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IDEMPOTENT GENERATORS OF INCIDENCE ALGEBRAS

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Abstract

The minimum number of idempotent generators is calculated for an incidence algebra of a finite poset over a commutative ring. This quantity equals either $\lceil \log_2 n \rceil$ or $\lceil \log_2 n \rceil + 1$, where n is the cardinality of the poset. The two cases are separated in terms of the embedding of the Hasse diagram of the poset into the complement of the hypercube graph.

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1. Introduction

The class of associative algebras generated by idempotents has many interesting properties. For example, a simple algebra $\mathcal{A}$ over a field $\mathbb{F}$ with char $\mathbb{F} \neq 2$ belongs to this class if $\mathcal{A}$ contains at least one nontrivial idempotent. This follows from a more general result about invariant subalgebras due to Amitsur [1, Theorem 2]. A self-contained proof was given by Brešar [2, Section 2]. Laffey [11] showed that a noncommutative simple algebra generated by two idempotents is always isomorphic to the algebra of 2×2 matrices over a simple extension of the ground field.

The structure of not necessarily simple algebras generated by two idempotents was described by Weiss [19] and Rowen and Segev [15]. Kawai [6] proved that a commutative algebra over an algebraically closed field is generated by idempotents if and only if it is a homomorphic image of a group algebra with certain properties. Brešar [3] showed that a finite-dimensional algebra is zero product determined if and only if it can be generated by idempotents. Hu and Xiao [5] obtained a homological description of finite-dimensional algebras generated by idempotents. Algebras generated by two quadratic elements were described by Drensky *et al.* [4]. Similar problems have been investigated successfully in functional analysis for Banach algebras including C^*-algebras (see [10, 14, 16, 19]).

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If we know that an algebra is generated by a finite set of idempotents, it is a natural task to determine its minimum cardinality. This quantity will be denoted by ν. Laffey [11] proved that $\nu \le 3$ for the $\mathbb{F}$-algebra of $n \times n$ matrices $M_n(D)$, where D is a finite-dimensional division $\mathbb{F}$-algebra and its centre $Z(D)$ is a separable extension of the field $\mathbb{F}$. Krupnik [9] calculated ν precisely for $M_n(\mathbb{F})$. Kelarev *et al.* [7] determined ν for the algebra of upper-triangular matrices over a commutative ring. This result was generalised in [18] to complete block triangular matrix algebras over an infinite field.

Throughout the paper, $\mathcal{R}$ is a commutative associative ring with the identity $1 \neq 0$. A subalgebra $\mathcal{A}$ of the matrix algebra $M_n(\mathcal{R})$ is called a *structural matrix algebra* if it has a basis of some matrix units E_{ij} as an $\mathcal{R}$-module and all diagonal matrix units $\{E_{ii}\}_{i=1}^n$ belong to this basis. If additionally no two symmetric matrix units E_{ij}, E_{ji} are in this basis simultaneously, $\mathcal{A}$ is called an *incidence algebra*. Then the set of all such pairs (i, j) for which E_{ij} belongs to the basis is a partial order $\preceq$ on the set $\mathcal{N} = \{1, \dots, n\}$. Conversely, for any partial order $\preceq$ on $\mathcal{N}$, all possible $\mathcal{R}$-linear combinations of the matrices $\{E_{ij} \mid i \preceq j\}$ constitute an incidence algebra

$$\mathcal{A}_n(\preceq, \mathcal{R}) = \left\{ \sum_{i \preceq j} r_{ij} E_{ij} : r_{ij} \in \mathcal{R} \right\} \subseteq M_n(\mathcal{R}).$$

For example, if one takes the standard linear order $\le$, the corresponding incidence algebra will be the algebra of all upper-triangular matrices $T_n(\mathcal{R})$. In contrast, if $\preceq$ is the trivial partial order (equality), then $\mathcal{A}_n(=, \mathcal{R})$ coincides with the algebra of all diagonal matrices $D_n(\mathcal{R})$.

According to [7], the minimum number of idempotent generators ν of $T_n(\mathcal{R})$ equals $\lceil \log_2 n \rceil$ in almost all cases. The exceptional cases are $n = 2, 3, 4$, when $\nu = \lceil \log_2 n \rceil + 1$. We will generalise this result to all incidence algebras of finite posets. In other words, the following problem will be considered.

PROBLEM 1.1. Given an incidence algebra $\mathcal{A}$ of a finite poset over a unital commutative ring $\mathcal{R}$, find the minimum number ν of idempotent matrices which generate $\mathcal{A}$ as a unital $\mathcal{R}$-algebra.

The solution to this problem is obtained in Theorem 5.6. It turns out that again ν equals either $\lceil \log_2 n \rceil$ or $\lceil \log_2 n \rceil + 1$, but delimitation between these two cases is more complicated. It relies on embedding the Hasse diagram of the poset into the complement of a hypercube graph. In the particular case of $\nu(T_n(\mathcal{R}))$, the choice between the two formulae can be interpreted in terms of the existence of a Hamiltonian path in a certain graph (see the proof of Corollary 5.7).

The paper is organised as follows. In Section 2, necessary definitions and known results are given. In Section 3, it is shown that given idempotent generators of an incidence algebra, the Hasse diagram of the poset can be embedded in the complement of a hypercube graph. Conversely, Section 4 is devoted to a construction of idempotent

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generators when the Hasse diagram is represented as a subgraph of the complement of a hypercube. In Section 5, the solution to Problem 1.1 is obtained in Theorem 5.6. To formulate this result, new types of graphs are introduced in Definition 5.3.

2. Preliminaries

Let $\mathcal{A}$ be an associative $\mathcal{R}$-algebra with the identity $1_{\mathcal{A}}$. A subset $S \subseteq \mathcal{A}$ is said to *generate $\mathcal{A}$ as a unital ring* if any element $a \in \mathcal{A}$ can be represented as a sum of some products of elements from the set $S \cup \{1_{\mathcal{A}}\}$, that is, $a = \sum_{i=1}^{n} s_{i,1} \cdot s_{i,2} \cdot \dots \cdot s_{i,k_i}$, where $s_{i,j} \in S \cup \{1_{\mathcal{A}}\}$, some of the $s_{i,j}$ may coincide and $k_i \in \mathbb{N}$. We denote $\mathcal{R}1_{\mathcal{A}} = \{r \cdot 1_{\mathcal{A}} \mid r \in \mathcal{R}\}$. A subset $S \subseteq \mathcal{A}$ *generates $\mathcal{A}$ as a unital algebra* if $\mathcal{A}$ can be generated by $S \cup \mathcal{R}1_{\mathcal{A}}$ as a unital ring. Generators of incidence algebras over finite posets were described in [13, Theorem] and [8, Theorem 1.1].

Consider a partial order $\preceq$ on $\mathcal{N} = \{1, \dots, n\}$ and a bijection $\sigma : \mathcal{N} \to \mathcal{N}$. Then we can introduce a new partial order $\preceq_\sigma$ given by $i \preceq_\sigma j$ whenever $\sigma^{-1}(i) \preceq \sigma^{-1}(j)$. So the posets $(\mathcal{N}, \preceq)$ and $(\mathcal{N}, \preceq_\sigma)$ are isomorphic.

PROPOSITION 2.1 [17, Proposition 1.2.7], [8, Section 2]. *For any incidence algebra $\mathcal{A}_n(\preceq, \mathcal{R})$, there exists a bijection $\sigma : \mathcal{N} \to \mathcal{N}$ such that $\mathcal{A}_n(\preceq_\sigma, \mathcal{R}) \subseteq T_n(\mathcal{R})$. Moreover, $\mathcal{A}_n(\preceq_\sigma, \mathcal{R}) = P^{-1}\mathcal{A}_n(\preceq, \mathcal{R})P$, where $(P)_{ij} = 1$ if $j = \sigma(i)$ and $(P)_{ij} = 0$ otherwise.*

If $i \prec j$ and there is no k such that $i \prec k \prec j$, then j is said to *cover* i. This relation will be denoted by $i \prec\!\!: j$. The definition of an incidence algebra implies the following result.

LEMMA 2.2. *Let $A, B \in \mathcal{A}_n(\preceq, \mathcal{R})$ and $i \prec\!\!: j$. Then $(AB)_{ij} = (A)_{ii}(B)_{ij} + (A)_{ij}(B)_{jj}$.*

For an incidence algebra $\mathcal{A} = \mathcal{A}_n(\preceq, \mathcal{R})$, we denote by

$$\mathcal{Z} = \mathcal{Z}(\mathcal{A}) = \{A \in \mathcal{A} \mid (A)_{ii} = 0,\ i = 1, \dots, n\},$$

the set of matrices with zero diagonals. By Proposition 2.1, $\mathcal{Z}$ is a two-sided ideal. Let $\mathcal{Z}^k$ be the set of all possible sums of products $A_{i_1} \cdot \dots \cdot A_{i_m}$ for any $m \leq k$ and all (possibly repeating) matrices $A_{i_j} \in \mathcal{Z}$. We have $\mathcal{Z}^n = (O_n)$ since $P^{-1}\mathcal{Z}(\mathcal{A})P \subseteq \mathcal{Z}(T_n(\mathcal{R}))$ by Proposition 2.1.

Consider the ideal $\mathcal{Z}^2$. Lemma 2.2 implies that $(A)_{ij} = 0$ for all $i \prec\!\!: j$ and any $A \in \mathcal{Z}^2$. Conversely, for any $i, j \in \mathcal{N}$ satisfying $i \prec j$ and $i \not\prec\!\!: j$, one may find a chain $i = i_1 \prec\!\!: i_2 \prec\!\!: \dots \prec\!\!: i_k = j$ of length $k \geq 3$ and so $E_{ij} = E_{i_1 i_2} \cdot (E_{i_2 i_3} \cdot \dots \cdot E_{i_{k-1} i_k})$ belongs to $\mathcal{Z}^2$. It follows that

$$\mathcal{Z}^2 = \{A \in \mathcal{Z}(\mathcal{A}) \mid (A)_{ij} = 0 \text{ for all } i \prec\!\!: j\}. \tag{2.1}$$

The word 'graph' will mean an undirected graph without loops and multiple edges, that is, a pair $G = (V, E)$ with sets of vertices V and edges E; each edge is a set of two distinct vertices. A graph is complete if any two of its vertices are joined by an edge. Given an arbitrary graph G and any complete subgraph G', the set of vertices

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of G' is called a clique. Each clique with the maximum number of vertices is said to be a maximum clique and its cardinality is the clique number $\omega = \omega(G)$ of the graph G.

The *complement* $\overline{G} = (V, \overline{E})$ of G is the graph with the same set of vertices V such that for all $x, y \in V$, we have $\{x, y\} \in \overline{E}$ if and only if $\{x, y\} \notin E$. Thus, $E \cap \overline{E} = \varnothing$ and $(V, E \cup \overline{E})$ is a complete graph.

For a partial order $\preceq$ on $\mathcal{N} = \{1, \dots, n\}$, the *(undirected) Hasse diagram* is the graph with the set of vertices $\mathcal{N}$ such that $x, y \in \mathcal{N}$ are joined by an edge if and only if either $x \lessdot y$ or $y \lessdot x$.

Given $m \in \mathbb{N}$, then $\{0, 1\}^m$ denotes the set of all possible 2^m tuples of zeros and ones of length m. We will denote by Q_m the *m-hypercube graph*. Its set of vertices is $\{0, 1\}^m$. Two tuples are joined by an edge if and only if they differ in precisely one coordinate. So two tuples are joined in $\overline{Q_m}$ if and only if they differ at least in two coordinates.

PROPOSITION 2.3 (Folklore). *The graph $\overline{Q_m}$ has the following properties:*

(1) *each vertex of $\overline{Q_m}$ has degree $2^m - m - 1$;*
(2) *the clique number $\omega(\overline{Q_m}) = 2^{m-1}$;*
(3) *there are exactly two maximum cliques C_0, C_1 in $\overline{Q_m}$;*
(4) *each vertex from C_0 is joined precisely to $2^{m-1} - m$ vertices from C_1 and vice versa;*
(5) *for $m = 1, 2$, the graph $\overline{Q_m}$ is disconnected; if $m \geq 3$, there exists a Hamiltonian cycle in $\overline{Q_m}$, that is, a cycle that visits each vertex precisely once (except for the starting vertex, which it visits twice);*
(6) *the graph $\overline{Q_m}$ is naturally isomorphic to a subgraph of $\overline{Q_{m+1}}$.*

PROOF. (1) Any vertex of the m-hypercube has degree m.

(2)–(4) Let C_0 and C_1 be the sets of all tuples that contain even and odd numbers of ones, respectively. Then $|C_0| = |C_1| = 2^{m-1}$. Two tuples with the same parity of the number of ones are joined in $\overline{Q_m}$ and so C_0 and C_1 are indeed cliques.

Let v be a vertex in C_0, then $\deg v - (|C_0| - 1) = 2^m - m - 1 - 2^{m-1} + 1 = 2^{m-1} - m$, that is, v is joined to $2^{m-1} - m$ vertices from C_1. The symmetric property holds for any $v \in C_1$. This proves item (4).

Consider an arbitrary clique C in $\overline{Q_m}$. The union $C_0 \cup C_1$ contains all vertices of the graph. If $C \subseteq C_0$ or $C \subseteq C_1$, then clearly $|C| \leq 2^{m-1}$. Assume that $C \cap C_0 \neq \varnothing$ and $C \cap C_1 \neq \varnothing$. Then $|C| \leq 2^{m-1} - m$ according to item (4). Thus, C_0, C_1 are maximum cliques. For the same reasons, they are the only maximum cliques in $\overline{Q_m}$. This proves items (2) and (3).

(5) The graph $\overline{Q_1}$ consists of two isolated vertices, while $\overline{Q_2}$ is the union of two segments. Let $m \geq 3$. We set $v = (0 \dots 0000)$, $v' = (0 \dots 0110)$, $w = (0 \dots 0111)$, $w' = (0 \dots 0001)$. A possible cycle starts at v, visits each vertex of C_0 and finishes at v'. After that, it comes to w', visits all vertices of C_1 and finishes at w. Eventually, the cycle returns to v.

(6) An embedding sends a tuple $(a_1 \dots a_m)$ to $(a_1 \dots a_m 0)$ for all $a_i \in \{0, 1\}$. $\square$

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3. A construction of an embedding of the Hasse diagram

Given idempotent matrices which generate an incidence algebra, it will be shown that the Hasse diagram of the poset is isomorphic to a subgraph of the complement of a hypercube. We assume first in the proof that the base ring $\mathcal{R}$ is a field and then we turn to the general case.

LEMMA 3.1. *Consider an incidence algebra $\mathcal{A} = \mathcal{A}_n(\preceq, \mathcal{R})$. Let $\mathcal{A}$ be generated by distinct idempotent matrices $A_1, \ldots, A_u$ as a unital $\mathcal{R}$-algebra for some $u \in \mathbb{N}$. Then the Hasse diagram of $\preceq$ is isomorphic to a subgraph of the complement $\overline{Q_m}$ of the m-hypercube for any $m \geq u$.*

PROOF

Case 1. Assume that $\mathcal{R} = \mathbb{F}$ is a field. If $m = 1$, then $u = 1$, and the linear span of the identity matrix I_n and A_1 coincides with $\mathcal{A}$. Hence, $\dim \mathcal{A} \leq 2$, and we have either $n = 1$ or $n = 2$ and $\mathcal{A} = D_2(\mathbb{F})$. In these cases, the Hasse diagram consists of one or two isolated points and so it can be embedded in $\overline{Q_1}$. Henceforth, we shall assume that $m \geq 2$.

Consider the tuple of matrices $(A_1, \ldots, A_m)$, where $A_{u+1} = A_1, \ldots, A_m = A_1$. Proposition 2.1 implies that there is no loss of generality in assuming that $\mathcal{A} \subseteq T_n(\mathcal{R})$ and so all the matrices $A_1, \ldots, A_m$ are upper-triangular. Since they are idempotent, their diagonal entries must be idempotent elements of the field $\mathbb{F}$. So the tuple $((A_1)_{ii}, \ldots, (A_m)_{ii})$ belongs to $\{0, 1\}^m$ for any $i \in \mathcal{N} = \{1, \ldots, n\}$. Consider the map $f : \mathcal{N} \longrightarrow \{0, 1\}^m$ given by $f(i) = ((A_1)_{ii}, \ldots, (A_m)_{ii})$. Item 1 of [13, Theorem] guarantees that f is injective.

We will show that f is an embedding of the Hasse diagram into the graph $\overline{Q_m}$. Assume in contrast that there exists a pair $i_* \prec\!\!: j_*$ such that the tuples $f(i_*)$ and $f(j_*)$ differ exactly in one position. In other words, one can find $k_* \in \{1, \ldots, m\}$ such that $(A_{k_*})_{i_*i_*} \neq (A_{k_*})_{j_*j_*}$, but for any $k \in \{1, \ldots, m\} \setminus \{k_*\}$, we have $(A_k)_{i_*i_*} = (A_k)_{j_*j_*}$.

Since $A_k^2 = A_k$ and $i_* \prec\!\!: j_*$, Lemma 2.2 implies $(1 - (A_k)_{i_*i_*} - (A_k)_{j_*j_*}) \cdot (A_k)_{i_*j_*} = 0$. When $k \neq k_*$, the entries $(A_k)_{i_*i_*}$ and $(A_k)_{j_*j_*}$ are equal and belong to $\{0, 1\}$. Consequently, $1 - (A_k)_{i_*i_*} - (A_k)_{j_*j_*} \neq 0$ and we have $(A_k)_{i_*j_*} = 0$ for any $k \neq k_*$.

Item 2 of [13, Theorem] implies that there exists B in the linear span $\langle A_1, \ldots, A_m \rangle$ such that $(B)_{i_*j_*} \neq 0$ and $(B)_{i_*i_*} = (B)_{j_*j_*}$. We can write $B = \lambda_1 A_1 + \cdots + \lambda_m A_m$ for some $\lambda_1, \ldots, \lambda_m$ from $\mathbb{F}$. Then at least one of the matrices $A_1, \ldots, A_m$ must have a nonzero i_*, j_*-entry. The only possibility is that $(A_{k_*})_{i_*j_*} \neq 0$ and $\lambda_{k_*} \neq 0$. Then

$$(B)_{i_*i_*} - (B)_{j_*j_*} = \sum_{k=1}^{m} \lambda_k ((A_k)_{i_*i_*} - (A_k)_{j_*j_*}) = \lambda_{k_*}((A_{k_*})_{i_*i_*} - (A_{k_*})_{j_*j_*}) \neq 0.$$

This contradicts the requirement that $(B)_{i_*i_*} = (B)_{j_*j_*}$.

Case 2. Let $\mathcal{R}$ be an arbitrary unital commutative ring. Consider a maximal ideal $\mathfrak{m} \lhd \mathcal{R}$ and the residue field $\mathbb{F} = \mathcal{R}/\mathfrak{m}$. Let $\pi : \mathcal{R} \to \mathcal{R}/\mathfrak{m}$ be the natural projection. We construct the surjective ring homomorphism $\pi_n : \mathcal{A}_n(\preceq, \mathcal{R}) \to \mathcal{A}_n(\preceq, \mathbb{F})$ given by

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$(\pi_n(A))_{ij} = \pi((A)_{ij})$ for all $i, j = 1, \ldots, n$. Note that π_n preserves scalar matrices, that is, $\pi_n(\mathcal{R}I_n) = \mathbb{F}I_n$.

Consider the matrices $\pi_n(A_1), \ldots, \pi_n(A_u)$. Though some of them may coincide, there is no loss of generality to assume that $\pi_n(A_1), \ldots, \pi_n(A_{u'})$ are all possible pairwise distinct matrices and $u' \le u$. Since the set $\{A_1, \ldots, A_u\} \cup \mathcal{R}I_n$ generates $\mathcal{A}_n(\preceq, \mathcal{R})$ as a ring, the image $\{\pi_n(A_1), \ldots, \pi_n(A_{u'})\} \cup \mathbb{F}I_n$ generates $\mathcal{A}_n(\preceq, \mathbb{F})$ as a ring. Thus, $\mathcal{A}_n(\preceq, \mathbb{F})$ is generated by idempotents $\pi_n(A_1), \ldots, \pi_n(A_{u'})$ as a unital $\mathbb{F}$-algebra. According to Case 1, the Hasse diagram of $\preceq$ is isomorphic to a subgraph of $\overline{Q_m}$ for all $m \ge u'$. In particular, we may take any $m \ge u$. □

4. A construction of idempotent generators

The following lemma demonstrates how to obtain idempotent matrices that generate the incidence algebra when we have an embedding of the Hasse diagram into the complement of a hypercube.

LEMMA 4.1. *Let the set $\mathcal{N} = \{1, \ldots, n\}$ be partially ordered by a relation $\preceq$. Assume that the Hasse diagram of $\preceq$ is isomorphic to a subgraph of $\overline{Q_m}$. Then there exists a natural number $u \le m$ such that the incidence algebra $\mathcal{A} = \mathcal{A}_n(\preceq, \mathcal{R})$ can be generated by some distinct idempotent matrices $A_1, \ldots, A_u$.*

PROOF. Let $f : \mathcal{N} \longrightarrow \{0, 1\}^m$ be an embedding of the Hasse diagram into the graph $\overline{Q_m}$. Given $i \in \mathcal{N}$, then $f(i) = (a_1^{(i)}, \ldots, a_m^{(i)})$ for some $a_k^{(i)} \in \{0, 1\}$. The definition of $\overline{Q_m}$ implies that for any pair $i \prec: j$, there exist indices $\eta \ne \theta$ such that $a_\eta^{(i)} \ne a_\eta^{(j)}$ and $a_\theta^{(i)} \ne a_\theta^{(j)}$. So we may consider $\eta = \eta(i, j)$ and $\theta = \theta(i, j)$ as functions of pairs $i \prec: j$.

Proposition 2.1 guarantees that there is no loss of generality in assuming that all matrices of $\mathcal{A}$ are upper-triangular. For any $k = 1, \ldots, m$, we introduce the matrix

$$(\widetilde{A}_k)_{ij} = \begin{cases} a_k^{(i)} & \text{if } i = j; \\ 1 & \text{if } i \prec: j \text{ and } \eta(i, j) = k; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widetilde{S}$ be the set of these matrices. Note that $\widetilde{S} \subseteq \mathcal{A}$. We need to show that $\widetilde{S}$ generates $\mathcal{A}$ as a unital $\mathcal{R}$-algebra. Conditions A, B of [8, Theorem 1.1] must be checked.

Since the map f is an embedding, it is injective. So for any pair $i \ne j$, the tuples $(a_1^{(i)}, \ldots, a_m^{(i)})$ and $(a_1^{(j)}, \ldots, a_m^{(j)})$ are distinct, that is, there exists an index $\xi = \xi(i, j)$ such that $a_\xi^{(i)} \ne a_\xi^{(j)}$. Therefore, $(\widetilde{A}_\xi)_{ii} - (\widetilde{A}_\xi)_{jj} = \pm 1$ and Condition A of [8, Theorem 1.1] holds since $1 \in \mathfrak{a}_{ij}(\widetilde{S})$.

Consider a pair $i \prec: j$. Let $\eta = \eta(i, j)$ and $\theta = \theta(i, j)$. Then

$$\begin{pmatrix} (\widetilde{A}_\eta)_{ii} & (\widetilde{A}_\eta)_{ij} \\ 0 & (\widetilde{A}_\eta)_{jj} \end{pmatrix} = \begin{pmatrix} x & 1 \\ 0 & 1 - x \end{pmatrix}, \quad \begin{pmatrix} (\widetilde{A}_\theta)_{ii} & (\widetilde{A}_\theta)_{ij} \\ 0 & (\widetilde{A}_\theta)_{jj} \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & 1 - y \end{pmatrix}$$

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for some $x, y \in \{0, 1\} \subseteq \mathcal{R}$. We set

$$\widetilde{B} = \begin{cases} \widetilde{A}_\eta + \widetilde{A}_\theta & \text{if } x \neq y, \\ \widetilde{A}_\eta - \widetilde{A}_\theta + I_n & \text{if } x = y. \end{cases}$$

Consequently,

$$\begin{pmatrix} (\widetilde{B})_{ii} & (\widetilde{B})_{ij} \\ 0 & (\widetilde{B})_{jj} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then Condition B of [8, Theorem 1.1] is satisfied since $1 \in \mathfrak{b}_{ij}(\widetilde{S})$.

Thus, [8, Theorem 1.1] guarantees that the set $\widetilde{S}$ generates $\mathcal{A}$ as a unital $\mathcal{R}$-algebra. However, the matrices $\widetilde{A}_1, \ldots, \widetilde{A}_m$ are not necessarily idempotent. We need to 'slightly' change their entries to make them idempotent generators.

Consider the ideal $\mathcal{Z}^2$ from (2.1). Let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{Z}^2$ be the natural projection onto the quotient algebra $\mathcal{A}/\mathcal{Z}^2$. We will prove that $\pi(\widetilde{A}_k^2 - \widetilde{A}_k) = 0$ for any $k = 1, \ldots, m$. Since the matrices are triangular and their diagonals contain only zeros and ones, we have $(\widetilde{A}_k^2)_{ii} - (\widetilde{A}_k)_{ii} = 0$ for all $i \in \mathcal{N}$. Fix a pair $i \lessdot j$. If $(\widetilde{A}_k)_{ij} = 0$, then $(\widetilde{A}_k^2)_{ij} = 0$ by Lemma 2.2. In the case $(\widetilde{A}_k)_{ij} = 1$, we have $k = \eta(i, j)$ and so $(\widetilde{A}_k)_{ii} \neq (\widetilde{A}_k)_{jj}$. Then one of the entries $(\widetilde{A}_k)_{ii}$, $(\widetilde{A}_k)_{jj}$ equals 1 and the other equals 0. By Lemma 2.2, it follows that $(\widetilde{A}_k^2)_{ij} = 1$. Thus, we have shown that $\widetilde{A}_k^2 - \widetilde{A}_k \in \mathcal{Z}^2$, which is equivalent to $\pi(\widetilde{A}_k^2 - \widetilde{A}_k) = 0$.

Since π is an algebra homomorphism, we obtain $\pi(\widetilde{A}_k)^2 = \pi(\widetilde{A}_k)$, that is, all the images $\pi(\widetilde{A}_1), \ldots, \pi(\widetilde{A}_m)$ are idempotents in the algebra $\mathcal{A}/\mathcal{Z}^2$. It is known that idempotents can be lifted modulo a nil ideal [12, Section 3.6, Proposition 1]. Hence, there exist matrices $A_1, \ldots, A_m$ from $\mathcal{A}$ such that $A_k^2 = A_k$ and $\pi(A_k) = \pi(\widetilde{A}_k)$ for all $k = 1, \ldots, n$. Let S denote the set of these matrices.

We will prove that S generates $\mathcal{A}$ as a unital $\mathcal{R}$-algebra. Since $A_k - \widetilde{A}_k \in \mathcal{Z}^2$ for any $k = 1, \ldots, m$, we have $(A_k)_{ii} = (\widetilde{A}_k)_{ii}$ for all $i \in \mathcal{N}$ and $(A_k)_{ij} = (\widetilde{A}_k)_{ij}$ for all pairs $i \lessdot j$. Then $\mathfrak{a}_{ij}(S) = \mathfrak{a}_{ij}(\widetilde{S}) = \mathcal{R}$ and $\mathfrak{b}_{ij}(S) = \mathfrak{b}_{ij}(\widetilde{S}) = \mathcal{R}$ in the terms of [8, Theorem 1.1]. Therefore, S generates $\mathcal{A}$ as a unital $\mathcal{R}$-algebra.

It remains to note that the matrices $A_1, \ldots, A_m$ are not necessarily pairwise distinct. So we may choose some $u \in \mathbb{N}$ distinct matrices among $A_1, \ldots, A_m$ as required in the lemma. □

5. The main result

In this section, we provide the solution to Problem 1.1 in Theorem 5.6. Before doing that, we need to unite the results from two previous sections in one theorem.

THEOREM 5.1. *Consider an incidence algebra $\mathcal{A} = \mathcal{A}_n(\preceq, \mathcal{R})$. Then the Hasse diagram of $\preceq$ is isomorphic to a subgraph of the complement $\overline{Q_m}$ of an m-hypercube if and only if there exists $u \in \{1, \ldots, m\}$ such that $\mathcal{A}$ can be generated by some distinct idempotent matrices $A_1, \ldots, A_u$.*

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PROOF. The implication (⇒) is given by Lemma 4.1, while the inverse (⇐) follows from Lemma 3.1. □

Hence, we need to consider subgraphs of $\overline{Q_m}$. The following lemma imposes restrictions on the possible number of vertices in such a subgraph.

LEMMA 5.2. *Consider an arbitrary graph G on n vertices and the complement $\overline{Q_m}$ of an m-hypercube. If $m < \lceil \log_2 n \rceil$, there are no subgraphs of $\overline{Q_m}$ isomorphic to G. When $m = \lceil \log_2 n \rceil + 1$, there exists a subgraph of $\overline{Q_m}$ isomorphic to G.*

PROOF. Assume that G is a subgraph of $\overline{Q_m}$. Then $\overline{Q_m}$ must have at least n vertices, so $2^m \geq n$ or $m \geq \lceil \log_2 n \rceil$.

Now let $m = \lceil \log_2 n \rceil + 1$. Proposition 2.3(2) implies that the graph $\overline{Q_m}$ contains a clique of cardinality $2^{m-1} = 2^{\lceil \log_2 n \rceil} \geq n$. Hence, any graph on n vertices can be embedded in $\overline{Q_m}$. □

The previous lemma fails to cover only the case when $m = \lceil \log_2 n \rceil$. In fact, $\overline{Q_{\lceil \log_2 n \rceil}}$ can contain G or not. So we can divide all graphs into two disjoint classes.

DEFINITION 5.3. A graph G on n vertices belongs to *the zeroth idempotent class* if it is isomorphic to a subgraph of the complement $\overline{Q_m}$ of the m-hypercube for $m = \lceil \log_2 n \rceil$. If a graph does not belong to the zeroth idempotent class, it is said to belong to *the first idempotent class*. The number of the class will be denoted by $\mathrm{idem}(G)$. So we have $\mathrm{idem}(G) \in \{0, 1\}$.

The following example guarantees that for any $m \in \mathbb{N}$, there exists a graph on $n = 2^m$ vertices that belongs to the first idempotent class. Moreover, this graph is the Hasse diagram of some poset.

EXAMPLE 5.4. If a graph G on n vertices has a vertex of degree greater than $2^{\lceil \log_2 n \rceil} - \lceil \log_2 n \rceil - 1$, then G belongs to the first class by virtue of Proposition 2.3(1). For instance, we may consider a partial order on the set $\{1, \ldots, 2^m\}$ such that all elements $2, 3, \ldots, 2^m$ are pairwise incomparable and 1 is the least element. Then the Hasse diagram of this poset belongs to the first class since the vertex 1 has degree $2^m - 1 > 2^m - m - 1$.

The zeroth idempotent class is closed under the disjoint union of graphs on 2^k vertices, but the first class is not.

EXAMPLE 5.5. The disjoint union $G_1 + G_2$ of two arbitrary graphs G_1, G_2 on 2^k vertices always belongs to the zeroth idempotent class. Indeed, the total number of vertices of $G_1 + G_2$ equals $n = 2^{k+1}$ and so $\lceil \log_2 n \rceil = k + 1$. Proposition 2.3(2)–(3) implies that the graph $\overline{Q_{\lceil \log_2 n \rceil}}$ has two disjoint cliques C_0, C_1 of cardinality 2^k. Hence, G_1 and G_2 can be embedded in C_0 and C_1, respectively.

The following theorem provides the solution to Problem 1.1.

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THEOREM 5.6. *Consider an incidence algebra $\mathcal{A} = \mathcal{A}_n(\preceq, \mathcal{R})$. Then the following quantities coincide.*

(1) *The minimum number ν of idempotent matrices that generate $\mathcal{A}$ as a unital $\mathcal{R}$-algebra.*
(2) *The minimum $m \in \mathbb{N}$ such that the Hasse diagram of $\preceq$ can be embedded in the complement $\overline{Q_m}$ of the m-hypercube.*
(3) *The sum $\lceil \log_2 n \rceil + \operatorname{idem}(\lessdot)$, where $\operatorname{idem}(\lessdot) \in \{0, 1\}$ is the number of the idempotent class of the Hasse diagram of $\preceq$ in terms of Definition 5.3.*

PROOF. Quantities (1) and (2) are equal by Theorem 5.1. Also, Lemma 5.2 implies that quantities (2) and (3) coincide. □

As a consequence, we can provide another proof of item (i) of [7, Theorem 6]. It turns out that idempotent generators of $T_n(\mathcal{R})$ are related to Hamiltonian paths in a graph.

COROLLARY 5.7 [7, Kelarev *et al.*]. *Let ν be the minimum number of idempotent generators of the algebra $T_n(\mathcal{R})$ of all $n \times n$ upper-triangular matrices. Then $\nu = \lceil \log_2 n \rceil$ if $n \geq 5$ and $\nu = \lceil \log_2 n \rceil + 1$ for $n = 2, 3, 4$.*

PROOF. The algebra $T_n(\mathcal{R})$ is an incidence algebra over the standard linear order $\leq$ on $\{1, \dots, n\}$. Then the (undirected) Hasse diagram of $\leq$ is the graph with edges $\{1, 2\}$, $\{2, 3\}$, ..., $\{n-1, n\}$. It can be embedded in $\overline{Q_{\lceil \log_2 n \rceil}}$ if and only if $\overline{Q_{\lceil \log_2 n \rceil}}$ contains a simple path on n vertices. Proposition 2.3(5) implies that $\overline{Q_m}$ has a Hamiltonian path whenever $m \geq 3$, that is, $\lceil \log_2 n \rceil \geq 3$, or $n \geq 5$. The graph $\overline{Q_1}$ does not have edges and $\overline{Q_2}$ does not contain a simple path on 3 vertices (and on 4 as well). It remains to apply Theorem 5.6. □

PROBLEM 5.8. Let $\mathcal{K}$ be a noncommutative associative ring with an identity. Given an incidence ring $\mathcal{A} = \mathcal{A}_n(\preceq, \mathcal{K})$, find the minimum cardinality ν of a set S of idempotent matrices such that its union with the set of scalar matrices $S \cup \mathcal{K} I_n$ generates $\mathcal{A}$ as a unital ring.

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