GREEN'S FUNCTIONS FOR SINGULAR ORDINARY DIFFERENTIAL OPERATORS

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1. There are several ways to approach the eigenfunction expansion problem for ordinary differential operators via the spectral theorem for self-adjoint linear operators in Hilbert space. One can examine the resolvent, which requires a detailed study of the Green's function (**4**, **5**, **7**), or one can use the spectral theorem for unbounded operators (**2**, **3**, **9**). Since the eigenfunction expansion theorem also requires some multiplicity theory, unless one is prepared to use a rather powerful form of the spectral theorem for unbounded operators, as in (**2**, **9**), the proof requires a good deal of work in addition to the spectral theorem.

This suggests that even though the natural setting of the problem is in terms of linear operators in Hilbert space, the most natural approach to the problem is the elementary one used in (**8**, **10**, **11**). This approach, requiring only the elementary properties of Hilbert space, clarifies the relationship between the spectral theorem and the eigenfunction expansion theorem. The treatment in (**11**) lays the groundwork for the study of Green's function, but does not go into this aspect of the problem. The purpose of this paper is to complete the approach of (**11**) by establishing the properties of the Green's function and its relation to the spectral matrix, without going back to the beginning of the problem as was necessary in (**4**). We also show the relation between self-adjoint boundary conditions and self-adjoint problems in the sense of (**8**, Chapter 7), and establish an integral expansion for the Green's function.

There should be no difficulty in extending this approach to the maximal symmetric case studied using a generalized spectral theorem (**1**, p. 127) as in (**6**, **7**).

2. Let L denote the formal ordinary differential operator

$$L = p_0 \frac{d^n}{dt^n} + p_1 \frac{d^{n-1}}{dt^{n-1}} + \cdots + p_n,$$

where p_k is a complex-valued function having $n - k$ continuous derivatives on an open interval $a < t < b$ $(k = 0, 1, \ldots, n)$ and $p_0 \neq 0$ on $a < t < b$. The

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cases $a = -\infty$, $b = \infty$, or both, are allowed. We assume that L is formally self-adjoint, i.e. that L coincides with its formal adjoint

$$L^+ = (-1)^n \frac{d^n}{dt^n} \bar{p}_0 + (-1)^{n-1} \frac{d^{n-1}}{dt^{n-1}} \bar{p}_1 + \cdots + \bar{p}_n.$$

We define the set D, consisting of all functions $f \in L^2(a, b)$ having continuous derivatives up to order $n - 1$ on (a, b), with $f^{(n-1)}$ absolutely continuous on (a, b), so that $f^{(n)}$ exists almost everywhere on (a, b), and such that $Lf \in L^2(a, b)$. According to Green's formula (**7**, p. 86), for every compact subinterval $\delta = [\alpha, \beta]$ of (a, b) and every pair of functions $u, v \in D$, we have

$$\int_\alpha^\beta [Lu(t)\bar{v}(t) - u(t)\overline{Lv}(t)]dt = [uv](\beta) - [uv](\alpha).$$

Here $[uv](t)$ is a bilinear form given by

$$[uv](t) = \sum_{m=1}^{n} \sum_{j+k=m-1} (-1)^j u^{(k)}(t)[a_{n-m}(t)\bar{v}(t)]^{(j)}.$$

We define

$$\langle uv \rangle_\delta = [uv](\beta) - [uv](\alpha).$$

It follows from Green's formula that

$$[uv](a) = \lim_{\alpha \to a+} [uv](\alpha) \quad \text{and} \quad [uv](b) = \lim_{\beta \to b-} [uv](\beta)$$

exist. Thus we may also define

$$\langle uv \rangle = [uv](b) - [uv](a).$$

By a homogeneous boundary condition on the open interval (a, b) we mean a condition of the form $\langle u\alpha \rangle = 0$ for a given $\alpha \in D$. We say that a set of p boundary conditions $\langle u\alpha_j \rangle = 0$ $(j = 1, \ldots, p)$ is linearly independent if and only if

$$\sum_{j=1}^{p} y_j \langle u\alpha_j \rangle = 0$$

for every $u \in D$ implies that $y_1 = \ldots = y_p = 0$. We also say that such a set of boundary conditions is self-adjoint if and only if

$$\langle \alpha_j \alpha_k \rangle = 0 \qquad (j, k = 1, \ldots, p). \tag{1}$$

We say that two sets of boundary conditions are equivalent if they are satisfied by the same set of functions.

Now we define the set $D_0 \subseteq D$, consisting of all functions $f \in D$ such that $\langle f\alpha \rangle = 0$ for every $\alpha \in D$. Corresponding to the sets D and D_0 we define two linear operators in the Hilbert space $L^2(a, b)$. We define the maximal operator T associated with the formal differential operator L as having domain D and being given by $Tu = Lu$ for $u \in D$. We define the minimal operator T_0 associated with the formal differential operator L as having domain D_0 and being given by $T_0 u = Lu$ for $u \in D_0$. Obviously T_0 is a restriction of T $T_0 \subseteq T$, Also,

$$(T_0 u, v) - (u, Tv) = \langle uv \rangle = 0 \qquad (u \in D_0, v \in D),$$

which implies that $T \subseteq T_0{}^*$. Here (,) denotes the usual inner product in $L^2(a, b)$, and $T_0{}^*$ is the adjoint operator, in the Hilbert space sense, of T_0. In fact $T = T_0{}^*$ and $T^* = T_0$ **(3)**, but we shall not need this more precise information.

Let $\mathfrak{E}(i)$ denote the set of solutions of $T_0{}^*u = iu$ in D, and let $\mathfrak{E}(-i)$ denote the set of solutions of $T_0{}^*u = -iu$ in D. Then it is known **(1**, p. 98**)** that the domain of $T_0{}^*$ is the direct sum of D_0, $\mathfrak{E}(i)$, and $\mathfrak{E}(-i)$. Thus, since $T \subseteq T_0{}^*$, we have

$$D \subseteq D_0 \oplus \mathfrak{E}(i) \oplus \mathfrak{E}(-i). \tag{2}$$

It is also known **(1**, p. 97**)** that T_0 has a self-adjoint extension if and only if the vector spaces $\mathfrak{E}(i)$ and $\mathfrak{E}(-i)$ have the same dimension. Our approach will not depend on the theory of self-adjoint operators in Hilbert space, and we shall make no use of this fact. However, we shall always make the obviously related assumption that the differential equations $Lu = iu$ and $Lu = -iu$ each have exactly ω linearly independent solutions in D. Clearly $0 \leqslant \omega \leqslant n$, and for a non-singular differential operator on a compact interval $\omega = n$.

We can now define what is meant by a self-adjoint boundary value problem on (a, b). We use the approach introduced in **(5)**. Let $y_1, y_2, \ldots, y_\omega$ be an orthonormal set of solutions of $Lu = iu$ in D, and let $z_1, z_2, \ldots, z_\omega$ be an orthonormal set of solutions of $Lu = -iu$ in D. Let $V = (v_{jk})$ be an $\omega \times \omega$ unitary matrix and let

$$\alpha_j = y_j - \sum_{k=1}^{\omega} v_{jk} z_k \qquad (j = 1, \ldots, \omega).$$

A self-adjoint boundary value problem associated with L on (a, b) consists of the differential equation

$$Lu = \lambda u \tag{3}$$

together with the boundary conditions

$$\langle u\alpha_j \rangle = 0 \qquad (j = 1, \ldots, \omega). \tag{4}$$

THEOREM 1. *The boundary conditions* (4) *are self-adjoint, and the boundary value problem* (3), (4) *is self-adjoint in the sense of* **(8**, *p.* 189**)**, *i.e. if D_A is the set of functions in D which satisfy the boundary conditions* (4), *then* $\langle fg \rangle = 0$ *for every pair of functions* $f, g \in D_A$.

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Proof. The proof that the boundary conditions (4) are self-adjoint is given in **(3**, Theorem 3**)**. It is also shown in **(3)** that if $f, g \in D_A$, we can write

$$f = f_0 - \sum_{j=1}^{\omega} \left(\sum_{k=1}^{\omega} b_k \bar{v}_{jk} \right) \alpha_j, \tag{5}$$

$$g = g_0 - \sum_{j=1}^{\omega} \left(\sum_{k=1}^{\omega} c_k \bar{v}_{jk} \right) \alpha_j, \tag{6}$$

where $f, g \in D_0$, and where b_k, c_k are complex constants $(k = 1, \ldots, \omega)$. The relation $\langle fg \rangle = 0$ now follows immediately from (5), (6), and (1).

It is not difficult to see that the problem (3), (4) corresponds to a self-adjoint operator in the Hilbert space sense. We define D_A to be the set of functions in D which satisfy the boundary conditions (4). Then there is a self-adjoint operator A with domain D_A defined by $Au = Lu$ for $u \in D_A$. We shall make no explicit use of this fact, but it is the link between our approach and the approach via spectral theory to the eigenfunction expansion problem.

Having defined a self-adjoint boundary value problem on the interval (a, b), we can use the argument of **(8**, Chapter 10**)**, which treated the case $\omega = 0$, to establish the existence of a spectral matrix and the expansion theorem. It has been pointed out **(11)** that no change is needed to treat the more general problem considered here. As our notation differs slightly from that used in **(4, 5, 7, 8, 10, 11)**, we begin with an outline of the known results.

We treat the singular boundary value problem on (a, b) by defining a self-adjoint boundary value problem in a set of compact subintervals $\delta = [\alpha, \beta]$ of (a, b) as in **(5)** and then letting $\delta \to (a, b)$. For functions $u, v \in L^2(\delta)$ we define the inner product $(u, v)_\delta = \int_\delta u(t)\bar{v}(t)\,dt$ and the norm $||u||_\delta = (u, u)_\delta^{\frac{1}{2}}$. We choose the subintervals δ so that each contains a point c, and we define a fundamental set of solutions $\phi_1(t, \lambda), \ldots, \phi_n(t, \lambda)$ of (3) by the initial conditions

$$\phi_j^{(k-1)}(c, \lambda) = \delta_{jk} \qquad (j, k = 1, \ldots, n). \tag{7}$$

Then the following results are known **(8**, Chapter 7**)** for the non-singular boundary value problem on the subinterval δ.

THEOREM 2. *There exists a unique Hermitian matrix $\rho_\delta = (\rho_{\delta jk})$ whose elements are step functions with discontinuities at the eigenvalues of the self-adjoint boundary value problem on δ. If $\Delta = (\lambda_1, \lambda_2]$, where $\lambda_2 > \lambda_1$, the matrix $\rho_\delta(\Delta) = \rho_\delta(\lambda_2) - \rho_\delta(\lambda_1)$ is positive semi-definite. The total variation of $\rho_{\delta jk}$ on every finite interval is finite. If we define the transform g_δ of $f \in L^2(\delta)$ by*

$$g_{\delta j}(\lambda) = \int_\delta f(t)\bar{\phi}_j(t, \lambda)dt \qquad (j = 1, \ldots, n),$$

then

$$||f||_\delta^2 = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} g_{\delta j}(\lambda) g_{\delta k}(\lambda) d\rho_{\delta jk}(\lambda).$$

Also

$$f(t) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} g_{\delta j}(\lambda)\phi_k(t, \lambda)d\rho_{\delta jk}(\lambda),$$

with the integral converging to f in the norm of $L^2(\delta)$.

Using the results of Theorem 2 and proceeding as in **(8**, Chapter 10**)**, we prove the existence of a limiting matrix ρ, called a spectral matrix, for the singular boundary value problem on (a, b).

THEOREM 3. *Let $\{\delta\}$ be a set of compact subintervals of (a, b) tending to (a, b). Then $\{\delta\}$ contains a sequence $\{\delta_r\}$ tending to (a, b) as $r \to \infty$ such that*

$$\rho(\lambda) = \lim_{r\to\infty} \rho_{\delta_r}(\lambda)$$

exists on $-\infty < \lambda < \infty$. The limit matrix ρ is Hermitian; $\rho(\Delta) = \rho(\lambda) - \rho(\mu)$ is positive semi-definite if $\Delta = (\lambda_1, \lambda_2]$, $\lambda_1 < \lambda_2$; and the total variation of ρ on every finite interval is finite.

For any limiting matrix ρ, we define $L^2(\rho)$ to be the Hilbert space of vectors $g(\lambda) = (g_1(\lambda), \ldots, g_n(\lambda))$ measurable with respect to ρ and such that

$$||g||^2 = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} g_j(\lambda)\bar{g}_k(\lambda)d\rho_{jk}(\lambda) < \infty. \tag{8}$$

The inner product in $L^2(\rho)$ is given by

$$(g^{(1)}, g^{(2)}) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} g_j^{(1)}(\lambda)\bar{g}_k^{(2)}(\lambda)d\rho_{jk}(\lambda).$$

Now, proceeding as in **(8**, Chapter 10**)**, as indicated for our more general problem in **(11)**, we obtain the following results.

THEOREM 4. *Let ρ be any limiting matrix given by Theorem 3. If $f \in L^2(a, b)$, then there exists $g \in L^2(\rho)$ such that if*

$$g_{\delta j}(\lambda) = \int_\delta f(t)\bar{\phi}_j(t, \lambda)dt \qquad (\delta \subset (a, b); j = 1, \ldots, n),$$

then $||g - g_\delta|| \to 0$ as $\delta \to (a, b)$. In terms of this transform g, we have the Parseval equality

$$||f|| = ||g||, \tag{9}$$

and the expansion

$$f(t) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} g_j(\lambda)\phi_k(t, \lambda)d\rho_{jk}(\lambda), \tag{10}$$

where the integral converges to f in the norm of $L^2(a, b)$. We represent the transform g by

$$g_j(\lambda) = \int_a^b f(t)\bar{\phi}_j(t, \lambda)dt \qquad (j = 1, \ldots, n). \tag{11}$$

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The inverse transform theorem and the uniqueness of the spectral matrix are derived from the following theorem **(11)**. The main point of this paper is to show that the properties of the Green's function can be established from this theorem without the need to return to the study of the subintervals δ as in **(8**, Chapter 10**)**.

THEOREM 5. *If* $f \in L^2(a, b)$ *is expressed in terms of* $g \in L^2(\rho)$ *as in Theorem* 4 *and if* $\operatorname{Im} \lambda \neq 0$, *then there exists a unique solution* u *of the non-homogeneous equation*

$$Lu = \lambda u + f \tag{12}$$

which satisfies the boundary conditions (4). *This solution is given by*

$$u(t) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} (\sigma - \lambda)^{-1} g_j(\sigma) \phi_k(t, \sigma) d\rho_{jk}(\sigma). \tag{13}$$

In the course of proving Theorem 5, we define

$$f_\Delta(t) = \int_\Delta \sum_{j,k=1}^{n} g_j(\sigma) \phi_k(t, \sigma) d\rho_{jk}(\sigma), \tag{14}$$

$$u_\Delta^{(t)} = \int_\Delta \sum_{j,k=1}^{n} (\sigma - \lambda)^{-1} g_j(\sigma) \phi_k(t, \sigma) d\rho_{jk}(\sigma), \tag{15}$$

where $\Delta = (\lambda_1, \lambda_2]$, and we show that

$$||f - f_\Delta|| \to 0, \quad ||u - u_\Delta|| \to 0 \qquad \text{as } \Delta \to (-\infty, \infty).$$

Also, we show that

$$Lu_\Delta = \lambda u_\Delta + f_\Delta,$$

and that u_Δ satisfies the boundary conditions (4).

We shall make use of these facts in the next section.

3. As we know from the theory of non-singular boundary value problems **(8**, Chapter 7**)**, we can construct a Green's function for a self-adjoint boundary value problem on a compact subinterval of (a, b). It is possible **(8**, Chapter 10**)** to find a sequence of compact subintervals and a corresponding sequence of Green's functions which converges to a limit function. This limit function is a Green's function for the singular boundary value problem on (a, b). We shall use a different approach to study the Green's function for the singular boundary value problem by using Theorem 5.

We define

$$G_\Delta(t, \tau, \lambda) = \int_\Delta \sum_{j,k=1}^{n} (\sigma - \lambda)^{-1} \bar{\phi}_j(\tau, \sigma) \phi_k(t, \sigma) d\rho_{jk}(\sigma), \tag{16}$$

for $a < t, \tau < b$, $\operatorname{Im} \lambda \neq 0$, $\Delta = (\lambda_1, \lambda_2]$. Then we have

$$(17) \qquad \begin{aligned} u_\Delta(t) &= \int_\Delta \sum_{j,k=1}^n (\sigma - \lambda)^{-1} g_j(\sigma)\phi_k(t, \sigma) d\rho_{jk}(\sigma) \\ &= \int_\Delta \sum_{j,k=1}^n (\sigma - \lambda)^{-1} \left[\int_a^b f(\tau)\bar{\phi}_j(\tau, \sigma) d\tau \right] \phi_k(t, \sigma) d\rho_{jk}(\sigma) \\ &= \int_a^b \left[\int_\Delta \sum_{j,k=1}^n (\sigma - \lambda)^{-1} \bar{\phi}_j(\tau, \sigma)\phi_k(t, \sigma) d\rho_{jk}(\sigma) \right] f(\tau) d\tau \\ &= \int_a^b G_\Delta(t, \tau, \lambda) f(\tau) d\tau. \end{aligned}$$

For $\operatorname{Im} \lambda \neq 0$, $a < t, \tau < b$, we now define

$$(18) \qquad \begin{aligned} H_\Delta(t, \tau, \lambda) &= G_\Delta(t, \tau, \lambda) - G_\Delta(t, \tau, \bar{\lambda}), \\ P_{\Delta jk}(\lambda) &= \frac{\partial^{j+k-2}}{\partial t^{j-1} \partial \tau^{k-1}} H_\Delta(c, c, \lambda) \qquad (j, k = 1, \ldots, n). \end{aligned}$$

THEOREM 6. *The matrix $P_\Delta(\lambda)$ is related to the spectral matrix $\rho(\lambda)$ by*

$$(19) \qquad P_{\Delta jk} = 2i \operatorname{Im} \lambda \int_\Delta \frac{d\rho_{jk}(\sigma)}{|\sigma - \lambda|^2},$$

$$(20) \qquad \rho_{jk}(\mu_2) - \rho_{jk}(\mu_1) = \frac{1}{2\pi i} \lim_{r \to 0+} \int_{\mu_1}^{\mu_2} P_{\Delta jk}(\mu + ir) d\mu \qquad (j, k = 1, \ldots, n)$$

provided $\mu_1, \mu_2 \in \Delta$.

Proof. If $\lambda = \mu + ir$, then

$$\begin{aligned} H_\Delta(t, \tau, \lambda) &= \int_\Delta \sum_{p,q=1}^n \left(\frac{1}{\sigma - \lambda} - \frac{1}{\sigma - \bar{\lambda}} \right) \bar{\phi}_p(\tau, \sigma)\phi_q(t, \sigma) d\rho_{pq}(\sigma) \\ &= 2ir \int_\Delta \sum_{p,q=1}^n \bar{\phi}_p(\tau, \sigma)\phi_q(t, \sigma) \frac{d\rho_{pq}(\sigma)}{(\sigma - \mu)^2 + r^2}. \end{aligned}$$

If we differentiate $j - 1$ times with respect to t and $k - 1$ times with respect to τ, then setting $t = \tau = c$, and using (7) and (18), we obtain

$$P_{\Delta jk}(\lambda) = 2ir \int_\Delta \frac{d\rho_{jk}(\sigma)}{(\sigma - \mu)^2 + r^2},$$

which is equivalent to (19).

To obtain (20), we write

$$\begin{aligned} \frac{1}{2\pi i} \lim_{r \to 0+} \int_{\mu_1}^{\mu_2} P_{\Delta jk}(\mu + ir) d\mu &= \frac{1}{\pi} \lim \int_{\mu_1}^{\mu_2} \left[\int_\Delta \frac{r d\rho_{jk}(\sigma)}{(\sigma - \mu)^2 + r^2} \right] d\mu \\ &= \frac{1}{\pi} \lim_{r \to 0+} \int_\Delta \left[\int_{\mu_1}^{\mu_2} \frac{r d\mu}{(\sigma - \mu)^2 + r^2} \right] d\rho_{jk}(\sigma) \\ &= \frac{1}{\pi} \lim_{r \to 0+} \int_\Delta \left[\tan^{-1} \left(\frac{\mu_2 - \sigma}{r} \right) \right. \\ &\qquad \left. - \tan^{-1} \left(\frac{\mu_1 - \sigma}{r} \right) \right] d\rho_{jk}(\sigma) \\ &= \int_{\mu_1}^{\mu_2} d\rho_{jk}(\sigma) = \rho_{jk}(\mu_2) - \rho_{jk}(\mu_1), \end{aligned}$$

provided $\mu_1, \mu_2 \in \Delta$.

It can be shown (**1**, pp. 177–181), that there exists a Green's function $G(t, \tau, \lambda)$ for $a < t, \tau < b$, $\operatorname{Im} \lambda \neq 0$ which is in the class $L^2(a, b)$ as a function of t for each fixed τ and $\operatorname{Im} \lambda \neq 0$, and also as a function of τ for each fixed t and $\operatorname{Im} \lambda \neq 0$. This function has partial derivatives up to order $n - 1$ with respect to t, which also belong to $L^2(a, b)$ as functions of t, and

$$\partial^{n-1} G(t, \tau, \lambda)/\partial t^{n-1}$$

is continuous except for a simple discontinuity at $t = \tau$. The function $G(t, \tau, \lambda)$ has the symmetry property

$$G(t, \tau, \lambda) = \bar{G}(\tau, t, \bar{\lambda}) \tag{21}$$

for $a \subset t, \tau \subset b$, $\operatorname{Im} \lambda \neq 0$. The unique solution u of the non-homogeneous equation (12), for any given $f \in L^2(a, b)$, which satisfies the boundary conditions (4) can be written

$$u(t) = \int_a^b G(t, \tau, \lambda) f(\tau) d\tau. \tag{22}$$

The proofs of these statements depend on the variation of constants formula for linear differential equations and on elementary properties of Hilbert space, but not on any facts concerning linear operators. In particular, they do not depend on the spectral theorem.

Because of the symmetry relation (21), not only the partial derivatives $\partial^{p-1} G(t, \tau, \lambda)/\partial t^{p-1}$ belong to $L^2(a, b)$ as functions of t for fixed (τ, λ), but also the partial derivatives $\partial^{p-1} G(t, \tau, \lambda)/\partial \tau^{p-1}$ with respect to the second variable.

In order to relate the Green's function $G(t, \tau, \lambda)$ to the function $G_\Delta(t, \tau, \lambda)$ defined by (16), we need the following lemmas.

LEMMA 1. *Let $f \in L^2(a, b)$ have transform $g \in L^2(\rho)$, and let u be the unique solution of* (12) *which satisfies the boundary conditions* (4), *i.e.*

$$u(t) = \int_a^b G(t, \tau, \lambda) f(\tau) d\tau.$$

If $v \in L^2(\rho)$ is the transform of u, then

$$v_j(\sigma) = g_j(\sigma)/(\sigma - \lambda) \qquad (j = 1, \ldots, n). \tag{23}$$

Proof. For any function $h(t) \in L^2(a, b)$ which vanishes outside a compact subinterval of (a, b), Green's formula shows that

$$\int_a^b Lh(t) \bar{\phi}_j(t, \sigma) dt = \int_a^b h(t) \overline{L\phi_j}(t, \sigma) dt$$

$$= \sigma \int_a^b h(t) \bar{\phi}_j(t, \sigma) dt \qquad (j = 1, \ldots, n).$$

Thus the transform in $L^2(\rho)$ of Lh is the product of the transform of h and the independent variable in $L^2(\rho)$. This can be extended to all $h \in D$ which satisfy the boundary conditions (4) by a standard density argument. Using this fact we may now take transforms in the equation

$$Lu = \lambda u + f$$

to obtain

$$\sigma v_j(\sigma) = \lambda v_j(\sigma) + g_j(\sigma) \qquad (j = 1, \ldots, n),$$

from which (23) follows immediately.

LEMMA 2. *The transform in* $L^2(\rho)$ *of* $\partial^{p-1}G(t, \tau, \lambda)/\partial\tau^{p-1}$, *considered as a function of t for fixed* (τ, λ), *is given by*

$$(\sigma - \lambda)^{-1}\bar{\phi}_j^{(p-1)}(\tau, \sigma) \qquad (j, p = 1, \ldots, n).$$

Also, for $j, k = 1, \ldots, m$, *and* $\operatorname{Im} \lambda = 0$,

$$\int_{-\infty}^{\infty} \frac{d\rho_{jk}(\sigma)}{|\sigma - \lambda|^2} < \infty. \tag{24}$$

Proof. Let f be the function in $L^2(a, b)$ with the value $1/s$ on the interval $(\tau, \tau + s)$ and the value zero elsewhere. Its transform g is given by

$$g_j(\sigma) = \frac{1}{s}\int_{\tau}^{\tau+s} \bar{\phi}_j(t, \sigma)dt \qquad (j = 1, \ldots, n).$$

Let

$$u(t) = \int_a^b G(t, x, \lambda)f(x)dx = \frac{1}{s}\int_{\tau}^{\tau+s} G(t, x, \lambda)dx. \tag{25}$$

By Lemma 1, the transform v of u is given by

$$v_j(\sigma) = \int_a^b \left[\frac{1}{s}\int_{\tau}^{\tau+s} G(t, x, \lambda)dx\right]\bar{\phi}_j(t, \sigma)dt = \frac{g_j(\sigma)}{\sigma - \lambda} \quad (j = 1, \ldots, n). \tag{26}$$

We observe that (25), (26) imply that

$$\lim_{s\to 0+} g_j(\sigma) = \bar{\phi}_j(\tau, \sigma), \qquad \lim_{s\to 0+} u(t) = G(t, \tau, \lambda), \qquad \lim_{s\to 0+} v_j(\sigma) = \frac{\bar{\phi}_j(\tau, \sigma)}{\sigma - \lambda}. \tag{27}$$

Since $G(t, \tau, \lambda)$, considered as a function of t, belongs to $L^2(a, b)$, Theorem 4 shows that the integral

$$\int_a^b G(t, \tau, \lambda)\bar{\phi}_j(t, \sigma)dt$$

converges for $j = 1, \ldots, n$. Thus, using (27),

$$\begin{aligned} \int_a^b G(t, \tau, \lambda)\bar{\phi}_j(t, \sigma)dt &= \int_a^b \lim_{s\to 0+} u(t)\bar{\phi}_j(t, \sigma)dt \\ &= \lim_{s\to 0+}\int_a^b u(t)\bar{\phi}_j(t, \sigma)dt \\ &= \lim_{s\to 0+} v_j(\sigma) = \frac{\bar{\phi}_j(\tau, \sigma)}{\sigma - \lambda} \qquad (j = 1, \ldots, n). \end{aligned} \tag{28}$$

Since the partial derivatives of G with respect to τ of order up to $n - 1$ belong to $L^2(a, b)$, Theorem 4 shows that the integrals

$$\int_a^b \frac{\partial^{p-1}}{\partial \tau^{p-1}} G(t, \tau, \lambda) \bar{\phi}_j(t, \sigma) dt \qquad (p = 1, \ldots, n)$$

converge. Therefore we may differentiate (28) with respect to τ, obtaining

$$(29) \qquad \int_a^b \frac{\partial^{p-1}}{\partial \tau^{p-1}} G(t, \tau, \lambda) \bar{\phi}_j(t, \sigma) dt = \frac{\bar{\phi}_j^{(p-1)}(\tau, \sigma)}{\sigma - \lambda} \qquad (j, p = 1, \ldots, n).$$

If f^1 and f^2 are two functions in $L^2(a, b)$ with transforms g^1 and g^2 in $L^2(\rho)$ respectively, then it is an easy consequence of the Parseval equality (9) that

$$(30) \qquad \int_a^b f^1(t) \bar{f}^2(t) dt = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} g_j^1(\sigma) \bar{g}_k^2(\sigma) d\rho_{jk}(\sigma).$$

We apply (30) to the functions

$$f^1(t) = \frac{\partial^{p-1}}{\partial x^{p-1}} G(t, \tau, \lambda), \qquad f^2(t) = \frac{\partial^{q-1}}{\partial x^{q-1}} G(t, \tau, \lambda)$$

whose transforms we have just calculated. We obtain

$$\infty > \int_a^b \frac{\partial^{p-1}}{\partial x^{p-1}} G(t, \tau, \lambda) \frac{\partial^{q-1}}{\partial x^{q-1}} \bar{G}(t, \tau, \lambda) dt$$

$$= \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} |\sigma - \lambda|^{-2} \bar{\phi}_j^{(p-1)}(\tau, \sigma) \phi_k^{(q-1)}(\tau, \sigma) d\rho_{jk}(\sigma)$$

$$(p, q = 1, \ldots, n).$$

Now we set $\tau = c$ and use (7) to obtain

$$\int_{-\infty}^{\infty} \frac{d\rho_{pq}(\sigma)}{|\sigma - \lambda|^2} < \infty \qquad (p, q = 1, \ldots, n),$$

as desired. This completes the proof of Lemma 2.

THEOREM 7. *The Green's function* $G(t, \tau, \lambda)$ *is given by*

$$(31) \qquad G(t, \tau, \lambda) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} (\sigma - \lambda)^{-1} \bar{\phi}_j(\tau, \sigma) \phi_k(t, \sigma) d\rho_{jk}(\sigma)$$

$$= \lim_{\Delta \to (-\infty, \infty)} G_\Delta(t, \tau, \lambda),$$

for $0 < t, \tau < b$, $\operatorname{Im} \lambda \neq 0$, *with the integral converging in the norm of* $L^2(a, b)$ *as a function of* t *for fixed* (τ, λ), *and also in the norm of* $L^2(a, b)$ *as a function of* τ *for fixed* (t, λ). *The equation* (31) *may be differentiated* $n - 1$ *times with respect to either* t *or* τ.

Proof. By Lemma 2 the function $\partial^{p-1} G(t, \tau, \lambda)/\partial \tau^{p-1}$ has the transform

$$(\sigma - \lambda)^{-1} \bar{\phi}_j^{(p-1)}(\tau, \sigma) \qquad (p = 1, \ldots, n).$$

We may now apply Theorem 4 to give the expansion of this function, namely

$$(32)\quad \frac{\partial^{p-1}}{\partial\tau^{p-1}} G(t, \tau, \lambda) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} (\sigma - \lambda)^{-1} \bar{\phi}_j^{(p-1)}(\tau, \sigma)\phi_k(\tau, \sigma) d\rho_{jk}(\sigma) \qquad (p = 1, \ldots, n),$$

with the integral converging in the norm of $L^2(a, b)$ as a function of t. The convergence in the norm of $L^2(a, b)$ as a function of τ and the possibility of differentiating $G(t, \tau, \lambda)$ under the integral sign with respect to t are immediate consequences of the symmetry relation (21).

For a non-singular self-adjoint boundary value problem on a finite interval with a sequence of eigenvalues $\{\lambda_k\}$ and corresponding orthonormal eigenfunctions $\{u_k(t)\}$, we have

$$(33)\qquad G(t, \tau, \lambda) = \sum_{k=0}^{\infty} (\lambda_k - \lambda)^{-1} u_k(t)\bar{u}_k(\tau) \qquad (\operatorname{Im} \lambda \neq 0),$$

with the series converging to $G(t, \tau, \lambda)$ in the norm of $L^2[(a, b) \times (a, b)]$ (**8**, p. 202). When expressed in terms of the spectral matrix, (33) takes the form (31). Thus Theorem 6 is the natural generalization of (33) to the singular case. The expansion (31) cannot be expected to converge in the norm of $L^2L[(a, b) \times (a, b)]$ in general, however, because this would imply that $G(t, \tau, \lambda) \in L^2[(a, b) \times (a, b)]$, from which it would follow that the boundary value problem (3), (4) has a pure point spectrum just as in the non-singular case.

The convergence of the integral

$$\int_{-\infty}^{\infty} \frac{d\rho_{jk}(\sigma)}{|\sigma - \lambda|^2}$$

demonstrated in Lemma 2 shows that we may let $\Delta \to (-\infty, \infty)$ in (19). We see that $P_\Delta(\lambda)$ converges to a matrix $P(\lambda)$ as $\Delta \to (-\infty, \infty)$, and

$$(34)\qquad P_{jk}(\lambda) = 2i \operatorname{Im} \lambda \int_{-\infty}^{\infty} \frac{d\rho_{jk}(\sigma)}{|\sigma - \lambda|^2} \qquad (j, k = 1, \ldots, n).$$

Then we may let $\Delta \to (-\infty, \infty)$ in (20), obtaining

$$(35)\qquad \rho_{jk}(\mu_2) - \rho_{jk}(\mu_1) = \frac{1}{2\pi i} \lim_{r \to 0+} \int_{\mu_1}^{\mu_2} P_{jk}(\mu + ir) d\mu \qquad (j, k = 1, \ldots, n).$$

The convergence of $G_\Delta(t, \tau, \lambda)$ to $G(t, \tau, \lambda)$ shows that $P_{jk}(\lambda)$ is given by

$$H(t, \tau, \lambda) = G(t, \tau, \lambda) - G(t, \tau, \bar{\lambda}),$$

$$P_{jk}(\lambda) = \frac{\partial^{j+k-2}}{\partial t^{j-1} \partial \tau^{k-1}} H(c, c, \lambda) \qquad (j, k = 1, \ldots, n).$$

analogous to (18). The formulae (34), (35) are known as the Titchmarsh-Kodaira formulae (**8**, p. 280), relating the Green's function to the spectral matrix.

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