, ac, ad; q)_r q^r}{(abcd; q)_r(az, a/z; q)_{r+1}}; z = 0\Big]. \tag{6.1}$$

When $0 \leq r < j + k$ in the above sum we have

$$\begin{aligned}
&\mathrm{Res}\Big[(1-z^{-2})\frac{(az, a/z; q)_{j+k}}{(az, a/z; q)_{r+1}}; z = 0\Big]\\
&= \mathrm{Res}[(1-z^{-2})(azq^{r+1}, aq^{r+1}/z; q)_{j+k-r-1}; z = 0]\\
&= \sum_{u=0}^{j+k-r-1}\sum_{v=0}^{j+k-r-1}\frac{(q^{r+1-k-j}; q)_u(q^{r+1-j-k}; q)_v}{(q; q)_u(q; q)_v}(aq^{j+k})^{u+v}\,\mathrm{Res}[(1-z^{-2})z^{u-v}; z = 0]\\
&= \sum_{u=0}^{j+k-r-1}\sum_{v=0}^{j+k-r-1}\frac{(q^{r+1-k-j}; q)_u(q^{r+1-j-k}; q)_v}{(q; q)_u(q; q)_v}(aq^{j+k})^{u+v}\,\mathrm{Res}[(1-1/z^2)z^{u-v}; z = 0]\\
&= 0,
\end{aligned}$$

since the residue is zero except when $u = v - 1$ and $u = v + 1$ where the residue is 1 and $-1$, repectively but the rest of the summand is symmetric in $u$ and $v$. On the other hand the term $r = j + k$ contributes a multiple of

$$\mathrm{Res}\{(1-z^{-2})/[(1-azq^{j+k})(1-aq^{j+k}/z)]; z = 0\} = q^{-j-k}/a. \tag{6.2}$$

Thus the contribution of the pole at $z = 0$ to the second integral in (5.17) is

$$(ab, ac, ad; q)_{j+k}/(abcd; q)_{j+k}. \tag{6.3}$$

Using (5.4) we see that the singularity at $z = aq^{j+k}$ contributes

$$\begin{aligned}
&\frac{(1-a^2bcq^{2j+2k+1})(ab, ac, ad; q)_{j+k+1}}{(1-a^2q^{2j+2k+1})(1-abq^{j+k})(1-acq^{j+k})(abcd; q)_{j+k+1}}\\
&\cdot {}_6\mathbf{W}_5(a^2bcq^{2j+2k+1}; bc, q, aq^{j+k+1}/d; q, adq^{j+k}),
\end{aligned} \tag{6.4}$$

to the first term on the right-hand side in (5.17). The ${}_6W_5$ in (6.4) can be summed by (II.20), p. 238 in [9] and its sum is

$$\frac{(a^2bcq^{2j+2k+2}, a^2q^{2j+2k+1}, adq^{j+k+1}, abcdq^{j+k}; q)_\infty}{(a^2q^{2j+2k+2}, a^2bcq^{2j+2k+1}, abcdq^{j+k+1}, adq^{j+k}; q)_\infty}.$$

Thus the first term on the right-hand side in (5.17) is

$$(ab, ac, ad; q)_{j+k}/(abcd; q)_{j+k}.$$

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Now the contribution to the second term on the right-hand side of (5.17) from the residue at $z = aq^{j+k}$ is easily seen to be

$$-(ab, ac, ad; q)_{j+k} / (abcd; q)_{j+k},$$

which, by (6.5) cancels out the first term on the right-hand side in (5.17). Thus

$$M_{j+k} = (ab, ac, ad; q)_{j+k} / (abcd; q)_{j+k},$$

and from (5.15) and (5.17) we obtain

$$\begin{aligned} &J_{m,n}(a,b,c,d) \\ &\quad = \frac{a^m (bc, bd; q)_m}{b^m (ac, ad; q)_m} \sum_{k=0}^{n} \frac{(q^{-n}, abcdq^{n-1}; q)_k}{(q, abcd; q)_k} q^k {}_3\Phi_2\left( \begin{matrix} q^{-m}, abcdq^{m-1}, abq^k \\ abcdq^k, ba \end{matrix} ; q, q \right), \end{aligned}$$

and the ${}_3\Phi_2$ can be summed by the $q$-analog of the Pfaff-Saalschütz theorem, [9, (1.7.2)]. Therefore

$$\begin{aligned} &J_{m,n}(a,b,c,d) \\ &= \frac{a^m (bc, bd; q)_m}{b^m (ac, ad; q)_m} \sum_{k=0}^{n} \frac{(q^{-n}, abcdq^{n-1}; q)_k}{(q, abcd; q)_k} q^k {}_3\Phi_2\left( \begin{matrix} q^{-m}, abcdq^{m-1}, abq^k \\ abcdq^k, ba \end{matrix} ; q, q \right) \\ &= \frac{a^m (bc, bd; q)_m}{b^m (ac, ad; q)_m} \sum_{k=0}^{n} \frac{(q^{-n}, abcdq^{n-1}; q)_k}{(q, abcd; q)_k} q^k \frac{(q^{1+k-m}, cd; q)_m}{(abcdq^k, q^{1-m}/ab; q)_m} \\ &= \frac{(bc, bd, cd; q)_m}{(ac, ad, q^{1-m}/(ab); q)_m} \frac{a^m}{b^m} \sum_{k=0}^{n-m} \frac{(q^{-n}, abcdq^{n-1}; q)_{m+k} (q^{k+1}; q)_m}{(q; q)_{m+k} (abcd; q)_{2m+k}} q^{k+m} \\ &= \frac{(bc, bd, cd; q)_m}{(ac, ad, q^{1-m}/(ab); q)_m} \frac{a^m}{b^m} \frac{(q^{-n}, abcdq^{n-1}; q)_m}{(abcd; q)_{2m}} q^m {}_2\Phi_1\left( \begin{matrix} q^{m-n}, abcdq^{n+m-1} \\ abcdq^{2m} \end{matrix} ; q, q \right). \end{aligned}$$

Now by the $q$-analog of the Gauss summation theorem [9, (1.5.2)] the sum of the above ${}_2\Phi_1$ is $(q^{m-n+1}; q)_{n-m} / (abcdq^{2m}; q)_{n-m}$, which vanishes if $n > m$. Thus we have proved

$$J_{m,n}(a,b,c,d) = \delta_{m,n} / g_n, \tag{6.5}$$

where

$$g_n := \frac{(ab, ac, ad; q)_n (abcd; q)_{2n}}{(q, bc, bd, cd; q)_n (abcdq^{n-1}; q)_n a^{2n}},$$

and after a simple calculation we find that $g_n = h_n$ (of (5.9)) and the proof of the orthogonality relation (5.9) is now complete.

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