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ABELIAN GROUPS THAT ARE TORSION OVER THEIR ENDOMORPHISM RINGS

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Using Lambek torsion as the torsion theory, we investigate the question of when an Abelian group G is torsion as a module over its endomorphism ring E. Groups that are torsion modules in this sense are called $\mathcal{L}$-torsion. Among the classes of torsion and truly mixed Abelian groups, we are able to determine completely those groups that are $\mathcal{L}$-torsion. However, the case when G is torsion free is more complicated. Whereas no torsion-free group of finite rank is $\mathcal{L}$-torsion, we show that there are large classes of torsion-free groups of infinite rank that are $\mathcal{L}$-torsion. Nevertheless, meaningful definitive criteria for a torsion-free group to be $\mathcal{L}$-torsion have not been found.

1. INTRODUCTION

All modules considered here are unitary left modules. The ring of endomorphisms of an Abelian group G is denoted by E or by $E(G)$, if the more elaborate notation is needed for clarity. By virtue of a natural endowment, G becomes a member of E-Mod, that is, G is a left module over its endomorphism ring. If G is in the torsion class of the Lambek torsion theory on E-Mod, we say that G is $\mathcal{L}$-torsion or, if the meaning is clear, simply torsion over E.

The properties of Abelian groups as modules over their endomorphism rings have been investigated in a number of papers including [**1, 2, 5, 8, 9, 10, 11, 12, 13, 14**]. Topics studied include the question of when an Abelian group is projective, flat, finitely generated, or cyclic as a module over its endomorphism ring. Here, we investigate the same question for torsion: when is an Abelian group torsion over its endomorphism ring? For torsion-free Abelian groups, this question was considered by Faticoni in [**4**]. In that paper, a variation of Corner's well-known construction [**3**] was used to produce a countable torsion-free group G which is torsion over its endomorphism ring $E \cong \mathbb{Z}[x]$. Since in this case E is an integral domain, it was not necessary to designate the torsion theory employed because in this case all the usual torsion theories coincide with ordinary torsion; by ordinary torsion (for modules over an integral domain) we mean an element is torsion if it is annihilated by a nonzero scalar. But for general rings R, the

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determination of whether or not an R-module is torsion depends of course on the choice of the torsion theory for R-Mod. Moreover, the concept of an element being torsion when it is annihilated by a nonzero scalar does not in general yield a legitimate torsion theory. Hence, in this paper, as indicated earlier, we use Lambek torsion exclusively as the torsion theory for Abelian groups as modules over their endomorphism rings. For a complete treatment of torsion theories for R-Mod in general, we refer to [**6**].

When we say simply that an Abelian group G is torsion or torsion free, we mean that G is torsion or torsion free as a $\mathbb{Z}$-module, whereas when we say that G is $\mathcal{L}$-torsion we mean that G is torsion as a module over its endomorphism ring. Our objective therefore is to describe those groups that are $\mathcal{L}$-torsion. We are able to meet fully this objective for torsion and truly mixed Abelian groups, but only partial results are obtained for torsion-free groups. It perhaps may come as some surprise that torsion-free groups have more propensity for being torsion over their endomorphism rings than do torsion groups, notwithstanding the fact a nonzero torsion-free group of finite rank can never be $\mathcal{L}$-torsion. In Section 4, we demonstrate how to construct an abundance of countable torsion-free Abelian groups G that are torsion over their endomorphism rings; typically, $E(G)$ is a noncommutative ring in our construction.

Recall that the torsion class $\mathcal{T}$ for Lambek torsion in R-Mod is the class generated by the injective hull of R. Thus

$$\mathcal{T} = \{T \in R\text{-Mod} : \operatorname{Hom}(T, \overline{R}) = 0\}$$

where $\overline{R}$ denotes the injective hull of R considered as a left module over itself. The following characterisation of $\mathcal{T}$ will prove very useful. If the reader prefers, this characterisation can be taken as the definition of Lambek torsion, so we omit the proof.

Proposition 1. *An R-module M belongs to $\mathcal{T}$, the Lambek torsion class, if and only if for each $x \in M$ and each nonzero $c \in R$ there exists $r \in R$ such that $rx = 0$ but $rc \neq 0$.*

From Proposition 1, it quickly follows that an Abelian group G is $\mathcal{L}$-torsion if and only if given any $x \in G$ and any nonzero endomorphism φ of G, there exists an endomorphism π of G for which $\pi(x) = 0$ but $\pi\varphi \neq 0$. (We compose mappings from right to left.)

We shall investigate the Abelian groups that are $\mathcal{L}$-torsion by cases, distinguishing the cases where G is torsion, mixed, or torsion free. First, however, we establish the following general lemma.

Lemma 2. *If G has a nonzero cyclic endomorphic image as a $\mathbb{Z}$-module, then G cannot be $\mathcal{L}$-torsion.*

Proof: Suppose that $\varphi \neq 0$ is an endomorphism of G and that $\varphi(G) = \mathbb{Z}c$ for some nonzero $c \in G$. If π is any endomorphism of G such that $\pi(c) = 0$, then clearly $\pi\varphi = 0$. Therfore, G cannot be $\mathcal{L}$-torsion. ∎

2. Torsion groups that are $\mathcal{L}$-torsion

First, observe that the divisible p-primary group $\mathbb{Z}(p^\infty)$ is $\mathcal{L}$-torsion, for the endomorphism ring of $\mathbb{Z}(p^\infty)$ has no zero divisors. Thus, the criterion given in Proposition 1 simplifies to the existence of a nonzero endomorphism that annihilates a given element x of the group. This simple example is typical of the general situation.

Theorem 3. *A torsion Abelian group is $\mathcal{L}$-torsion if and only if it is divisible.*

Proof: Since a torsion group that is not divisible always has a nonzero cyclic summand, it follows from Lemma 2 that a torsion group G must be divisible in order to be $\mathcal{L}$-torsion.

Conversely, let G be a divisible torsion group and assume that $x \in G$ and a nonzero $\varphi \in E(G)$ are given. Since $\varphi(G)$ is a divisible group, it contains a divisible subgroup $D \cong \mathbb{Z}(p^\infty)$ for some prime p. Since D is a summand of G, there is an endomorphism ρ of G (indeed, a projection) that is the identity map on D and maps G onto D. Clearly then there exists a nonzero endomorphism π_0 of $D \cong \mathbb{Z}(p^\infty)$ for which $\pi_0\rho(x) = 0$, where $\pi_0\rho(G) = D$. Obviously, π_0 can be extended to an endomorphism of G (which we again call π_0). If we now set $\pi = \pi_0\rho$, we have the desired result because $\pi(x) = 0$ and

$$\pi\varphi(G) = \pi_0\rho\varphi(G) = \pi_0(D) = D$$

implies that $\pi\varphi \neq 0$. ∎

3. Mixed groups that are $\mathcal{L}$-torsion

In this section, we consider the truly mixed groups – those that are not torsion but have a nontrivial torsion subgroup. We completely determine which ones are $\mathcal{L}$-torsion.

The next example is illustrative of the general theory for mixed groups.

Example 4. Let $G = \mathbb{Z}(p^\infty) \oplus \mathbb{Q}$ for some prime p. If φ is a nonzero endomorphism of G, then $\varphi(G)$ is divisible and it is easy to see that there exists a homomorphism ρ from $\varphi(G)$ onto $G_t = \mathbb{Z}(p^\infty)$, the torsion subgroup of G. Since G_t is injective, ρ can be extended to a mapping from G onto G_t, so there is an endomorphism ρ of G with the property that $\rho\varphi(G) = G_t$. As in the proof of Theorem 3, if $x \in G$, there exists an endomorphism π_0 of G such that $\pi_0\rho(\varphi(G)) = G_t$ and $\pi_0\rho(x) = 0$. Thus, $\pi = \pi_0\rho$ has the desired features, and G is $\mathcal{L}$-torsion.

We say that a prime p is *relevant* for an Abelian group G if the p-primary component G_p of the torsion subgroup G_t of G is not zero.

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THEOREM 5. *Let G be an Abelian group with a nontrivial torsion subgroup G_t. Then G is $\mathcal{L}$-torsion if and only if*

(1) *G_t is divisible, and*

(2) *$A \cong G/G_t$ is p-divisible for each relevant prime p of G.*

PROOF: First suppose that G is $\mathcal{L}$-torsion. By Lemma 2, G_t must be divisible, for otherwise G has a nonzero cyclic summand (and therefore a nonzero cyclic endomorphic image). Let $G = G_t \oplus A$ and suppose p is any relevant prime for G. If A were not p-divisible, then $A/pA \neq 0$. Therefore, there would be a nonzero homomorphism from A/pA into G_t. Consequently, we would have a nonzero homomorphism from A into $G[p]$. Since A is a summand of G, this leads quickly to a nonzero endomorphism whose image is a cyclic subgroup of $G[p]$. But this is impossible since it would preclude G from being $\mathcal{L}$-torsion by Lemma 2.

Conversely, suppose that G satifies conditions (1) and (2) and that $x \in G$ and a nonzero $\varphi \in E(G)$ are given. We claim that there exists a subgroup $D \cong \mathbb{Z}(p^\infty)$ of G and an endomorphism ρ of G that maps both G and $\varphi(G)$ onto D. Since D is divisible, to prove this claim it suffices to demonstrate the existence of a homomorphism ρ from $\varphi(G)$ onto D, because such a ρ can always be extended to a homomorphism from G onto D. If $\varphi(G)$ has torsion, then it has a summand $D \cong \mathbb{Z}(p^\infty)$ for some p and we can take $\rho : \varphi(G) \to D$ to be the projection associated with a decomposition $\varphi(G) = D \oplus H$ of $\varphi(G)$. If $\varphi(G)$ is torsion free, it is a torsion-free subgroup that is p-divisible for any relevant prime p for G, and there is at least one such p. For a fixed relevant prime p, choose a subgroup $D \cong \mathbb{Z}(p^\infty)$ of G_t. Since $\varphi(G)$ is a nonzero torsion-free group that is p-divisible, there is a homomorphism ρ from $\varphi(G)$ onto D. Thus, we have verified the claim. Now, as in the proof of Theorem 3, there is an endomorphism π_0 of G such that $\pi_0(D) = D$ and $\pi_0\rho(x) = 0$. Therefore, $\pi = \pi_0\rho$ has the properties required to demonstrate that G is $\mathcal{L}$-torsion. ∎

REMARK. Even though Theorem 3 may be viewed as a special case of Theorem 5, the two cases are distinguished inasmuch as the proof of Theorem 5 uses the proof of Theorem 3.

4. TORSION-FREE GROUPS THAT ARE $\mathcal{L}$-TORSION

The first question here is the following. Can a nonzero torsion-free group be $\mathcal{L}$-torsion? The next two results suggest a negative answer.

PROPOSITION 6. *No torsion-free group that is $\mathcal{L}$-torsion can have a nonzero endomorphic image of finite rank.*

PROOF: Let G be a torsion-free group that is $\mathcal{L}$-torsion, and suppose to the contrary that G has a nonzero endomorphic image of finite rank. Among all nonzero

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endomorphisms of G, choose φ so that $\varphi(G)$ has smallest possible rank n. Select any nonzero $x \in \varphi(G)$. Since G is $\mathcal{L}$-torsion, there is an endomorphim π of G such that $\pi(x) = 0$, but $\pi\varphi \neq 0$. Clearly, $\pi\varphi(G)$ has rank less than n, a contradiction on the choice of φ. ∎

COROLLARY 7. *A nonzero torsion-free group of finite rank cannot be $\mathcal{L}$-torsion.*

The above results notwithstanding, there are many torsion-free groups of countable rank that are $\mathcal{L}$-torsion. Indeed, as mentioned in Section 1, a result in [**4**] provides such an example. However, the example constructed there could be considered as a somewhat atypical torsion-free group of infinite rank since its endomorphism ring is an integral domain. In this section, we show that torsion-free groups of infinite rank that are $\mathcal{L}$-torsion are not at all uncommon; in fact, suitable direct sums of what we call infinite rank torsion-free groups of type 1 always turn out to be $\mathcal{L}$-torsion.

Recall that a *cotype* of a torsion-free group G is the type of a torsion-free homomorphic image A of rank 1. We identify A with its type, and thus regard A itself as a cotype of G. Moreover, there is no loss in assuming that each cotype of G is a nonzero subgroup of $\mathbb{Q}$. For the statement of our next result, it will be convenient to have the following terminology.

DEFINITION 8: Suppose that G is a torsion-free group of finite rank and that P is the set consisting of those primes p such that G has a cotype with ∞ at p. Then, we say that G satisfies the *cotype condition* if P is not cofinite in the set of all primes.

PROPOSITION 9. *Suppose that G_n is a torsion-free group of finite rank n that satisfies the cotype condition. Then, G_n is contained as a pure subgroup in a torsion-free group G_{n+1} of rank $n+1$ such that G_{n+1} satisfies the cotype condition and no cotype of G_{n+1} is less than or equal to a cotype of G_n.*

PROOF: Let $\{x_1, x_2, \dots, x_n\}$ be a maximal $\mathbb{Z}$-independent subset of G_n. Thus,

$$G_n = \langle x_1, x_2, \dots, x_n \rangle_*$$

where H_* represents the purification in G_n of a subgroup H. Let P_n be the (possibly empty) set of primes p for which G_n has a cotype with ∞ at the prime p. Choose $n+1$ distinct primes $q_1, q_2, \dots, q_{n+1}$ not contained in P_n and set

$$G_{n+1} = \left\langle G_n, \frac{x_{n+1}}{q_{n+1}^k}, \frac{x_i + x_{n+1}}{q_i^k} \right\rangle_{k<\omega, 1 \leqslant i \leqslant n}$$

where x_{n+1} is a new generator. More precisely,

$$G_{n+1} = \langle G_n, y_k, z_{i,k} \rangle_{k<\omega, 1 \leqslant i \leqslant n}$$

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subject to the relations

$$y_0 = x_{n+1}, \quad q_{n+1} y_k = y_{k-1} \text{ if } k \geqslant 1,$$
$$z_{i,0} = x_i + x_{n+1} \text{ if } 1 \leqslant i \leqslant n, \quad q_i z_{i,k} = z_{i,k-1} \text{ if } k \geqslant 1 \text{ and } 1 \leqslant i \leqslant n.$$

Clearly, G_{n+1} has rank $n+1$. Note also that G_n is pure in G_{n+1} since G_{n+1}/G_n is torsion free. Moreover, G_{n+1} satisfies the cotype condition. Indeed, if B is a cotype of G_{n+1}, then the type of B can have ∞ only at primes in the set $P_{n+1} = P_n \cup \{q_1, q_2, \dots, q_{n+1}\}$ and P_{n+1} is not cofinite in the set of all primes.

It remains to show that no cotype of G_{n+1} is less than or equal to a cotype of G_n. Let A be a cotype of G_n and first observe that the cotype G_{n+1}/G_n of G_{n+1} is not less than or equal to A. Indeed, the coset $x_{n+1}+G_n$ is a nonzero element of G_{n+1}/G_n with infinite q_{n+1}-height, and $q_{n+1} \notin P_n$. Therefore, it is enough to show that no nonzero map from G_n into A can be extended to a map from G_{n+1} into A. So, suppose to the contrary that $\varphi : G_n \to A$ is a nonzero map that extends to G_{n+1}. Then, the extension must map x_{n+1} to 0 since x_{n+1} has infinite q_{n+1}-height in G_{n+1}, but no nonzero element of A has infinite q_{n+1}-height because $q_{n+1} \notin P_n$. Consequently, for $i \leqslant n$, x_i must also map to 0, for otherwise $x_i + x_{n+1}$ would not map to 0. But this is impossible since $x_i + x_{n+1}$ has infinite q_i-height in G_{n+1}, whereas no element of A has infinite q_i-height because $q_i \notin P_n$. Thus $\varphi = 0$, which contradicts the choice of φ. □

DEFINITION 10: Call a torsion-free group G of infinite rank *an infinite rank group of type 1* if no cotype of G is a cotype of a nonzero finite rank subgroup of G.

Our next result effectively establishes an abundance of infinite rank groups of type 1.

THEOREM 11. *Let G be the union of a sequence of subgroups G_n such that G_n and G_{n+1} satisfy the hypothesis and conclusion, respectively, of Proposition 9 for all $n < \omega$. Then G is an infinite rank group of type 1.*

PROOF: Assume that $B = G/H$ is a cotype of G, and suppose to the contrary that B is also a cotype of a finite rank subgroup F of G. Then, $F \subseteq G_n$ for some n and, by selecting n large enough, we may assume that G_{n+1} is not contained in H. Since $F \subseteq G_n$ and since B is a cotype of F, then B is less than or equal to a cotype of G_n. Indeed, because $B \subseteq \mathbb{Q}$, any epimorphism $\varphi : F \twoheadrightarrow B$ can be extended to a homomorphism from G_n into $\mathbb{Q}$ (whose image contains B). Therefore, G_{n+1} does not have a cotype less than or equal to B. However, this is impossible because $G_{n+1}/(G_{n+1} \cap H)$ is a cotype of G_{n+1} and

$$G_{n+1}/(G_{n+1} \cap H) \cong (G_{n+1} + H)/H \subseteq G/H = B.$$

Thus, G must be an infinite rank group of type 1. □

THEOREM 12. *Suppose G is an infinite rank group of type 1, and let $\mathcal{F}$ be the set of all finite rank pure subgroups of G. Then*

$$G' = \bigoplus_{F \in \mathcal{F}} \Big(\bigoplus_{\aleph_0} (G/F) \Big)$$

is a torsion-free group that is $\mathcal{L}$-torsion.

PROOF: Suppose that φ is a nonzero endomorphism of G'. We claim that the image $\varphi(G')$ must have infinite rank. Indeed, if this is not the case, there exist $F_1, F_2 \in \mathcal{F}$ such that φ induces a nonzero map

$$\varphi_{(F_1,F_2)} : G/F_1 \to G/F_2$$

with a nonzero finite rank torsion-free image. Thus, $\varphi_{(F_1,F_2)}(G/F_1) = A/F_2$, where A is a nonzero finite rank subgroup of G that contains F_2. Now note that if B is a cotype of A/F_2, then B is a cotype of A. This is because there is a composition of surjective maps

$$A \twoheadrightarrow A/F_2 \twoheadrightarrow B.$$

Likewise, there is a composition of surjective maps

$$G \twoheadrightarrow G/F_1 \twoheadrightarrow A/F_2 \twoheadrightarrow B$$

which shows that B is a cotype of G; that is, G and its finite rank subgroup A share the common cotype B. However, this contradicts the hypothesis that G is an infinite rank group of type 1, and thus establishes the claim.

Now suppose that $x \in G'$ and write

$$x = (x_1 + F_1) + (x_2 + F_2) + \cdots + (x_n + F_n)$$

where all the x_i's are in G and $F_1, F_2, \ldots, F_n$ are (not necessarily distinct) elements of $\mathcal{F}$. Observe that

$$G' = (G/F_1) \oplus (G/F_2) \oplus \cdots \oplus (G/F_n) \oplus H.$$

Also,

$$G' = \big(G/\langle F_1, x_1\rangle_*\big) \oplus \big(G/\langle F_2, x_2\rangle_*\big) \oplus \cdots \oplus \big(G/\langle F_n, x_n\rangle_*\big) \oplus K$$

(where $*$ indicates purification in G). Since $H \cong G' \cong K$ and since there is a natural map from G/F_i onto $G/\langle F_i, x_i\rangle_*$ for each i, there must be an endomorphism π of G' such that π has finite rank kernel and $\pi(x) = 0$. Because φ has an infinite rank image, $\pi\varphi \neq 0$. Therefore, G' is $\mathcal{L}$-torsion. ∎

Applications of Theorems 11 and 12 yield the following.

Corollary 13. *There exist countable torsion-free groups G such that G is $\mathcal{L}$-torsion, and $E(G)$ is a noncommutative ring with zero divisors.*

Corollary 14. *Every infinite rank group of type 1 is a direct summand of a torsion-free Abelian group that is $\mathcal{L}$-torsion.*

Remark. In connection with the preceding corollary, it is easily seen that an infinite rank torsion-free homomorphic image of an infinite rank group of type 1 is also such a group. Therefore, the group G' constructed in Theorem 12 is a direct sum of infinite rank groups of type 1.

In [7], the Continuum Hypothesis was invoked to construct an $\aleph_1$-free group that does not have $\mathbb{Z}$ as a homomorphic image. It may be of interest to observe that such groups can also be used to construct torsion-free groups that are $\mathcal{L}$-torsion.

Proposition 15. *Suppose that G is an $\aleph_1$-free group that does not have $\mathbb{Z}$ as a homomorphic image. If $\mathcal{F}$ denotes the set of all finite rank pure subgroups of G, then*

$$G' = \bigoplus_{F \in \mathcal{F}} \Big(\bigoplus_{\aleph_0} (G/F) \Big)$$

is $\mathcal{L}$-torsion.

Proof: The first part of the proof of Theorem 12 can be adapted to show that a nonzero endomorphism φ of G' must have an infinite rank image. Indeed, if this were not the case, $\mathbb{Z}$ would then be a homomorphic image of G. Moreover, for a given $x \in G'$, the construction of a suitable endomorphism π is done exactly as in Theorem 12. ∎

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