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# ON COMPLETE INTEGRAL CLOSURE AND ARCHIMEDEAN VALUATION DOMAINS

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**Abstract**

Suppose $D$ is an integral domain with quotient field $K$ and that $L$ is an extension field of $K$. We show in Theorem 4 that if the complete integral closure of $D$ is an intersection of Archimedean valuation domains on $K$, then the complete integral closure of $D$ in $L$ is an intersection of Archimedean valuation domains on $L$; this answers a question raised by Gilmer and Heinzer in 1965.

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All rings considered in this paper are assumed to be commutative and to contain a unity element. If $R$ is a subring of $S$, we assume that the unity element of $S$ belongs to $R$, and hence is the unity element of $R$.

If $R$ is a subring of $S$ and $s \in S$, then as is well known, $s$ is integral over $R$ if and only if $R[s]$ is a finitely generated $R$-module. On the other hand, if $R[s]$ is contained in a finitely generated $R$-submodule of $S$, then $s$ is said to be *almost integral* over $R$. Clearly $s$ is almost integral over $R$ if it is integral over $R$; the converse holds if $R$ is Noetherian, but not in general. We denote by $C(R, S)$ the set of elements of $S$ that are almost integral over $R$; $C(R, S)$ is a subring of $S$ containing $R$, and is called the *complete integral closure* of $R$ in $S$. If $C(R, S) = S$, we say that $S$ is *almost integral* over $R$, and at the opposite extreme where $C(R, S) = R$, we say that $R$ is *completely integrally closed* in $S$. If $S$ is the total quotient ring of $R$, then $C(R, S)$ is the *complete integral closure* of $R$, and if $C(R, S) = R$, then $R$ is said to be *completely integrally closed.*

The concept of almost integrality was first considered by Krull in his famous 1932 paper *Allgemeine Bewertungstheorie* [5], in which he introduced the notion





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of a general valuation and proved, among other things, that an integral domain is integrally closed if and only if it is an intersection of valuation domains [5, Satz 7]. (More precisely, Krull considers the concept in the case where $R$ is an integral domain and $S$ is the quotient field of $R$. In this case it is easy to show that $s \in S$ is almost integral over $R$ if and only if there exists a nonzero element $r \in R$ such that $rs^k \in R$ for each positive integer $k$.) Krull also proved [5, Satz 8] that a valuation domain $V$ is completely integrally closed if and only if the rank of $V$ is at most 1. (More generally, the complete integral closure of $V$ is either $V_P$ (if $V$ has a height-one prime ideal $P$) or the quotient field $F$ of $V$ (if $V$ has no height-one prime).) This led Krull to conjecture [5, p. 170] that a domain is completely integrally closed if and only if it is an intersection of valuation domains of rank at most 1 — that is, of Archimedean valuation domains. Nakayama disproved Krull's conjecture in [7, 8], and a simpler counterexample due to J. Ohm appears as Example 19.12 of [2]. On the other hand, Krull's conjecture is known to be correct for certain classes of domains; for example, it is valid for Noetherian domains [6, Theorem 33.10] and for domains of finite character [3, Proposition 5], [1, Theorem 5]. This paper answers a question related to the discussion above that arose in work that W. Heinzer and I did in connection with [3]. The question is the following:

(Q) *Suppose $D$ is an integral domain with quotient field $K$, and assume that the complete integral closure of $D$ is an intersection of Archimedean valuation domains on $K$. If $L$ is an extension field of $K$, is the complete integral closure of $D$ in $L$ an intersection of Archimedean valuation domains on $L$?*

We show in Theorem 4 that (Q) has an affirmative answer. Our general reference for results concerning almost integrality is Section 13 of [2]. We first resolve the case of (Q) where $L$ is algebraic over $K$; Lemma 1 and Theorem 2 deal with the algebraic case.

LEMMA 1. *Let $\{D_a\}_{a\in A}$ be a family of integrally closed domains with common quotient field $K$, and assume that $D = \bigcap_{a\in A} D_a$ also has quotient field $K$. Assume that $L$ is an algebraic extension field of $K$, and use bars to denote integral closure in $L$. Then $\overline{D} = \bigcap_{a\in A} \overline{D_a}$.*

PROOF. The inclusion $\overline{D} \subseteq \bigcap_a \overline{D_a}$ is clear. Suppose $t \in \bigcap \overline{D_a}$ and let $f(x)$ be the minimal polynomial for $t$ over $K$. Since $D_a$ is integrally closed for $a \in A$, it is known that $f(x) \in D_a[x]$ [2, Theorem 10.5], and hence $f(x) \in \bigcap_{a\in A} D_a[x] = (\bigcap_a D_a)[x] = D[x]$. Therefore $t \in \overline{D}$, as we wished to show.

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THEOREM 2. *Let $D$ be an integral domain with quotient field $K$ and let $L$ be an algebraic extension field of $K$. If the complete integral closure $E$ of $D$ is an intersection of Archimedean valuation domains on $K$, then the complete integral closure $C(D, L)$ of $D$ in $L$ is an intersection of Archimedean valuation domains on $L$.*

PROOF. Let $\overline{E}$ be the integral closure of $E$ in $L$. Since $E$ is completely integrally closed, $\overline{E}$ is the complete integral closure of $E$ in $L$ and is a completely integrally closed domain [2, Theorem 13.8]. Clearly $C(D, L) \subseteq \overline{E}$, but since $\overline{E}$ is integral over $E$ it follows that $\overline{E}$ is almost integral over $D$ [2, Corollary 13.2], and hence $\overline{E} \subseteq C(D, L)$. Therefore $C(D, L) = \overline{E}$. By hypothesis $E = \bigcap_{a\in A} V_a$, where each $V_a$ is an Archimedean valuation domain on $K$, and Lemma 1 shows that $\overline{E} = \bigcap_a \overline{V_a}$, where bars indicate integral closure in $L$. Now $\overline{V_a}$ is a Prüfer domain [2, Theorem 20.1] and $\dim \overline{V_a} = \dim V_a \leq 1$. Consequently, $\overline{V_a}$ is an intersection of Archimedean valuation domains on $L$, so $\overline{E} = \bigcap_a \overline{V_a}$ is also an intersection of Archimedean valuation domains on $L$. This completes the proof of Theorem 2.

Suppose $R$ is a subring of $S$ and $S$ is a subring of $T$. In contrast with the situation for integrality, almost integrality is not transitive — that is, $T$ need not be almost integral over $R$ even if $T/S$ and $S/R$ are almost integral. For example, if $D$ is an integral domain with quotient field $K$, the complete integral closure $D^{(1)}$ of $D$ need not be completely integrally closed [3, Example 1]; in fact, Hill [4] has given an example of a Bezout domain $D$ such that $D^{(n)} < D^{(n+1)}$ for each positive integer $n$, where $D^{(n+1)}$ is defined inductively as the complete integral closure of $D^{(n)}$. Another contrast between the properties of integrality and almost integrality is that the inclusion $C(R, S) \subseteq C(R, T) \bigcap S$ may be proper [3, Example 2]. However, it follows from part (b) of [3, Proposition 2] that $C(R, S) = C(R, T) \cap S$ if $S$ is a field. We use this equality in the proof of Proposition 3.

PROPOSITION 3. *Suppose $D$ is an integral domain with quotient field $K$ and $T$ is an extension ring of $K$. If $\Delta$ is the integral closure of $K$ in $T$, then $C(D, T) = C(D, \Delta)$.*

PROOF. By the result from [3] just cited, $C(D, \Delta) = C(D, T) \bigcap \Delta$. Hence, to prove Proposition 3, it suffices to show that $C(D, T) \subseteq \Delta$. This is straightforward: if $t \in C(D, T)$ then clearly $t \in C(K, T)$, and since $K$ is Noetherian, $t$ is integral over $K$ — that is, $t \in \Delta$.

THEOREM 4. *Let $D$ be an integral domain with quotient field $K$ and let $L$ be an extension field of $K$. If the complete integral closure of $D$ is an intersection of Archimedean valuation domains on $K$, then the complete integral closure of $D$ in $L$ is an intersection of Archimedean valuation domains on $L$.*

PROOF. Let $\Delta$ be the algebraic closure of $K$ in $L$. Proposition 3 shows that $C(D, L) = C(D, \Delta)$, and Theorem 2 shows that $C(D, \Delta)$ is an intersection of

Archimedean valuation domains on $\Delta$. Hence the proof is complete if $\Delta = L$. If $\Delta < L$, we consider two cases.

CASE 1. $C(D, L) = \Delta$. In this case we show that if $t \in L - \Delta$, then there exists a rank-one valuation domain on $L$ that contains $\Delta$, but not $t$. Thus, since $t^{-1}$ is transcendental over $\Delta$, then $V = \Delta[t^{-1}]_{t^{-1}V[t^{-1}]}$ is a rank-one valuation domain on $\Delta(t)$ that does not contain $t$. The valuation domain $V$ admits an extension to a rank-one valuation domain $W$ on $L$ [2, Proposition 20.5], and $t \notin W$ since $t \notin W \bigcap \Delta(t)$. Therefore $\Delta$ is the intersection of a family of rank-one valuation domains on $L$. This resolves Case 1.

CASE 2. $C(D, L) < \Delta$. In this case there exists a family $\{V_a\}_{a\in A}$ of rank-one valuation domains on $\Delta$ such that $C(D, L) = \bigcap_{a\in A} V_a$. For each $a \in A$, let $W_a$ be a rank-one extension of $V_a$ to $L$ and, by Case 1, let $\{U_b\}_{b\in B}$ be a family of rank-one valuation domains on $L$ such that $\Delta = \bigcap_{b\in B} U_b$. If $\mathscr{F} = \{W_a\}_a \bigcup \{U_b\}_b$, then $\mathscr{F}$ is a family of rank-one valuation domains on $L$ and $\bigcap \mathscr{F} = (\bigcap_a W_a) \cap \Delta = \bigcap_a (W_a \cap \Delta) = \bigcap_a V_a = C(D, L)$.

We remark that the converse of Theorem 4 is also valid: If $C(D, L) = \bigcap_{b\in B} W_b$, where each $W_b$ is an Archimedean valuation domain on $L$, then $C(D, K) = C(D, L) \bigcap K = \bigcap_{b\in B}(W_b \bigcap K)$, where each $W_b \bigcap K$ is an Archimedean valuation domain on $K$ [2, Theorem 19.16]. Thus, for any extension field $L$ of $K$, the condition that $C(D, L)$ is an intersection of Archimedean valuation domains on $L$ depends only upon $D$, not on $L$.

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