Canad. Math. Bull.Vol. **36** (3), 1993 pp. 314–323

REPRESENTATIONS OF COMPACT RIGHT TOPOLOGICAL GROUPS

PAUL MILNES

ABSTRACT. Compact right topological groups arise naturally as the enveloping semigroups of distal flows. Recently, John Pym and the author established the existence of Haar measure μ on such groups, which invites the consideration of the regular representations. We start here by characterizing the continuous representations of a compact right topological group G, and are led to the conclusion that the right regular representation r is not continuous (unless G is topological). The domain of the left regular representation l is generally taken to be the topological centre

$$\Lambda(G) := \{s \in G \mid t \longmapsto ts, G \longrightarrow G, \text{ is continuous}\}$$

or a tractable subgroup of it, furnished with a topology stronger than the relative topology from G (the goals being to have l both defined and continuous). An analysis of l and r on $H = L^2(G)$ for some non-topological compact right topological groups G shows, among other things, that:

(i) for the simplest (perhaps) G generated by $\mathbb{Z}$, (l, H) decomposes into one copy of each irreducible representation of $\mathbb{Z}$ and c copies of the regular representation.

(ii) for the simplest (perhaps) G generated by the euclidean group of the plane $\mathbb{T} \times \mathbb{C}$, (l, H) decomposes into one copy of each of the continuous one-dimensional representations of $\mathbb{T} \times \mathbb{C}$ and c copies of each continuous irreducible representation U^a, $a > 0$.

(iii) when $\Lambda(G)$ is not dense in G, it can seem very reasonable to regard r as a continuous representation of a related compact topological group, and also, G can be almost completely "lost" in the measure space (G, μ).

Preliminaries. A *flow* (S, X) consists of a compact Hausdorff space X and a group S with identity e; each $s \in S$ determines a homeomorphism $x \longmapsto sx$ of X and the conditions $ex = x$ and $s\big(t(x)\big) = (st)x$ for all $s, t \in S$ and $x \in X$ are satisfied. So, S determines a subgroup (denoted here also by S) of the semigroup X^X of all transformations of X. The closure S^- of S in X^X is a subsemigroup of X^X called the *enveloping semigroup* of the flow. With the relative topology from X^X, S^- is a compact *right topological* semigroup, *i.e.*, for all $\eta \in S^-$, right multiplication by η, $\nu \longmapsto \nu\eta$, $S^- \longrightarrow S^-$, is continuous. The set

$$\Lambda(S^-) := \{\eta \in S^- \mid \nu \longmapsto \eta\nu, S^- \longrightarrow S^- \text{ is continuous}\}$$

is called the *topological centre* of S^-; here $S \subset \Lambda(S^-)$, so $\Lambda(S^-)$ is dense in S^-. The flow is called *distal* if $s_\alpha x_1 \longrightarrow x_0$ and $s_\alpha x_2 \longrightarrow x_0$ for net $\{s_\alpha\} \subset S$ and $x_0, x_1, x_2 \in X$ always implies $x_1 = x_2$. We quote a beautiful theorem of Ellis [5, or 6].

This research was supported in part by NSERC grant A7857.

Received by the editors February 14, 1992.

AMS subject classification: 22D10.

Key words and phrases: compact right topological group, Haar measure, regular representation.

https://doi.org/10.4153/CMB-1993-044-1 Published online by Cambridge University Press

1. THEOREM. *A flow* (S,X) *is distal if and only if its enveloping semigroup* S^- *is a group (i.e., a subgroup of* X^X*).*

For a distal flow (S,X), the compact right topological group $G := S^-$ is called the *Ellis group* of the flow. There is a powerful structure theorem for compact groups G that come from topological dynamics like this (*i.e.*, have dense topological centres); it developed over an extended period [7, 6, 14], the claim about normality of each subgroup L_ξ in G (rather than just in $L_{\xi-1}$ for successor ordinals ξ) having been established only recently [12, 13].

2. THE FURSTENBERG-ELLIS-NAMIOKA STRUCTURE THEOREM. *Let* G_1 *be a compact right topological group and suppose that*

$$(*) \qquad \Lambda(G_1) \text{ is dense in } G_1.$$

Let G be a closed subgroup of G_1. *Then G has a system of subgroups*

$$\{L_\xi \mid 0 \le \xi \le \xi_0\}$$

ordered by the set of ordinals less than or equal to an ordinal ξ_0 *and satisfying*

(i) *each* L_ξ *is a closed normal subgroup of G,* $L_0 = G$, *and* $L_{\xi_0} = \{e\}$;

(ii) *for* $\xi < \xi_0$, $L_\xi \supset L_{\xi+1}$, *and the function*

$$(sL_{\xi+1}, tL_{\xi+1}) \mapsto stL_{\xi+1}, \quad G/L_{\xi+1} \times L_\xi/L_{\xi+1} \longrightarrow G/L_{\xi+1}$$

is continuous for the quotient topologies; and

(iii) *for each limit ordinal* $\xi \le \xi_0$, $L_\xi = \bigcap_{\eta<\xi} L_\eta$.

3. REMARKS. (i) The proof of the theorem shows that every compact right topological group G has a smallest compact normal subgroup K such that G/K is a compact topological group; this conclusion does not depend on the hypothesis $(*)$. The important conclusion that does depend on $(*)$ is $K \neq G$; indeed, in Example (e) (below) $K = G$, and G does not have a system of subgroups as in the theorem. The situation is different for Example (d); it has a system of subgroups as in the theorem, but the proof of the theorem does not produce it.

(ii) The system of subgroups in the conclusion of Theorem 2 can be used to establish the existence of Haar measure μ for groups G satisfying the hypotheses of the theorem. μ is the unique probability measure on G that is invariant under all right translations; it is also invariant under all continuous left translations [12, 13]. Example (e) shows that the existence of the system of subgroups is not a necessary condition for the existence of Haar measure. (We mention that Haar measure exists for all compact right topological groups we have encountered.) When μ exists, we have the Banach spaces $L^p(G) := L^p(G,\mu)$ as usual. However, $L^1(G)$ is unlikely to be an algebra, since the definition of convolution $f*$

https://doi.org/10.4153/CMB-1993-044-1 Published online by Cambridge University Press

g requires some strong condition (*e.g.*, separate continuity) on multiplication m: $(s,t) \longmapsto st$; but m is often not even measurable.

Representations. Let G be a compact right topological group, and let (π, H) be a representation of G, π is a homomorphism of G into the group $\mathcal{U} = \mathcal{U}(H)$ of unitary operators on a Hilbert space H. We note first that weak (operator) continuity of π is equivalent to strong continuity of π. This is proved in the same way as for topological groups. So, by "continuity of π", we mean continuity in either of these senses; in particular, the continuity of π requires each coefficient $s \longmapsto \big(\pi(s)\eta, \nu\big)$, $\eta, \nu \in H$, to be continuous. Since inversion can be involved by such a trivial maneuver, *i.e.*, $\big(\pi(s)\eta, \nu\big) = \big(\eta, \pi(s^{-1})\nu\big)$, and since inversion in G is not continuous, one sees that continuity of π may be a rare phenomenon. As for topological groups, a continuous representation π of G is a direct sum of irreducible finite dimensional representations. This is a consequence of the fact that multiplication in $\mathcal{U}$ is separately continuous for the weak operator topology, so in that topology the image $\pi(G)$ is a compact Hausdorff group with separately continuous multiplication, *i.e.*, $\pi(G)$ is a compact topological group by Ellis' theorem [4]. The result for compact topological groups [3; 15.1.3] now gives the desired conclusion. This line of arguing also tells us that π factors through G/K, where K is as in Remark 3(i). So, a faithful representation cannot be continuous (unless G is topological), and the only continuous representation of the group in Example (e) is the trivial one.

Let (π, H) be a representation of a compact topological group G' by uniformly bounded operators on a Hilbert space H. It is proved in [9; p. 162] that there is an equivalent scalar product on H, for which π is a unitary representation. The proof does not work for compact right topological groups, and so it is conceivable that such a group has a uniformly bounded representation that is not equivalent to a unitary representation. We do not know an example.

We now turn our attention to compact right topological groups G with Haar measure μ, and consider the regular representations of G on $H = L^2(G)$. Because of the asymmetry of continuity of the multiplication in G, the left and right regular representations have to be treated separately.

The right regular representation, r: $s \longmapsto R_s$, where $R_s f(t) = f(ts)$ for $s \in G$ and $f \in H$, is a faithful representation of G, so it cannot be continuous (unless G is topological). For the left regular representation, l: $s \longmapsto L_{s^{-1}}$, $L_{s^{-1}} f(t) = f(s^{-1}t)$ for $f \in H$, the domain of l cannot be all of G, since the discontinuous left translations may fail to be measurable. One choice for the domain of l is $\Lambda(G)$. It seems that l: $\Lambda(G) \longrightarrow \mathcal{U}(H)$ is seldom continuous if $\Lambda(G)$ has the relative topology from G. In the setting where G comes from a distal flow (S, X) and we have

$$S \longrightarrow \Lambda(G) \subset G = S^{-} \subset X^X,$$

we can, and generally shall, consider S to be the domain of l; viewed like this, l is continuous at least if the map from S into $\Lambda(G)$ is continuous and S is first countable (Lebesgue dominated convergence theorem).

https://doi.org/10.4153/CMB-1993-044-1 Published online by Cambridge University Press

Examples.

EXAMPLE (a) [15, OR 2; 1.3.40]. Let $\mathbb{T}$ be the circle group, and let E be the set of all endomorphisms of $\mathbb{T}$, $E = \mathbb{T}_d{}^\wedge \cong \mathbb{Z}^{\mathcal{AP}}$, the almost periodic compactification of the integers $\mathbb{Z}$ [1, or 2; 4.3.18]. (A general reference for abelian harmonic analysis is [8].) Then $G = \mathbb{T} \times E$ with multiplication

$$(w,h)(w_1,h_1) = \big(ww_1 h \circ h_1(e^{2i}), hh_1\big)$$

is a compact right topological group with

$$\Lambda(G) = \{(w,h) \mid h = (\,)^n \text{ for some } n \in \mathbb{Z}\}$$

($(\,)^n\colon \mathbb{T} \longrightarrow \mathbb{T}$ being the character $w \longmapsto w^n$). $\Lambda(G) = Z(G)$, the algebraic centre of G; also, $\Lambda(G)$ is dense in G, as is the copy of the integers $\big\{\big(e^{in^2}, (\,)^n\big) \mid n \in \mathbb{Z}\big\} \subset \Lambda(G)$ [15]. Haar measure μ on G is given by

$$\int_G f\,d\mu = \iint f(w,h)\,dw\,dh,$$

just integration with respect to the product of the Haar measures on the compact topological groups $\mathbb{T}$ and E, which is the same as the Haar measure on the compact, abelian, direct product, topological group $G' = \mathbb{T} \times E$. Thus, an orthonormal basis for

$$H = L^2(G) \quad \big(= L^2(G')\big)$$

is given by $\hat{G}' = \hat{\mathbb{T}} \times \hat{E} = \mathbb{Z} \times \mathbb{T}$, $(n_1, v_1) \in \mathbb{Z} \times \mathbb{T}$ corresponding to

$$f_1 \in H, \quad f_1(w,h) = w^{n_1} h(v_1);$$

the orthogonality comes from the fact that the integral of a character χ over a compact abelian topological group is zero unless $\chi = 1$, the trivial character, in which case the integral is 1.

Considering the right regular representation $r\colon (w_1,h_1) \longmapsto R_{(w_1,h_1)}$ of G, we have

$$R_{(w_1,h_1)} f_1(w,h) = \big(ww_1 h \circ h_1(e^{2i})\big)^{n_1} (hh_1)(v_1),$$

and if $f_2 \sim (n_2, v_2) \in \mathbb{Z} \times \mathbb{T}$, then

$$|(R_{(w_1,h_1)} f_1, f_2)| = \left|\iint w^{n_1} w_1^{n_1} h\big(h_1(e^{2in_1})\big) h(v_1) h_1(v_1) \overline{w^{n_2} h(v_2)}\,dw\,dh\right|$$
$$= \begin{cases} 1 & \text{if and only if } n_1 = n_2 \text{ and } h_1(e^{2in_1})v_1 = v_2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus r decomposes into subrepresentations on

$$H = \Big(\bigoplus_{v \in \mathbb{T}} \mathbb{C}(0,v)\Big) \oplus \Big(\bigoplus_{n \neq 0} \overline{\mathrm{sp}}(n,\mathbb{T})\Big).$$

(Here $\overline{\mathrm{sp}}(n,\mathbb{T})$ denotes the closed linear span in H of $(n,\mathbb{T}) := \{(n,v) \mid v \in \mathbb{T}\}$.) The subrepresentations on the one-dimensional subspaces $\mathbb{C}(0,v)$ are continuous; they factor through $G/K \cong E$, where $K = \mathbb{T} \times \{1\} \cong \mathbb{T}$ is the normal subgroup of G as in Remark 3(i). For fixed $n \neq 0$, set $[v] = (n, v)$. Then the subrepresentation of r on $\overline{\text{sp}}\,(n, \mathbb{T})$ is given by

$$R_{(w_1,h_1)}[v] = w_1^n h_1(v)[h_1(e^{2in})v];$$

it is induced from the representation $(w, 1) \longmapsto w^n$ of the subgroup $\mathbb{T} \times \{1\} \subset G$ (since $L_{(w_0,1)}[v] = w_0^n[v]$). With $E_n := \{h_1 \in E \mid h_1(e^{2in}) = 1\}$, the map $h_1 \longmapsto R_{(1,h_1)}[v]$ injects $E/E_n \cong \mathbb{T}$ onto an orthonormal basis for $\overline{\text{sp}}(n, \mathbb{T})$. These subrepresentations of r are not continuous.

The left regular representation l is not continuous on $\Lambda(G)$ or even on

$$\mathbb{Z} \cong \left\{ \left(e^{in^2}, (\ )^n\right) \mid n \in \mathbb{Z} \right\} \subset \Lambda(G),$$

if these groups are given the relative topology from G. So, let $\mathbb{Z}$ have the discrete topology and consider $l: \mathbb{Z} \longrightarrow \mathcal{U}(H)$. As above for fixed $n \neq 0$, let

$$[v] = (n, v) \in (n, \mathbb{T}) \subset \mathbb{Z} \times \mathbb{T} = \hat{\mathbb{T}} \times \hat{E}.$$

Then the map

$$m \longmapsto l(m)[v] = L_{(e^{im^2},(\ )^m)^{-1}}[v] = R_{(e^{im^2},(\ )^{-m})}[v]$$

injects $\mathbb{Z}$ onto an orthonormal set with closed linear span $J \subset \overline{\text{sp}}(n, \mathbb{T})$, say. The subrepresentation $m \longmapsto l(m)|_J$ is just (isomorphic to) the regular representation of $\mathbb{Z}$, and the representation (l, H) of $\mathbb{Z}$ is a direct product of c copies of it along with the c continuous one-dimensional representations.

EXAMPLE (b) [2; 1.3.40]. Let $G = \mathbb{T} \times \mathbb{T} \times E$ with ($\mathbb{T}$ and E as in Example (a) and) multiplication

$$(v, w, h)(v_1, w_1, h_1) = \big(vv_1 h(w_1), ww_1, hh_1\big),$$

a compact right topological group. $\Lambda(G) = \mathbb{T} \times \mathbb{T} \times \mathbb{Z}$ is a Heisenberg group,

$$(v, w, n)(v_1, w_1, n_1) = (vv_1 w_1^n, ww_1, n + n_1),$$

and $\Lambda(G)^- = G$. As for Example (a), Haar measure μ on G is just the product of the Haar measures on the component compact abelian topological groups, $d\mu = dv\,dw\,dh$, so an orthonormal basis for $H = L^2(G)$ is given by the members of

$$\hat{\mathbb{T}} \times \hat{\mathbb{T}} \times \hat{E} = \mathbb{Z} \times \mathbb{Z} \times \mathbb{T},$$

$(k_1, m_1, u_1) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{T}$ corresponding to $f_1 \in H, f_1(v, w, h) = v^{k_1} w^{m_1} h(u_1)$.

Considering the right regular representation r, we get

$$R_{(v_1,w_1,h_1)} f_1(v, w, h) = \big(vv_1 h(w_1)\big)^{k_1} (ww_1)^{m_1} hh_1(u_1) = g_1(v, w, h),$$

say, and if $f_2 \sim (k_2, m_2, u_2) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{T}$, then

$$|(g_1, f_2)| = \left| \iiint g_1(v, w, h) \overline{v^{k_2} w^{m_2} h(u_2)}\, dv\, dw\, dh \right|$$
$$= \begin{cases} 1 & \text{if and only if } k_1 = k_2,\ m_1 = m_2, \text{ and } w_1^{k_1} u_1 = u_2, \\ 0, & \text{otherwise.} \end{cases}$$

https://doi.org/10.4153/CMB-1993-044-1 Published online by Cambridge University Press

So, r decomposes into subrepresentations on

$$H = \left(\bigoplus\{\mathbb{C}(0,m,u) \mid m \in \mathbb{Z}, u \in \mathbb{T}\}\right) \oplus \left(\bigoplus\{\overline{\mathrm{sp}}(k,m,\mathbb{T}) \mid k \neq 0, m \in \mathbb{Z}\}\right).$$

The subrepresentations on the one-dimensional subspaces $\mathbb{C}(0,m,u)$ are continuous; they factor through the abelian quotient group $G/K \cong \mathbb{T} \times E$, where $K = \mathbb{T} \times \{1\} \times \{1\} \cong \mathbb{T}$ is the normal subgroup of G as in Example 3(i). For fixed $k \neq 0$ and $m \in \mathbb{Z}$, set

$$[u] = (k,m,u) \in (k,m,\mathbb{T}) \subset \mathbb{Z} \times \mathbb{Z} \times \mathbb{T}.$$

Then the subrepresentation of r on $\overline{\mathrm{sp}}(k,m,\mathbb{T})$ is induced from the representation

$$(v,w,1) \longmapsto v^k w^m$$

of the abelian normal subgroup $\mathbb{T} \times \mathbb{T} \times \{1\} \cong \mathbb{T}^2$ of G (since $L_{(v_0,w_0,1)}[u] = v_0^k w_0^m[u]$). Also, the map

$$w_1 \longmapsto R_{(1,w_1,1)}[u] = w_1^m[w_1^k u]$$

injects $\{w_1 = e^{i\theta} \mid 0 \leq \theta < 2\pi/k\}$ onto an orthonormal basis for $\overline{\mathrm{sp}}(k,m,\mathbb{T})$. These subrepresentations of r are not continuous.

The decomposition of $H = L^2(G)$ for the left regular representation $l\colon \mathbb{T} \times \mathbb{T} \times \mathbb{Z} \longrightarrow \mathcal{U}(H)$ is quite different from that for $r\colon G \longmapsto \mathcal{U}(H)$ (partly because $\mathbb{T} \times \mathbb{T} \times \mathbb{Z}$ is not abelian). For $f_i \sim (k_i, m_i, u_i)$, $i = 1,2$, as above, we have

$$L_{(v_1,w_1,n_1)} f_1(v,w,h) = (v_1 v w^{n_1})^{k_1}(w_1 w)^{m_1} u_1^{n_1} h(u_1) = g_1(v,w,h),$$

say, and

$$\begin{aligned} |(g_1,f_2)| &= \left|\iiint g_1(v,w,h)\overline{v^{k_2}w^{m_2}h(u_2)}\,dv\,dw\,dh\right| \\ &= \begin{cases} 1 & \text{if and only if } k_1 = k_2,\ n_1k_1 + m_1 = m_2, \text{ and } u_1 = u_2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The decomposition of H for l is $H =$

$$\left(\bigoplus\{\mathbb{C}(0,m,u) \mid m \in \mathbb{Z}, u \in \mathbb{T}\}\right) \oplus \left(\bigoplus\left\{\bigoplus_{m=0}^{k-1} \overline{\mathrm{sp}}\{(k, nk+m, u) \mid n \in \mathbb{Z}\} \mid k \in \mathbb{N}, u \in \mathbb{T}\right\}\right).$$

l is not continuous if $\mathbb{T} \times \mathbb{T} \times \mathbb{Z}$ has the relative topology from G, but is continuous if $\mathbb{T} \times \mathbb{T} \times \mathbb{Z}$ has its usual topology.

It seems interesting to note that $\varphi\colon (w,h) \longmapsto (w, h(e^{2i}), h)$ is a continuous isomorphism of the group $\mathbb{T} \times E$ of Example (a) into $G = \mathbb{T} \times \mathbb{T} \times E$. The image $\varphi(\mathbb{T} \times E)$ is normal in $\mathbb{T} \times \mathbb{T} \times E$, which is therefore an extension of $\varphi(\mathbb{T} \times E)$ by $(\mathbb{T} \times \mathbb{T} \times E)/\varphi(\mathbb{T} \times E) \cong \mathbb{T}$.

EXAMPLE (c). Let $\mathbb{C}$ be the complex numbers with dual group $\hat{\mathbb{C}} \cong \mathbb{C}$ and almost periodic compactification $\mathbb{C}^{\mathcal{AP}} \cong \mathbb{C}_d{}^{\wedge}$. The canonical map $\mathbb{C} \longrightarrow \mathbb{C}^{\mathcal{AP}}$ sends z to the character $\zeta \longmapsto e^{2\pi i \mathrm{Re}(z\zeta)}$; we will often identify $z \in \mathbb{C}$ with its image in $\mathbb{C}^{\mathcal{AP}}$. Then $G = \mathbb{T} \times \mathbb{C}^{\mathcal{AP}}$ with multiplication

$$(w,h)(w_1,h_1) = (ww_1, R_{w_1}hh_1)$$

https://doi.org/10.4153/CMB-1993-044-1 Published online by Cambridge University Press

is a compact right topological group. If $S = \mathbb{T} \times \mathbb{C}$ is the euclidean group of the plane with multiplication

$$(w, z)(w_1, z_1) = (ww_1, zw_1 + z_1),$$

the map ψ: $(w, z) \longmapsto (w, e^{2\pi i \mathrm{Re}(z\cdot)})$ is a continuous isomorphism of S onto $\Lambda(G)$. (We note that G may be viewed as a subgroup of $\mathbb{T} \times \mathbb{T}^{\mathbb{T}}$ (as in [2; 1.3.40]), since a character in $\mathbb{C}^{\mathcal{AP}}$ is completely determined by its restriction to $\mathbb{T}$.)

Haar measure μ on G is the product of the Haar measures on the component compact abelian topological groups, $d\mu = dw\, dh$. (We don't know if μ is uniquely determined by left translation invariance here; it is for Examples (a) and (b).) An orthonormal basis for $H = L^2(G)$ is given by the members of $\hat{\mathbb{T}} \times (\mathbb{C}^{\mathcal{AP}})^\wedge = \mathbb{Z} \times \mathbb{C}$, $(n_1, z_1) \in \mathbb{Z} \times \mathbb{C}$ corresponding to $f_1 \in H, f_1(w, h) = w^{n_1} h(z_1)$.

Considering the right regular representation r, we have

$$R_{(w_1,h_1)} f_1(w, h) = (ww_1)^{n_1} R_{w_1} h(z_1) h_1(z_1) = g_1(w, h),$$

say, and if $f_2 \sim (n_2, z_2) \in \mathbb{Z} \times \mathbb{C}$, then

$$|(g_1, f_2)| = \left| \iint w^{n_1} w_1^{n_1} h(z_1 w_1) h_1(z_1) \overline{w^{n_2} h(z_2)}\, dw\, dh \right|$$
$$= \begin{cases} 1 & \text{if and only if } n_1 = n_2, \text{ and } z_1 w_1 = z_2, \\ 0, & \text{otherwise.} \end{cases}$$

So, r decomposes into subrepresentations on

$$H = \Big(\bigoplus_{n \in \mathbb{Z}} \mathbb{C}(n, 0) \Big) \oplus \Big(\bigoplus \{ \overline{\mathrm{sp}}(n, a\mathbb{T}) \mid n \in \mathbb{Z}, a > 0 \} \Big).$$

The subrepresentations on the one-dimensional subspaces $\mathbb{C}(n, 0)$ are continuous; they factor through $G/K \cong \mathbb{T}$, where $K = \{1\} \times \mathbb{C}^{\mathcal{AP}}$ is the normal subgroup of G as in Remark 3(i). Fix $n \in \mathbb{Z}$ and $a > 0$, and set $[v] = (n, av)$ for $v \in \mathbb{T}$. Then the subrepresentation of r on $\overline{\mathrm{sp}}(n, a\mathbb{T})$ is given by

$$R_{(w_1,h_1)}[v] = w_1^n h_1(av)[vw_1],$$

and is induced from the representation $(w, 1) \longmapsto w^n$ of the subgroup $\mathbb{T} \times \{1\} \subset G$ (since $L_{(w_0,1)}[v] = w_0^n[v]$). These subrepresentations of r are not continuous.

The left regular representation l: $\mathbb{T} \times \mathbb{C} \longrightarrow \mathcal{U}(H)$ is continuous, and the decomposition of H for it is quite different from that for r (as in Example (b)). We still have continuous one-dimensional subrepresentations of l on each $\mathbb{C}(n, 0)$. Now fix $z \neq 0$ and consider the subspace

$$H_z = \overline{\mathrm{sp}}\{(n, z) \mid n \in \mathbb{Z}\} = L^2(\mathbb{T}) \otimes \{z\} \subset H.$$

If $f_1 \in L^2(\mathbb{T})$ and $F_1 = f_1 \otimes z \in H_z$, then $F_1(w, h) = f_1(w) h(z)$ and

$$L_{(w_1,z_1)} F(w, h) = f(w_1 w) e^{2\pi i \mathrm{Re}(z_1 wz)} h(z).$$

Thus $L_{(w_1,z_1)}F_1 \in H_z$, and if $F_2 = f_2 \otimes z \in H_z$ for $f_2 \in L^2(\mathbb{T})$, we have

$$(L_{(w_1,z_1)}F_1, F_2) = \iint L_{(w_1,z_1)}F_1(w,h)\overline{F_2(w,h)}\,dw\,dh = \int f(w_1 w)e^{2\pi i \mathrm{Re}(z_1 wz)}\overline{f_2(w)}\,dw.$$

This makes it clear that the subrepresentation (l, H_z) of $S = \mathbb{T} \times \mathbb{C}$ is unitarily equivalent to the irreducible representation $U^{|z|}$ of S on $L^2(\mathbb{T})$ [17; p. 153]. (To verify the details of this, bear in mind that $l(w_1, z_1) = L_{(w_1,z_1)^{-1}}$, that $(w, z) \longmapsto (\overline{w}, \overline{z})$ is an automorphism of $\mathbb{T} \times \mathbb{C}$, and that the euclidean group of the plane is displayed in [17] as $\mathbb{C} \times \mathbb{T}$,

$$(z', w')(z, w) = (z' + w'z, w'w).\,)$$

Also, the subrepresentation (l, H_z) may be regarded as induced from the representation $(1, z_1) \longmapsto e^{2\pi i \mathrm{Re}(z_1 z)}$ of the subgroup $\{1\} \times \mathbb{C} \subset \mathbb{T} \times \mathbb{C}$ [17].

In summary, (l, H) is a direct product of all the continuous irreducible representations of $S = \mathbb{T} \times \mathbb{C}$; it contains one copy of each of the one dimensional representations and c copies of each of the representations U^a, $0 < a \in \mathbb{R}$. We point out that, as in [17; p. 153, and 11], Bessel functions arise in this context. For example, if $(0, z) \in \mathbb{Z} \times \mathbb{C}$ with $z \neq 0$ corresponds to $f \in H_z, f(w, h) = h(z)$ (as above), and if we write $w = e^{i\theta}$, then

$$\begin{aligned}\mathfrak{F}(w_1, z_1) &:= (L_{(w_1,z_1)}f, f) = \iint f(w_1 w, e^{2\pi i \mathrm{Re}(z_1 w\,\cdot\,)}h)\overline{f(w,h)}\,dw\,dh \\ &= \iint e^{2\pi i \mathrm{Re}(z_1 wz)}h(z)\overline{h(z)}\,dw\,dh = \frac{1}{2\pi}\int_0^{2\pi} e^{2\pi i |z_1 z| \cos\theta}\,d\theta = J_0(|z_1 z|),\end{aligned}$$

J_0 being the Bessel function of order 0. Note that $\mathfrak{F}(w_1, z_1)$ is independent of w_1 and the arguments of z and z_1.

The last two examples are quite unlike the previous ones in that $\Lambda(G)$ is no longer dense in G; the reader will see what a difference this makes.

EXAMPLE (d). For this example, we need a discontinuous automorphism φ of $\mathbb{T}$ satisfying $\varphi^2 = 1$. (Regard $\mathbb{T}$ as the direct sum of the subgroup

$$\{w \in \mathbb{T} \mid w^n = 1 \text{ for some } n \in \mathbb{N}\}$$

and c copies of $\mathbb{Q}$; then a suitable φ just interchanges the coordinates of two fixed copies of $\mathbb{Q}$. A property of φ is that, for any $(v_1, v_2) \in \mathbb{T} \times \mathbb{T}$, there is a sequence $\{u_n\} \subset \mathbb{T}$ with $u_n \longrightarrow v_1$ and $\varphi(u_n) \longrightarrow v_2$. See [16, or 10] for more details.) Then $G := \mathbb{T} \times \{1, \varphi\}$ with multiplication

$$(u, \epsilon)(v, \delta) = \big(u\epsilon(v), \epsilon\delta\big)$$

is a compact right topological group. Here $\Lambda(G) = \mathbb{T} \times \{1\}$, so G does not satisfy the hypotheses of the structure theorem (Theorem 2). Nonetheless, G has a system of subgroups as in the conclusion of the structure theorem, and Haar measure μ on G is just Lebesgue measure on $\mathbb{T}$, divided by two, on each of $\mathbb{T} \times \{1\}$ and $\mathbb{T} \times \{\varphi\}$. We need some notation to discuss $r\colon G \longrightarrow \mathcal{U}(H)$ for $H = L^2(G)$. Let $f_\gamma \in H$ denote the function $f \in L^2(\mathbb{T})$ supported on $\mathbb{T} \times \{\gamma\} \subset G$. Then

$$R_{(v,\delta)}f_\gamma(u, \epsilon) = f_\gamma\big(u\epsilon(v), \epsilon\delta\big) = (R_{\gamma\delta(v)}f)_{\gamma\delta}(u, \epsilon),$$

since the middle term can be different from 0 only if $\epsilon\delta = \gamma$, *i.e.*, $\epsilon = \gamma\delta$. So,

$$(v, 1) \longmapsto R_{(v,1)}f_\varphi = (R_{\varphi(v)}f)_\varphi \text{ and } (v, \varphi) \longmapsto R_{(v,\varphi)}f_1 = (R_{\varphi(v)}f)_\varphi$$

are not continuous (unless $f = 0$).

One can hardly be surprised by these discontinuities, given the discontinuity in the definition of the group. However, this representation can be viewed as a continuous representation of a related compact topological group, the semidirect product

$$G_1 = (\mathbb{T} \times \mathbb{T}) \times \{1, \varphi\}$$

with multiplication $(u_1, u_\varphi, \epsilon)(v_1, v_\varphi, \delta) = (u_1 v_\epsilon, u_\varphi v_{\epsilon\varphi}, \epsilon\delta)$. (It seems best for notation to let $\{1, \varphi\}$ denote the two element group for G_1, as well as for G; for G_1, φ is the automorphism of $\mathbb{T} \times \mathbb{T}$ that interchanges coordinates. See [16, or 10] for how one arrives at the compact topological group G_1 from the compact right topological group G.) The map

$$\theta: (v, \delta) \longmapsto \big(v, \varphi(v), \delta\big)$$

is a discontinuous isomorphism of G onto a dense subgroup of G_1. Also,

$$\pi: G_1 \longrightarrow \mathcal{U}(H), \quad \pi(v_1, v_\varphi, \delta)f_\gamma = (R_{v_{\gamma\delta}}f)_{\gamma\delta},$$

is a continuous representation of G_1 and $r = \pi \circ \theta: G \longrightarrow G_1 \longrightarrow \mathcal{U}(H)$.

The last example is perhaps even more striking. Not only does it seem more reasonable to think of r as a continuous representation of a related compact topological group, but the measure space (G, μ) can be simplified beyond recognition.

EXAMPLE (e) [12]. Let G be the semidirect product $\{\pm 1\} \times \mathbb{T}$ with multiplication

$$(\epsilon, u)(\delta, v) = (\epsilon\delta, u^\delta v).$$

Give G the topology for which a typical basic neighbourhood of $(1, e^{ia})$ or $(-1, e^{ib})$, where $a < b$, is

$$A := \{(1, e^{ia}), (-1, e^{ib})\} \cup \{(\epsilon, e^{i\theta}) \mid \epsilon = \pm 1, a < \theta < b\};$$

these basic neighbourhoods are open and closed.

(G, τ) is a compact, Hausdorff, right topological group and $\Lambda(G)$ is trivial, consisting only of $(1, 1)$, the identity of G. Thus G does not satisfy the hypotheses of the structure theorem; furthermore, G does not have a system of subgroups as in the conclusion of that theorem. Nonetheless, G has a (unique) Haar measure. For, a right invariant probability measure μ on G must assign measure $\min\{1, (b-a)/2\pi\}$ to the basic neighbourhood A. Also, every open set $B \subset G$ is the union of a countable number of sets of the form A. (The argument for this claim goes hardly beyond that required for the analogous claim about open sets of real numbers.) If we identify $\mathbb{T}_1 := \{1\} \times \mathbb{T}$ and $\mathbb{T}_2 := \{-1\} \times \mathbb{T}$ with $\mathbb{T}$ in the obvious way, it follows that the symmetric difference $(B \cap \mathbb{T}_1) \triangle (B \cap \mathbb{T}_2)$ is countable. Accordingly, we must have $\mu(B) = \eta(B \cap \mathbb{T}_1)$ for all open $B \subset G$, and hence, by regularity, for all Borel $B \subset G$. The equation in the last line can be used to define Haar measure μ on G in terms of η.

From the observations of the last paragraph, we may conclude that the map $(\epsilon, u) \longmapsto u$, while not one-to-one, effects an isomorphism between the measure spaces (G, μ) and $(\mathbb{T}, \eta)$, so $H := L^2(G) \cong L^2(\mathbb{T})$. To understand $r: G \longrightarrow \mathcal{U}(H)$, we identify $f \in H$ with $F \in L^2(\mathbb{T})$, $F(u) = f(\epsilon, u)$ almost everywhere. Then $R_{(1,v)}f \sim R_v F$ and $R_{(-1,v)}f \sim R_v \check{F}$, where $\check{F}(u) = F(u^{-1})$. Of course, r is not continuous, but it is continuous if G has the product topology (for which G is a compact topological group).

The examples lead us to close with a question: can one recover the topology of G, or some vestige of it, from the image $r(G) \subset \mathcal{U}(H)$?

REFERENCES

1. H. Anzai and S. Kakutani, *Bohr compactifications of a locally compact abelian group I, II*, Proc. Imp. Acad. Tokyo **19**(1943), 476–480, 533–539.

2. J. F. Berglund, H. D. Junghenn and P. Milnes, *Analysis on Semigroups: Function Spaces, Compactifications, Representations*, Wiley, New York, 1989.

3. J. Dixmier, *Les C^*-algèbras et leurs représentations*, Gauthier-Villars, Paris, 1964.

4. R. Ellis, *Locally compact transformation groups*, Duke Math. J. **24**(1957), 119–126.

5. ———, *Distal transformation groups*, Pacific J. Math. **9**(1958), 401–405.

6. ———, *Lectures on Topological Dynamics*, Benjamin, New York, 1969.

7. H. Furstenberg, *The structure of distal flows*, Amer. J. Math. **85**(1963), 477–515.

8. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, New York, 1963.

9. L. H. Loomis, *An Introduction to Abstract Harmonic Analysis*, Van Nostrand, New York, 1953.

10. P. Milnes, *Distal compact right topological groups* Acta Math. Hung., to appear..

11. P. Milnes and A. L. T. Paterson, *Ergodic sequences and a subspace of $B(G)$*, Rocky Mountain J. Math. **18**(1988), 681–694.

12. P. Milnes and J. Pym, *Haar measure for compact right topological groups*, Proc. Amer. Math. Soc., **114**(1992), 387–393.

13. ———, *Homomorphisms of minimal and distal flows*, to appear.

14. I. Namioka, *Right topological groups, distal flows and a fixed point theorem*, Math. Systems Theory **6**(1972), 193–209.

15. ———, *Ellis groups and compact right topological groups*. In: Contemporary Mathematics, Conference in Modern Analysis and Probability, Amer. Math. Soc. **26**(1984), 295–300.

16. W. Ruppert, *Über kompakte rechtstopologische Gruppen mit gleichgradig stetigen Linkstranslationen*, Sitzungsberichten der Österreichischen Akademie der Wissenschaften Mathem.-naturw. Klasse, Abteilung II **184**(1975), 159–169.

17. M. Sigiura, *Unitary Representations and Harmonic Analysis*, Wiley, New York, 1975.

University of Western Ontario
London, Ontario
N6A 5B7