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FITTING CLASSES BASED ON GROUPS OF NILPOTENT LENGTH THREE WITH OPERATOR-ISOMORPHIC MINIMAL NORMAL SUBGROUPS

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Abstract

In this paper a technique for constructing Fitting Classes is applied to certain groups of nilpotent length three which have non-unique minimal normal subgroups. A characterisation of the minimal Fitting Class of some of these groups is also given.

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Some recent work on minimal Fitting classes deals with groups of nilpotent length three or more which are monolithic, that is, which have unique minimal normal subgroups (see Bryce [5], Bryce, Cossey and Ormerod [6] and McCann [2], [3] and [4]). The study of Fitting classes based on monolithic groups goes back to Dark [7], a paper to which the above are indebted.

This paper applies methods of Fitting class construction, which are similar to those in the above papers, to certain non-monolithic groups. The groups in question are of nilpotent length three and have non-unique minimal normal subgroups which are operator-isomorphic with respect to conjugation. They are semi-direct products of elementary abelian r-groups (where r is prime) by certain r'-groups, given in matrix form. Facts about general linear groups, especially Lemma 1.4, will play an important role in this paper. We note here that all groups dealt with will be finite.

In Section 1 we define the groups on which the Fitting classes will be based. Some basic results from the theory of linear groups will also be given.

Section 2 deals with the "basic" Fitting class construction, while in Section 3 we look at the minimal Fitting class of an even more restricted type of (non-monolithic) group. The final section gives some results which indicate why the groups dealt with have been so restrictively chosen.

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1. Notation and preliminary results

The notation and conventions of this section will be used throughout the rest of this paper. We let p, q and r be primes and α, m and t be natural numbers which satisfy:

(i) $p \neq 2, p \neq r$;
(ii) $q \neq r$;
(iii) $p|(q-1)$;
(iv) $p|t$;
(v) t is the *minimal* natural number such that $q^{\alpha}|(r^t - 1)$;
(vi) $1 \leq m < p$.

We let F be the field with r^t elements and denote the additive group of F by U. Thus U can be considered as a vector space of dimension t over Z_r (the field of order r, that is, the prime field of F), or as an elementary abelian r-group of rank t. We let the multiplicative group of F be generated by γ_1, say.

By the theory of finite fields the automorphism group of F is cyclic of order t, so by (iv) there is an element, δ_1, of order p in $\mathrm{Aut}(F)$. Multiplication by γ_1 in F induces an invertible linear transformation on U, as does the "natural" action of δ_1. We identify $\mathrm{Aut}(U)$ with $\mathrm{GL}(t, r)$, the general linear group of invertible $t \times t$ matrices over Z_r. In addition we identify γ_1 and δ_1 with the transformations induced by them on U. So we consider γ_1 and δ_1 as $t \times t$ matrices over Z_r (relative to some fixed basis of U). By (v) there exists an element $\omega_1 \in \langle \gamma_1 \rangle$ such that $o(\omega_1) = q^{\alpha}$, and, also by (v), U is irreducible under $\langle \omega_1 \rangle$. It may be seen that δ_1 normalises $\langle \gamma_1 \rangle$ and hence also $\langle \omega_1 \rangle$, but δ_1 does *not* centralize either of these two groups.

In order to construct groups with operator-isomorphic minimal normal subgroups we take the tensor-product of the above representation of $\langle \gamma_1, \delta_1 \rangle$ with the identity representation of suitable degree. Since we shall be dealing directly with general linear groups, we define the tensor-product in terms of matrices. Namely we define the $mt \times mt$ matrices (when m is as in (vi)) γ, ω and δ by:

$$\gamma = \underbrace{\begin{pmatrix} \gamma_1 & & & 0 \\ & \gamma_1 & & \\ & & \ddots & \\ 0 & & & \gamma_1 \end{pmatrix}}_{m} ; \omega = \underbrace{\begin{pmatrix} \omega_1 & & & 0 \\ & \omega_1 & & \\ & & \ddots & \\ 0 & & & \omega_1 \end{pmatrix}}_{m}$$

and

$$\delta = \underbrace{\begin{pmatrix} \delta_1 & & & 0 \\ & \delta_1 & & \\ & & \ddots & \\ 0 & & & \delta_1 \end{pmatrix}}_{m} .$$

That is γ, ω and δ have $t \times t$ blocks, equal to γ_1, ω_1 and δ_1 respectively, along the main diagonal and zero entries elsewhere.

We define V to be the direct sum $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$, where, for $i = 1, \ldots, m$, $V_i \cong U$. We let $\langle \gamma, \delta \rangle$ operate on V with the "natural" action, that is $V_1, \ldots, V_m$ are operator-isomorphic $\langle \gamma, \delta \rangle$-submodules of V and $\langle \gamma, \delta \rangle$ acts on V_i as $\langle \gamma_1, \delta_1 \rangle$ on U (for $i = 1, \ldots, m$).

For ease of notation we further define $\mathbf{K}$ to be the class of groups which satisfy: $K \cong V \rtimes \langle \omega, \delta \rangle$, where V and $\langle \omega, \delta \rangle$ are defined as above for some suitable p, q, r, α, m and t which satisfy (i), ..., (vi). We will construct Fitting classes to show that if K_1 and K_2 are elements of $\mathbf{K}$, then $K_1 \in \mathbf{Fit}(K_2)$, the minimal Fitting class containing K_2, if and only if $K_1 \cong K_2$. To construct these classes we take some fixed $\langle \omega, \delta \rangle$ and V, identify $\mathrm{Aut}(V)$ with $\mathrm{GL}(mt, r)$ and examine the centralizers of certain subgroups which have ω as an element in $\mathrm{GL}(mt, r)$. The following notation will be used:

$$C = C_{\mathrm{GL}(mt,r)}(\omega);$$
$$N = N_{\mathrm{GL}(mt,r)}(\langle \omega \rangle).$$

Note that since $\langle \omega \rangle$ is a characteristic subgroup of $\langle \omega, \delta \rangle$, we have

$$N_{\mathrm{GL}(mt,r)}(\langle \omega, \delta \rangle) \leq N,$$

and $C_{\mathrm{GL}(mt,r)}(\langle \omega, \delta \rangle) \leq C$. We also note the following elementary result from linear algebra.

LEMMA 1.1. *Let F be the field with r^t elements. Then $C \cong \mathrm{GL}(m, F)$.*

PROOF. Let $g \in \mathrm{GL}(mt, r)$. Then we can write g as

$$g = \begin{pmatrix} g_{11} & \cdots & g_{1m} \\ \vdots & & \vdots \\ g_{m1} & \cdots & g_{mm} \end{pmatrix},$$

where the g_{ij} are $t \times t$ block matrices, $(i, j = 1, \ldots, m)$. We then have $g\omega = \omega g$ if and only if

$$\begin{pmatrix} g_{11}\omega_1 & \cdots & g_{1m}\omega_1 \\ \vdots & & \vdots \\ g_{m1}\omega_1 & \cdots & g_{mm}\omega_1 \end{pmatrix} = \begin{pmatrix} \omega_1 g_{11} & \cdots & \omega_1 g_{1m} \\ \vdots & & \vdots \\ \omega_1 g_{m1} & \cdots & \omega_1 g_{mm} \end{pmatrix}.$$

So we see that ω_1 commutes with all of the g_{ij}. An application of Schur's lemma (Huppert [1, I, 10.5]) shows that $C_{\mathrm{End}(U)}(\omega_1) \cong F$. Thus C is isomorphic to the group of invertible $m \times m$ matrices over F.

Lemma 1.1 is useful in that it allows us to calculate the order of a Sylow p-subgroup of C from that of $\mathrm{GL}(m, F)$. Before we determine a particular Sylow p-subgroup of C we need some more notation. We define s to be the *least* non-negative integer such that $p | (r^{st} - 1)$, and we define β by means of $p^\beta \,\top\, (r^{st} - 1)$. Here the symbol $\top$ stands for "the largest power of p to divide". For our fixed m, as in (vi), we let $m = sk + \varepsilon$, for a suitable k and ε, with $0 \le \varepsilon < s$. We will retain this definition of s, β, k and ε in what follows.

We let

$$W_i = V_{s(i-1)+1} \oplus \cdots \oplus V_{si}, \quad \text{for } i = 1, \ldots, k,$$

and consider restrictions of $\langle \gamma, \delta \rangle$ to W_1, say. We define $F_{r^{st}}$ to be the field of order r^{st} and, as W_1 has order r^{st}, we consider W_1 to be the additive group of $F_{r^{st}}$. Now, $F_{r^{st}}$ has a subfield of order r^t so we embed F (as above) in $F_{r^{st}}$. In particular we have $\langle \gamma \rangle \le F_{r^{st}}^\times$, the multiplicative group of $F_{r^{st}}$. We let $F_{r^{st}}^\times$ operate by multiplication on W_1, and can check that, relative to a suitable basis, this action of γ on W_1 is given in $st \times st$ matrix form as

$$\gamma \sim \underbrace{\begin{pmatrix} \gamma_1 & & & 0 \\ & \gamma_1 & & \\ & & \ddots & \\ 0 & & & \gamma_1 \end{pmatrix}}_{s}$$

(where we identify $\mathrm{Aut}(W_1)$ with $\mathrm{GL}(st, r)$). By the definition of β and s, we have $p^\beta \,\top\, |F_{r^{st}}^\times|$, so, since $F_{r^{st}}^\times$ cyclic, there is an element, h, of order

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p^{β} in $\mathrm{Aut}(W_1)$ which centralises γ and thus also ω. We assume h to be given in $st \times st$ matrix from (relative to the above basis of W_1).

We let $C_1 = C_{\mathrm{Aut}(W_1)}(\omega)$. By Lemma 1.1 we have $C_1 \cong \mathrm{GL}(s, F)$. Now we have $|\mathrm{GL}(s, F)| = (r^{st} - 1)(r^{st} - r^{t}) \cdots (r^{st} - r^{(s-1)t})$, so we see that, by the definition of s and β, $p^{\beta} \top |C_1|$. If we consider δ to be restricted to W_1 (that is, we consider δ to be the $st \times st$ matrix:

$$\delta \sim \underbrace{\begin{pmatrix} \delta_1 & & & 0 \\ & \delta_1 & & \\ & & \ddots & \\ 0 & & & \delta_1 \end{pmatrix}}_{s}),$$

we see by Sylow's theorems that δ normalises some Sylow p-subgroup of C_1 (this because $\delta \in N$). Since, by comparison of orders, $\langle h \rangle$ is a Sylow p-subgroup of C_1, we may assume that, as matrices, δ normalises $\langle h \rangle$.

We define the $mt \times mt$ matrices h_i, $i = 1, \ldots, k$, as follows

$$h_i = \begin{pmatrix} I_{st} & & & & & & & 0 \\ & \ddots & & & & & & \\ & & I_{st} & & & & & \\ & & & h & & & & \\ & & & \uparrow & I_{st} & & & \\ & & & | & & \ddots & & \\ & & & | & & & I_{st} & \\ 0 & & & | & & & & I_{\varepsilon t} \end{pmatrix}$$
$$i\text{th block}$$

(where I_n is an $n \times n$ identity matrix for the natural number n). Then $\langle h_1, \ldots, h_k \rangle = \langle h_1 \rangle \times \cdots \times \langle h_k \rangle \cong C_{p^{\beta}} \times \cdots \times C_{p^{\beta}}$.

In addition, by the above identifications, we have $\langle h_1, \ldots, h_k \rangle \leq C$ and δ (considered once more as an $mt \times mt$ matrix) normalises each one of $\langle h_1 \rangle, \ldots, \langle h_k \rangle$. We let $P = \langle h_1, \ldots, h_k \rangle$. Since $\langle \omega \rangle$ is cyclic of order q^{α} and q is a prime with (by (iii) say) $q \neq 2$, N/C is cyclic. Thus $\langle \delta \rangle C/C$ is the unique subgroup of order p in N/C. The following result tells us something about $\langle \delta \rangle P$.

Lemma 1.2. *$\langle \delta \rangle P$ is a Sylow p-subgroup of $\langle \delta \rangle C$.*

Proof. We need only show that P is a Sylow p-subgroup of C. Since P has order $p^{k\beta}$, we must show, by Lemma 1.1, that $p^{k\beta} \top |\mathrm{GL}(m, F)|$. Note that the trivial case where $s > m$ (whence $k = 0$) is covered by taking

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$P = 1$. Now

$$\begin{aligned} |\mathrm{GL}(m, F)| &= (r^{tm} - 1)(r^{tm} - r^{t}) \cdots (r^{tm} - r^{t(m-1)}) \\ &= (r^{tm} - 1)(r^{t(m-1)} - 1) \cdots (r^{t} - 1) r^{m(m-1)t/2}. \end{aligned}$$

Let n be such that $1 \leq n \leq m$. Then $n = sd + e$, for suitable values d and e with $0 \leq e < s$. We can see that if p divides $r^{nt} - 1$, then p divides $r^{et} - 1$. By the choice of s, this implies that $p \mid (r^{nt} - 1)$ if and only if n is a multiple of s.

Suppose now that $n = fs \leq m$, for some suitable f. We show that $p^{\beta} \Vert (r^{nt} - 1)$. We have

$$r^{nt} - 1 = r^{tsf} - 1 = (r^{ts} - 1)(r^{ts(f-1)} + \cdots + r^{ts} + 1).$$

Since $r^{ts} \equiv 1 \bmod (p^{\beta})$, we have

$$\begin{aligned} r^{ts(f-1)} + \cdots + r^{ts} + 1 &\equiv \underbrace{1 + \cdots + 1}_{f \text{ times}} \bmod (p^{\beta}) \\ &\equiv f \bmod (p^{\beta}). \end{aligned}$$

But $f \leq m < p$. Thus p does *not* divide $r^{ts(f-1)} + \cdots + r^{ts} + 1$, so we conclude $p^{\beta} \Vert (r^{nt} - 1)$.

There are exactly k distinct values of n with $s \mid n$ and $0 < n \leq m$. We thus have $p^{k\beta} \Vert |\mathrm{GL}(m, F)|$.

We note that $W_1, \ldots, W_k, V_{sk+1}, \ldots, V_{sk+\varepsilon}$ are all invariant under $\langle \omega, \delta, P \rangle$. The next lemma shows how W_i may be decomposed under certain subgroups of $\langle \omega, P \rangle$.

LEMMA 1.3. *Let $1 \neq x \in P$. Then either*

(i) *W_i is irreducible under $\langle x, \omega \rangle$*

or

(ii) *x centralises W_i and, in particular, $V_{s(i-1)+1}, \ldots, V_{si}$ are all invariant under $\langle x, \omega \rangle$.*

PROOF. If $s = 1$ then $W_i = V_i$ and we are done. Now assume that $s > 1$ *and* that W_i is not irreducible under $\langle x, \omega \rangle$. In addition we assume, without loss of generality, that $C_{\langle x \rangle}(W_i) = 1$. Thus $\langle x, \omega \rangle$ is an r'-group of automorphisms of W_i. We apply Maschke's Theorem to find a subgroup, T, of W_i such that T is irreducible under $\langle x, \omega \rangle$ and that $C_{\langle x \rangle}(T) = 1$. Since T is invariant under $\langle \omega \rangle$, we see that $|T| = r^{dt}$, for some d with $1 \leq d < s$. Now $\langle x, \omega \rangle$ is abelian so, in particular, $\langle x \rangle$ must act fixpoint freely on T. Thus $o(x) \mid (|T| - 1)$, whence we have $p \mid (r^{dt} - 1)$, which contradicts our choice of s. We conclude that x centralises W_i.

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MAIN LEMMA 1.4. *Let g be a p-element of $\langle\delta\rangle C$ which does not centralize ω. Then $C_{\mathrm{Aut}(V)}(\langle g, \omega\rangle)$ is isomorphic to a subgroup of* $\mathrm{GL}(m, F_{r^n})$, *where F_{r^n} is the field with r^n elements, and n is such that $t = np$.*

PROOF. We demonstrate the result first in the case where there is a decomposition $V = U_1 \oplus \cdots \oplus U_m$, where U_j is a faithful irreducible $\langle\omega\rangle$-submodule, which is invariant under $\langle g, \omega\rangle$, for $j = 1, \ldots, m$.

By the definition of V and $\langle\omega\rangle$ we see that all the U_j are operator-isomorphic to V_1 with respect to $\langle\omega\rangle$. We assume that $U_1, \ldots, U_d$ are all $\langle g, \omega\rangle$-isomorphic but that U_1 is not $\langle g, \omega\rangle$-isomorphic to any of $U_{d+1}, \ldots, U_m$. Let $c \in C_{\mathrm{Aut}(V)}(\langle g, \omega\rangle)$. Then for any j, U_j^c is $\langle g, \omega\rangle$-isomorphic to U_j and we infer that c normalises the groups $S = U_1 \oplus \cdots \oplus U_d$ and $T = U_{d+1} \oplus \cdots \oplus U_m$. We identify $\mathrm{Aut}(S)$ and $\mathrm{Aut}(T)$ with the "natural" subgroups of $\mathrm{Aut}(V)$. Thus

$$C_{\mathrm{Aut}(V)}(\langle g, \omega\rangle) = C_{\mathrm{Aut}(S)}(\langle g, \omega\rangle) \times C_{\mathrm{Aut}(T)}(\langle g, \omega\rangle).$$

By considering the restriction of $\langle g, \omega\rangle$ to U_1 and identifying U_1 with the additive group of F, the field with $r^t (= r^{np})$ elements, we see, using say [3, III.3], that $C_{\mathrm{End}(U_1)}(\langle g, \omega\rangle)$ is isomorphic to F_{r^n}, the subfield of F formed by those elements fixed by the Galois automorphism of order p. As in Lemma 1.1 we have $C_{\mathrm{Aut}(S)}(\langle g, \omega\rangle) \cong \mathrm{GL}(d, F_{r^n})$. We assume inductively that $C_{\mathrm{Aut}(T)}(\langle g, \omega\rangle) \mathrel{\tilde{\leq}} \mathrm{GL}(m-d, F_{r^n})$ and conclude that

$$\begin{aligned} C_{\mathrm{Aut}(V)}(\langle g, \omega\rangle) &= C_{\mathrm{Aut}(S)}(\langle g, \omega\rangle) \times C_{\mathrm{Aut}(T)}(\langle g, \omega\rangle), \\ &\mathrel{\tilde{\leq}} \mathrm{GL}(d, F_{r^n}) \times \mathrm{GL}(m-d, F_{r^n}) \\ &\mathrel{\tilde{\leq}} \mathrm{GL}(m, F_{r^n}). \end{aligned}$$

For the general case we may assume, by Sylow's theorems, that $g \in \langle\delta\rangle P$. Then $W_1, \ldots, W_k, V_{sk+1}, \ldots, V_m$ are all invariant under $\langle g, \omega\rangle$. If none of $W_1, \ldots, W_k$ is irreducible under $\langle g^p, \omega\rangle$, then by Lemma 1.3 g^p centralises $W_1, \ldots, W_k$ (since $g^p \in P$). Since P already centralises $V_{sk+1}, \ldots, V_m$, we see that $g^p = 1$, that is, $o(g) = p$. Now let U be a submodule of W_i which is irreducible under $\langle g, \omega\rangle$. By a result of Zassenhaus (see, for example [2, II.5] for a proof), either U is irreducible under $\langle\omega\rangle$ or U is the direct sum of p $\langle\omega\rangle$-submodules. In the latter case, since all $\langle\omega\rangle$-submodules of V have order greater than or equal to r^t, the contradiction $|U| \geq r^{pt} > r^{mt} = |V|$, would arise. Thus, applying Maschke's Theorem, we find a decomposition $W_i = U_{i_1} \oplus \cdots \oplus U_{i_s}$ for $i = 1, \ldots, k$, where the U_{i_j} are invariant under $\langle g, \omega\rangle$ and are irreducible under $\langle\omega\rangle$. This brings us back to the case already dealt with, so we now assume that W_1, say, is irreducible under $\langle g^p, \omega\rangle$ (and so also under $\langle g, \omega\rangle$).

We assume an enumeration such that $W_1, \dots, W_d$ are operator-isomorphic with regard to $\langle g, \omega\rangle$, for some $d \le k$, but that W_1 is not operator-isomorphic to any of $W_{d+1}, \dots, W_k, V_{sk+1}, \dots, V_m$. As above, we let $S = W_1 \oplus \cdots \oplus W_d$ and $T = W_{d+1} \oplus \cdots \oplus W_k \oplus V_{sk+1} \oplus \cdots \oplus V_m$ and have

$$C_{\mathrm{Aut}(V)}(\langle g, \omega\rangle) = C_{\mathrm{Aut}(S)}(\langle g, \omega\rangle) \times C_{\mathrm{Aut}(T)}(\langle g, \omega\rangle).$$

For notational convenience, we assume $C_{\langle g\rangle}(W_1) = 1$. We let $o(g) = p^{f+1}$ and let $x = g^p$, so xw has order $p^f q^\alpha$. In addition we let $F_{r^{st}}$ be the field with r^{st} elements, identify W_1 with the additive group of $F_{r^{st}}$ and $x\omega$ with an element of order $p^f q^\alpha$ in the multiplicative group of $F_{r^{st}}$. Applying Schur's Lemma as in Lemma 1.1, we see that $C_{\mathrm{End}(W_1)}(x\omega)$ is isomorphic to $F_{r^{st}}$. As above we let $F_{r^{sn}}$ be the subfield of order r^{sn} in $F_{r^{st}}$ which comprises those elements fixed by the Galois automorphism of order p and see that

$$C_{\mathrm{End}(W_1)}(\langle g, \omega\rangle) = C_{\mathrm{End}(W_1)}(\langle x\omega, \delta\rangle) \cong F_{r^{sn}}.$$

Since $W_1, \dots, W_d$ are $\langle g, \omega\rangle$-isomorphic, we see, as in Lemma 1.1, that

$$C_{\mathrm{Aut}(S)}(\langle g, \omega\rangle) \cong \mathrm{GL}(d, F_{r^{sn}}).$$

Since F_{r^n} can be identified with a subfield of $F_{r^{sn}}$, we see that $\mathrm{GL}(d, F_{r^{sn}})$ can be embedded in $\mathrm{GL}(sd, F_{r^n})$. We again inductively assume that the group $C_{\mathrm{Aut}(T)}(\langle g, \omega\rangle)$ can be embedded in $\mathrm{GL}(m - sd, F_{r^n})$, whence

$$\begin{aligned} C_{\mathrm{Aut}(V)}(\langle g, \omega\rangle) &= C_{\mathrm{Aut}(S)}(\langle g, \omega\rangle) \times C_{\mathrm{Aut}(T)}(\langle g, \omega\rangle) \\ &\lesssim \mathrm{GL}(sd, F_{r^n}) \times \mathrm{GL}(m - sd, F_{r^n}) \\ &\lesssim \mathrm{GL}(m, F_{r^n}). \end{aligned}$$

COROLLARY 1.5. *Let g be as in Lemma 1.4. Then there exists no element in $C_{\mathrm{Aut}(V)}(\langle g, \omega\rangle)$ which satisfies $o(y) = q^\alpha$.*

PROOF. By Lemma 1.4 $C_{\mathrm{Aut}(V)}(\langle g, \omega\rangle)$ is isomorphic to a subgroup of $\mathrm{GL}(m, F_{r^n})$. But $\mathrm{GL}(m, F_{r^n})$ can be embedded as a subgroup of $\mathrm{GL}(mn, r)$. If there exists an element y of $\mathrm{GL}(mn, r)$ with $o(y) = q^\alpha$, then we see that $q^\alpha | (r^e - 1)$ for some e with $e \le mn$. But $mn < pn = t$ and we have a contraction to condition (v).

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2. A Fitting class construction

For this section we let $K = V \rtimes \langle \omega, \delta \rangle$ be a *fixed* element of $\mathbf{K}$ relative to the fixed primes and natural numbers p, q, r, α, m and t which satisfy (i), ..., (vi) of Section 1. We recall that a Fitting class, $\mathbf{F}$, is a set of groups with the following properties:

1. If G is an element of $\mathbf{F}$, then so is every isomorphic copy of G;
2. If $G_1 \trianglelefteq G \in \mathbf{F}$, then $G_1 \in \mathbf{F}$;
3. If $G = G_1 G_2$ with $G_j \trianglelefteq G$ and $G_j \in \mathbf{F}$, for $j = 1, 2$, then $G \in \mathbf{F}$.

For the set of primes π we also recall that the π-residual of the group G is

$$0^\pi(G) = \bigcap_{N \trianglelefteq G,\, G/N \text{ a } \pi\text{-group}} N ,$$

and the π-radical of G is

$$0_\pi(G) = \langle N | N \trianglelefteq G,\ N \text{ is a } \pi\text{-group} \rangle .$$

(We use p to denote the set $\{p\}$, where p is a prime $-p'$ denotes the set of all primes except p.) Note that if $G_1 \trianglelefteq G$, then $0^\pi(G_1) \leq 0^\pi(G)$ and $0_\pi(G_1) = G_1 \cap 0_\pi(G)$, while if $G = G_1 G_2$ where G_1 and G_2 are normal in G, then $0^\pi(G) = 0^\pi(G_1)0^\pi(G_2)$.

In order to construct a "non-trivial" Fitting class which contains K we use the following lemma which deals with certain normal products of groups isomorphic to $V \rtimes \langle \omega \rangle$.

Lemma 2.1. *Let H be a group and let $A_1 B_1, \dots, A_f B_f$ be subgroups of H such that for $i = 1, \dots, f$:*

(i) *$A_i B_i \cong V \rtimes \langle \omega \rangle$, with $A_i \cong V$ and $B_i \cong \langle \omega \rangle$;*
(ii) *$A_i B_i \trianglelefteq H$;*
(iii) *let $C_i = C_H(A_i)$ then $B_i C_i / C_i$ is a Sylow q-subgroup of H/C_i.*

Then, for a suitable enumeration, there exists an $e \leq f$ such that

$$\langle A_1 B_1, \dots, A_f B_f \rangle = A_1 B_1 \times \cdots A_e B_e \times Q,$$

where Q is a q-group.

Proof. We use induction on f. The lemma is obviously true for $f = 1$. We now assume

$$\langle A_1 B_1, \dots, A_{f-1} B_{f-1} \rangle = A_1 B_1 \times \cdots \times A_{e_1} B_{e_1} \times Q_1 ,$$

for a suitable q-group Q_1, and $e_1 \le f-1$. If $A_fB_f \cap \langle A_1B_1, \dots, A_{f-1}B_{f-1}\rangle = 1$ then $\langle A_1B_1, \dots, A_fB_f\rangle = A_fB_f \times \langle A_1B_1, \dots, A_{f-1}B_{f-1}\rangle$ and, apart from reordering the indices, we are finished.

Suppose now that $A_fB_f \cap \langle A_1B_1, \dots, A_{f-1}B_{f-1}\rangle \neq 1$. Then, since every minimal normal subgroup of A_fB_f is contained in A_f, we have $A_f \cap (A_1 \times \cdots \times A_{e_1}) \neq 1$. If A_f is not a subgroup of $A_1 \times \cdots \times A_{e_1}$ then, by Maschke's Theorem say, we have $A_f = \widehat{A}_f \times \widetilde{A}_f$, where $\widehat{A}_f = A_f \cap (A_1 \times \cdots \times A_{e_1})$ and $\widetilde{A}_f$ is a complement to $\widehat{A}_f$ which is invariant under $B_1 \times \cdots \times B_{e_1}$. We see that

$$[\widetilde{A}_f, B_1 \times \cdots \times B_{e_1}] \le \widetilde{A}_f \cap \widehat{A}_f = 1.$$

Now, no element of $A_1 \times \cdots \times A_{e_1}$ is centralised by $B_1 \times \cdots \times B_{e_1}$ so there is some $b \in B_1 \times \cdots \times B_{e_1}$ which does not centralize $\widehat{A}_f$ (but does centralize $\widetilde{A}_f$). Since $\langle \omega \rangle$ operates fix-point freely on V we see that $bC_f \notin B_fC_f/C_f$. But clearly B_fC_f/C_f is a *normal* Sylow q-subgroup of H/C_f (by (iii)), so we have a contradiction. Thus $A_f \le A_1 \times \cdots \times A_{e_1}$. Since $A_f = 0_r(A_fB_f)$, we have

$$\begin{aligned} A_f &= [A_f, B_f] \\ &\le [A_1 \times \cdots \times A_{e_1}, B_f] \\ &= [A_1, B_f] \times \cdots \times [A_{e_1}, B_f] \\ &= (A_1 \cap A_f) \times \cdots \times (A_{e_1} \cap A_f). \end{aligned}$$

We may suppose that $A_1 \cap A_f \neq 1$. If in addition, say, $A_2 \cap A_f \neq 1$, then we see that

$$C_{q^\alpha} \times C_{q^\alpha} \cong B_1 \times B_2 \,\tilde{\le}\, H/C_f,$$

which again contradicts (iii). Thus we conclude $A_f = A_1$.

We let Q_2 be a Sylow q-subgroup of $C_{A_1B_1B_f}(A_1)$. By (iii) and comparison of orders we see that $A_1B_1B_f = A_1B_1 \times Q_2$. Note that Q_2 is normal in H, since it is characteristic in $A_1B_1B_f$. We let $Q = Q_1Q_2$ and then have $Q \trianglelefteq H$ and $Q \cap \langle A_1B_1, \dots, A_{e_1}B_{e_1}\rangle = 1$, whence

$$\langle A_1B_1, \dots, A_fB_f\rangle = A_1B_1 \times \cdots \times A_{e_1}B_{e_1} \times Q.$$

The main steps in the proof of the following result are analogous to those of Construction IV.1 of [4]. It is hoped that the more complicated situation dealt with here will justify the somewhat extreme length of the proof.

CONSTRUCTION 2.2. *Let* **F** *be the class of groups which satisfy the following conditions*:

(i) $0^p\{0^{p'}(G)\} = \langle A_1B_1, \dots, A_fB_f, R\rangle$, *for a suitable* f (*possibly* $f = 0$), *where*

(ii) $R = 0_r[0^p\{0^{p'}(G)\}]$ *and, for* $i = 1, \dots, f$,

(iii) $A_iB_i \trianglelefteq 0^{p'}(G)$;

(iv) $A_iB_i \cong V \rtimes \langle\omega\rangle$, *with* $A_i \cong V$ *and* $B_i \cong \langle\omega\rangle$;

(v) *let* $C_i = C_{0^{p'}(G)}(A_i)$, *then* B_iC_i/C_i *is a* (*normal*) *Sylow q-subgroup of* $0^{p'}(G)/C_i$;

(vi) $0_q[0^p\{0^{p'}(G)\}] = 1$.

Then $\mathbf{F}$ *is a Fitting class* (*which clearly contains* $K = V \rtimes \langle\omega, \delta\rangle$).

PROOF. We first show closure with regard to normal subgroups. Let $G_1 \trianglelefteq G \in \mathbf{F}$. By [4, II.6],

$$0^p\{0^{p'}(G_1)\} = \langle 0^p\{0^{p'}(G_1)\} \cap A_1B_1, \dots, 0^p\{0^{p'}(G_1)\} \cap A_fB_f, 0^p\{0^{p'}(G_1)\} \cap R\rangle,$$

where $A_1B_1, \dots, A_fB_f$ and R satisfies (i), ..., (vi). Note that $R_1 = 0^p\{0^{p'}(G_1)\} \cap R = 0_r[0^p\{0^{p'}(G_1)\}]$. By Lemma 2.1 with (vi), we may assume, without loss of generality, that

$$\langle A_1B_1, \dots, A_fB_f\rangle = A_1B_1 \times \cdots \times A_fB_f.$$

Now suppose that, say, $A_1B_1 \cap 0^p\{0^{p'}(G_1)\} \not\leq A_1$. We see then that there is a q-element b, say, with $1 \neq b \in A_1B_1 \cap 0^p\{0^{p'}(G_1)\}$. By considering the action of $\langle\omega\rangle$ on V, we have $A_1 = [A_1, b] \leq 0^p\{0^{p'}(G_1)\}$. Since $0^{p'}(G_1)$ is generated by p-elements, there must be a p-element, x, in $0^{p'}(G_1)$ such that x does not centralize B_1C_1/C_1 for otherwise the factor-group $0^{p'}(G_1)/(0^{p'}(G_1) \cap C_1)$ (which is isomorphic to $0^{p'}(G_1)C_1/C_1$) would have a non-trivial q-factor-group isomorphic to $\langle b\rangle$. Now, since B_1 is cyclic of order q^α (with $q \neq p$), we conclude that $A_1B_1 = A_1[B_1, x] \leq 0^{p'}(G_1)$, whence, in fact, $A_1B_1 \leq 0^p\{0^{p'}(G_1)\}$.

We can now assume the A_iB_i to be ordered such that $\langle A_1B_1, \dots, A_dB_d\rangle \leq 0^p\{0^{p'}(G_1)\}$, for some suitable d, and $A_jB_j \cap 0^p\{0^{p'}(G_1)\} \leq A_j \cap 0^p\{0^{p'}(G_1)\} \leq R_1$, for $j > d$, and since (vi) is trivially satisfied for G_1, we conclude that $G_1 \in \mathbf{F}$.

To show that $\mathbf{F}$ is closed with regard to normal products, we let $G = G_1G_2$ where G_1 and G_2 are normal in G and are elements of $\mathbf{F}$. We let

$$0^p\{0^{p'}(G_j)\} = \langle A_{j1}B_{j1}, \dots, A_{jf_j}B_{jf_j}, R_j\rangle, \text{ for } j = 1, 2,$$

where $R_j = 0_r[0^p\{0^{p'}(G_j)\}] \leq 0_r[0^p\{0^{p'}(G)\}]$ and $A_{j1}B_{j1}, \ldots, A_{jf_j}B_{jf_j}$ satisfies (iii), ..., (vi) in $0^{p'}(G_j)$. Again we assume, by Lemma 2.1 that

$$\langle A_{j1}B_{j1}, \ldots, A_{jf_j}B_{jf_j}\rangle = A_{j1}B_{j1} \times \cdots \times A_{jf_j}B_{jf_j},$$

and note that $A_{j1}B_{j1} \times \cdots \times A_{jf_j}B_{jf_j} = 0^{q'}[0^p\{0^{p'}(G_j)\}]$ is, in particular, a characteristic subgroup of G_j. We show that, say, $A_{11}B_{11}$ is a normal subgroup of $0^{p'}(G)$. Equivalently, we show that $0^{p'}(G_2)$ normalises $A_{11}B_{11}$. Note that, by the theorem of Krull-Remak-Schmidt (Huppert [1, p. 69]), if $x \in G$, then either $(A_{11}B_{11})^x = A_{11}B_{11}$ of $(A_{11}B_{11})^x$ is one of $A_{12}B_{12}, \ldots, A_{1f_1}B_{1f_1}$.

Now $A_{11}B_{11}$ is subnormal in $\langle A_{11}B_{11}, 0_r(G)\rangle$, is generated by q-elements and, by comparison of orders, contains a Sylow q-subgroup of the group $\langle A_{11}B_{11}, 0_r(G)\rangle$. Thus $A_{11}B_{11} = 0^{q'}(\langle A_{11}B_{11}, 0_r(G)\rangle)$, and so, in particular, R_2 (which is a subgroup of $0_r(G)$) normalises $A_{11}B_{11}$.

Suppose, without loss of generality, that $b \in B_{21}$ does *not* normalize $A_{11}B_{11}$. Since $q > p \geq 3$ we can assume that $(A_{11}B_{11})^b = A_{12}B_{12}$ and $(A_{12}B_{12})^b = A_{13}B_{13}$. We let $B_{ji} = \langle b_{ji}\rangle$ for $j = 1, 2$ and $i = 1, \ldots, f_j$. Then we can also assume that $(b_{11})^b = b_{12}$ and $(b_{12})^b = b_{13}$. Since $A_{21}B_{21}$ has defect *at most* two in $0^{p'}(G)$, we have $[b_{11}, b, b] = b_{11}b_{12}^{-2}b_{13} \in A_{21}B_{21}$. From above, $A_{11} \times \cdots \times A_{1f_1}$ normalises $A_{21}B_{21}$, so we have

$$A_{11} \times A_{12} \times A_{13} = [A_{11} \times A_{12} \times A_{13}, b_{11}b_{12}^{-2}b_{13}] \leq A_{21}B_{21},$$

that is, $A_{11} \times A_{12} \times A_{13} \leq A_{21}$, which is a contradiction to the order of A_{21}. We conclude that $0^p\{0^{p'}(G_2)\}$ normalises $A_{11}B_{11}$.

Finally we let x be a p-element of $0^{p'}(G_2)$ which does not normalize $A_{11}B_{11}$. Since $p \neq 2$ we again assume that $(A_{11}B_{11})^x = A_{12}B_{12}$, $(A_{12}B_{12})^x = A_{13}B_{13}$, $(b_{11})^x = b_{12}$ and $(b_{12})^x = b_{13}$ (where the b_{ji} are as above).

Now, $b_{11}^{-1}b_{12} = [b_{11}, x]$ is a q-element of $0^{p'}(G_2)$, and so is contained in $0^p\{0^{p'}(G_2)\}$. By Sylow's theorems we may assume that

$$b_{11}^{-1}b_{12} \in \langle b_{21}\rangle \times \cdots \times \langle b_{2f_2}\rangle,$$

so $b_{11}^{-1}b_{12} = (b_{21})^{e_{21}} \cdots (b_{2f_2})^{e_{2f_2}}$, for suitable powers e_{2i}. By considering the action of $\langle\omega\rangle$ on V, we see that for $e_{2i} \not\equiv 0 \bmod(q^\alpha)$, we have

$$A_{2i} = [A_{2i}, (b_{2i})^{e_{2i}}] = [A_{2i}, b_{11}^{-1}b_{12}] \leq A_{11} \times A_{12}$$

(since the $A_{ji}B_{ji}$ have been shown to be normal in $0^p\{0^{p'}(G)\}$).

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Since we then have $A_{2i} \leq A_{11} \times A_{12}$, we can have $e_{2i} \not\equiv 0 \bmod (q^\alpha)$ for at most two values of i. Thus we may assume that $b_{11}^{-1} b_{12} = (b_{12})^{e_{21}} (b_{22})^{e_{22}}$. But then

$$\begin{aligned} A_{11} \times A_{12} &= [A_{11} \times A_{12}, b_{11}^{-1} b_{12}] \\ &= [A_{11} \times A_{12}, A_{21}B_{21} \times A_{22}B_{22}] \\ &= A_{21} \times A_{22} \quad \text{(by comparison of orders)}. \end{aligned}$$

Now $A_{21} \times A_{22}$ is normal in $0^{p'}(G_2)$ (since the A_{2i} are), so we have that $A_{13} = (A_{12})^x \leq A_{21} \times A_{22}$, whence $A_{11} \times A_{12} \times A_{13} \leq A_{21} \times A_{22}$, which, by comparison of orders, is again a contradiction. We conclude that $0^{p'}(G_2)$ normalises $A_{11}B_{11}$, that is, the $A_{ji}B_{ji}$ are normal in $0^{p'}(G)$.

We let $C_{11} = C_{0^{p'}}(A_{11})$ and show that (v) holds for $B_{11}C_{11}/C_{11}$. To do this we first look at the case where there is an element $x \in 0^{p'}(G_2)$ such that $[A_{11}B_{11}, x] \nleq A_{11}$.

Then, since $A_{11}B_{11}/A_{11} \cong C_{q^\alpha}$, we have, in fact, $A_{11}B_{11} = [A_{11}B_{11}, x] \leq 0^{p'}(G_2)$. Using commutator arguments as above, we may assume $A_{11}B_{11} = A_{21}B_{21}$ and also $B_{11} = B_{21}$. Now a Sylow q-subgroup of $0^{p'}(G)/C_{11}$ is contained in $0^{q'}\{0^{p'}(G)\}C_{11}/C_{11}$ and

$$\begin{aligned} 0^{q'}\{0^{p'}(G)\}C_{11}/C_{11} &= (0^{q'}\{0^{p'}(G_1)\}C_{11}/C_{11})(0^{q'}\{0^{p'}(G_2)\}C_{11}/C_{11}) \\ &= ((B_{11} \times \cdots \times B_{1f_1})C_{11}/C_{11})((B_{21} \times \cdots \times B_{2f_2})C_{11}/C_{11}) \\ &= (B_{11}C_{11}/C_{11})(B_{21}C_{11}/C_{11}) \\ &= B_{11}C_{11}/C_{11}, \end{aligned}$$

so $B_{11}C_{11}/C_{11}$ is a Sylow q-subgroup of $0^{p'}(G)/C_{11}$, as desired.

Now we can assume that $[0^{p'}(G_2), A_{11}B_{11}] \leq A_{11}$. Suppose also that some q-element of $0^{p'}(G_2)$ does not centralize A_{11}. Say, without loss of generality, that B_{21} does not centralize A_{11}. Then we have, by normality, $1 \neq [A_{11}, B_{21}] \leq A_{11} \cap A_{21}B_{21} = A_{11} \cap A_{21}$. Again by considering the action of $\langle \omega \rangle$ on V, we see that $C_{B_{11}}(A_{11} \cap A_{21}) = 1$ and $C_{B_{21}}(A_{11} \cap A_{21}) = 1$. We let $\widetilde{C} = C_{0^{p'}(G)}(A_{11} \cap A_{21})$, and have $A_{11} \leq \widetilde{C}$, so

$$[B_{11}\widetilde{C}/\widetilde{C}, 0^{p'}(G_2)\widetilde{C}/\widetilde{C}] \leq A_{11}\widetilde{C}/\widetilde{C} = 1_{0^{p'}(G)/\widetilde{C}}.$$

Thus $B_{11}\widetilde{C}/\widetilde{C}$ centralises $0^{p'}(G_2)\widetilde{C}/\widetilde{C}$.

Since $\langle \omega \rangle$ acts operator-isomorphically on $V_1, \ldots, V_m$ (as in Section 1) we see that $A_{11} \cap A_{21}$ can be decomposed as

$$A_{11} \cap A_{21} = V_{11} \oplus \cdots \oplus V_{1e}, \quad \text{for some } e \text{ with } e \leq m,$$

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where $V_{1i} \cong U$ (as in Section 1), for $i = 1, \dots, e$ and V_{1i} and V_{1j} are operator-isomorphic with regard to B_{21} for all i and j (this follows from (iv)).

Now, $0^{p'}(G_2)\widetilde{C}/\widetilde{C}$ is generated by p-elements, so there is a p-element in $0^{p'}(G_2)$, g say, such that $g\widetilde{C}$ does not centralize $B_{21}\widetilde{C}/\widetilde{C}$, and we may assume that $g^p\widetilde{C}$ does centralize $B_{21}\widetilde{C}/\widetilde{C}$. We let $\widetilde{V} = A_{11} \cap A_{21}$, $\tilde{g} = gC$, $\langle\tilde{\omega}\rangle = B_{21}\widetilde{C}/\widetilde{C}$ and $\langle\tilde{y}\rangle = B_{11}\widetilde{C}/\widetilde{C}$, and we identify $\tilde{g}$, $\tilde{\omega}$ and $\tilde{y}$ with the elements of $\mathrm{Aut}(\widetilde{V})$ which they induce.

Replacing m with e, as above, we see that $\langle\tilde{g}, \tilde{\omega}\rangle$ and $\widetilde{V}$ satisfy the hypotheses of Lemma 1.4. But then we have a contradiction to Corollary 1.5, since $\tilde{y}$ centralises $\langle\tilde{g}, \tilde{\omega}\rangle$, $o(\tilde{y}) = q^\alpha$ and $C_{\langle\tilde{y}\rangle}(\widetilde{V}) \cong C_{B_{11}}(\widetilde{V}) = 1$. We conclude that all q-elements of $0^{p'}(G_2)$ centralize A_{11}. Thus the Sylow q-subgroup of $0^{p'}(G)/C_{11}$ are contained in $0^p\{0^{p'}(G_1)\}C_{11}/C_{11} = B_{11}C_{11}/C_{11}$ and again we see that $B_{11}C_{11}/C_{11}$ is the (normal) Sylow q-subgroup of $0^{p'}(G)/C_{11}$.

We have thus far demonstrated (iii), (iv) and (v) for the $A_{ji}B_{ji}$. Now $0_q[0^p\{0^{p'}(G)\}] \le 0^{q'}[0^p\{0^{p'}(G)\}] = \langle A_{ji}B_{ji} | j = 1, 2;\ i = 1, \dots, f_j\rangle$ and we apply Lemma 2.1 to see that

$$0^{q'}[0^p\{0^{p'}(G)\}] = A_1B_1 \times \cdots \times A_fB_f \times Q,$$

for a suitable value f, where the A_dB_d are among the $A_{ji}B_{ji}$ and Q is a q-group. Clearly $Q = 0_q[0^p\{0^{p'}(G)\}]$. Since G_1 and G_2 are elements of $\mathbf{F}$, we have

$$Q \cap 0^{p'}(G_j) = 0_q[0^{p'}(G_j)] = 0_q[0^p\{0^{p'}(G_j)\}] = 1,$$

so $[Q, 0^{p'}(G_j)] = 1$ (for $j = 1, 2$), whence $Q \le Z\{0^{p'}(G)\}$. Now we see that, for $R = 0_r[0^p\{0^{p'}(G)\}]$,

$$Q \cap \langle A_1B_1 \times \cdots \times A_fB_f, R\rangle = Q \cap (A_1B_1 \times \cdots \times A_fB_f) = 1,$$

so, if we let $E = \langle A_1B_1 \times \cdots \times A_fB_f, R\rangle$ then we see that QE/E is in the centre of $0^{p'}(G)/E$. In addition $0^{p'}(G)/QE$ is a p-group, since $QE = 0^p\{0^{p'}(G)\}$. But now $0^{p'}(G)/E$ is nilpotent and so has a factor-group isomorphic to Q. Since Q is a p'-group we conclude $Q = 1$. Thus

$$0^p\{0^{p'}(G)\} = \langle A_1B_1, \dots, A_fB_f, R\rangle$$

and properties (ii), ..., (vi) are satisfied.

By applying the above construction in the respective cases we see that if K_1 and K_2 are elements of $\mathbf{K}$, then $K_1 \in \mathbf{Fit}(K_2)$ if and only if $K_1 \cong K_2$. This type of result seems to indicate that the question as to whether two groups generate the same minimal Fitting class may generally be more easily solved in the case where both groups are extensions of elementary abelian r-groups by r'-linear groups, where the linear groups in question are of nilpotent length greater than or equal to two.

3. Some Fitting classes which are minimal

We once more let p, q, r, α, m and t be fixed primes and natural numbers which satisfy (i), ..., (vi) of Section 1. In order to construct some minimal Fitting classes we place two extra conditions on these numbers, namely

(vii) $p^2 \top (q-1)$,

(viii) $p \top |\mathrm{GL}(m, F)|$ (equivalently: $p \top |\mathrm{GL}(m, F_{r^n})|$, where $t = pn$), where F, as in Section 1, is the field with r^t elements. Since $p \neq r$, (viii) requires that p does not divide any of the numbers $r^t - 1, \ldots, r^{mt} - 1$. To reassure ourselves that we are not dealing with the empty set we may check, for example, that the following satisfy conditions (i), ..., (viii): $p = 5$, $q = 11$, $r = 3$, $\alpha = 1$, $t = 5$, $m = 2$ (note: $m = 3$ will also do, but not $m = 4$).

We let the group $K^* = V^* \rtimes \langle \omega^*, \delta^* \rangle$ be constructed in a manner analogous to the group K of Section 1, relative to the above more restricted values of p, q, r, α, m, and t. We note that, by Lemma 1.1 condition (viii) implies that $p \top |C^*|$, where, analogously to Lemma 1.1, $C^* = C_{\mathrm{Aut}(V^*)}(\omega^*)$.

We recall that if G is a (finite) group then $\mathbf{Fit}(G)$, the minimal Fitting class which contains G (or Fitting class generated by G), is

$$\mathbf{Fit}(G) = \cap\{\mathbf{F} : \mathbf{F} \text{ a Fitting class with } G \in \mathbf{F}\}.$$

We note that it is well known that if ε is a *prime* which divides the order of the soluble group G, then $\mathbf{S}_\varepsilon \subseteq \mathbf{Fit}(G)$, where $\mathbf{S}_\varepsilon$ is the (Fitting) class of all finite ε-groups.

CONSTRUCTION 3.1. $\mathbf{Fit}(K^*)$ *is the class of groups which satisfy*

(i) $0^p(G) \in \mathbf{Fit}(V^* \rtimes \langle \omega^* \rangle)$,

(ii) $0^p\{0^{p'}(G)\} = A_1B_1 \times \cdots \times A_fB_f$, *for a suitable* f (*with possibly* $f = 0$), *where, for* $i = 1, \ldots, f$,

(iii) $A_iB_i \trianglelefteq 0^{p'}(G)$

and

(iv) $A_iB_i \cong V^* \rtimes \langle \omega^* \rangle$, *with* $A_i \cong V^*$ *and* $B_i \cong \langle \omega^* \rangle$.

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PROOF. We note first that, for the case $m = 1$, that is, where V^* is irreducible under $\langle\omega^*\rangle$, this result is contained in Bryce [5, Example 5.3]. In addition we easily see that K^* is an element of the above class.

The class of groups which satisfy (i) is easily seen to be a Fitting class, so we show that the groups that satisfy (ii), (iii) and (iv) also form a Fitting class. We deal first with normal products. We let $G = G_1G_2$, where, for $j = 1, 2$, $G_j \trianglelefteq G$ and such that (ii), (iii) and (iv) hold for G_j. We see, in particular, that $0^p\{0^{p'}(G_j)\} = 0^{q'}[0^p\{0^{q'}(G_j)\}]$, whence $0^p\{0^{p'}(G)\} = 0^{q'}[0^p\{0^{p'}(G)\}]$.

We now apply Construction 2.2 (relative to the above p, q, r, α, m and t) and see that, since the groups which satisfy (ii), (iii) and (iv) form a subset of $\mathbf{F}$ (as in Construction 2.2), we have $G \in \mathbf{F}$. Since $0^p\{0^{p'}(G)\}$ is generated by q-elements, Construction 2.2 also shows that $0^p\{0^{p'}(G)\} = \langle A_1B_1, \dots, A_fB_f\rangle$ for a suitable f, where the A_iB_i satisfy (iii), (iv) and (v) of II.2. In addition $0_q[0^p\{0^{p'}(G)\}] = 1$, so by Lemma 2.1, we may assume that $0^p\{0^{p'}(G)\} = A_1B_1 \times \cdots \times A_fB_f$, and this shows that G satisfies (ii), (iii) and (iv) (of Construction 3.1).

This proof is unusual in that closure with regard to normal subgroups is as complicated to demonstrate as closure with regard to normal products. We let $G_1 \trianglelefteq G$, where $0^p\{0^{p'}(G)\} = A_1B_1 \times \cdots \times A_fB_f$, and (ii), (iii) and (iv) are satisfied. As in the proof of Construction 2.2, we have

$$0^p\{0^{p'}(G_1)\} = (0^p\{0^{p'}(G_1)\} \cap A_1B_1) \times \cdots \times (0^p\{0^{p'}(G_1)\} \cap A_fB_f),$$

and either $A_iB_i \leq 0^p\{0^{p'}(G_1)\}$ or $0^p\{0^{p'}(G_1)\} \cap A_iB_i \leq A_i$.

Suppose for some i that $0^p\{0^{p'}(G_1)\} \cap A_iB_i \leq A_i$. We then have that $[0^{p'}(G_1), A_iB_i] \leq A_i$, since otherwise, as in the proof of Construction 2.2, we would have, by commutators, that A_iB_i *is* a subgroup of $0^p\{0^{p'}(G_1)\}$. Thus if x is any p-element of $0^{p'}(G_1)$ and θ_x is the automorphism of A_i induced by conjugation with x, then θ_x commutes with B_i (considered as a subgroup of $\mathrm{Aut}(A_i)$). Since $p \nmid |C^*|$ (for C^* as above), we must have $\theta_x = 1_{\mathrm{Aut}(A_i)}$. Thus A_i is centralised by all p-elements of $0^{p'}(G_1)$ (which is generated by p-elements) and we conclude that $0^p\{0^{p'}(G_1)\} \cap A_iB_i \leq Z\{0^{p'}(G_1)\}$.

We put $Z = 0^p\{0^{p'}(G_1)\} \cap A_iB_i$, and let

$$S = 0^p\{0^{p'}(G_1)\} \cap (A_1B_1 \times \cdots \times A_{i-1}B_{i-1} \times A_{i+1}B_{i+1} \times \cdots \times A_fB_f).$$

Thus $0^p\{0^{p'}(G_1)\} = S \times Z$ and $0^{p'}(G_1)/(S \times Z)$ is a p-group, so $0^{p'}(G_1)/S$

is nilpotent. The Sylow r-subgroup of $0^{p'}(G_1)/S$ is isomorphic to Z, so, by nilpotency, $0^{p'}(G_1)/S$ has a factor-group isomorphic to Z. Since $0^{p'}(G_1)$ has no non-trivial p'-factor-groups, we conclude $Z = 1$. For a suitable enumeration of the A_iB_i we can now assume that

$$0^p\{0^{p'}(G_1)\} = A_1B_1 \times \cdots \times A_eB_e (e \le f),$$

and have verified that G_1 satisfies (ii), (iii) and (iv).

We denote by $\mathbf{X}$ the class of groups which satisfy (i), ..., (iv). Thus $\mathbf{X}$ is a Fitting class which has K^* as an element, so $\mathbf{Fit}(G^*) \subseteq \mathbf{X}$. Now let G be any element of $\mathbf{X}$. We show that G is an element of $\mathbf{Fit}(K^*)$. We have $G = 0^p(G)0^{p'}(G)$, and this is a normal product. By (i), $0^p(G) \in \mathbf{Fit}(V^* \rtimes \langle\omega^*\rangle) \subseteq \mathbf{Fit}(K^*)$, so it suffices to show that $0^{p'}(G) \in \mathbf{Fit}(K^*)$.

Let $D_i = A_1B_1 \times \cdots \times A_{i-1}B_{i-1} \times A_{i+1}B_{i+1} \times \cdots \times A_fB_f$, for $i = 1, \dots, f$ (where we are taking G to be the group of the statement of the construction). We first show that $0^{p'}(G)/D_i \in \mathbf{Fit}(K^*)$. For notational convenience we work "modulo" D_i, that is, we write $D_i = 1$. We let P be a Sylow p-subgroup of $0^{p'}(G)$ (so $0^{p'}(G) = A_iB_iP$), and write $\widehat{P} = C_P(A_i)$. As in the proof of Construction 2.2, we have $[P, A_iB_i] = A_iB_i$. In addition, considering B_i and $P/\widehat{P}$ as the "natural" subgroups of $\mathrm{Aut}(A_i)$, we see, by condition (viii), that no element of $P/\widehat{P}$ centralises B_i. By condition (vii), $p^2 \top (q-1)$, so we conclude that $P/\widetilde{P} \cong C_p$. Since $\widehat{P}$ centralises A_i we see that there is a decomposition $A_iB_i\widehat{P} = A_iB_i \times \widehat{P}$.

We now work modulo $\widehat{P}$. Then A_i is self-centralising in $0^{p'}(G)$ and $0^{p'}(G)$ has a Sylow p-subgroup, $\langle x\rangle$, which is cyclic of order p. We identify A_i with V^*, B_i with $\langle\omega^*\rangle$ and $\langle x\rangle$ with the "natural" subgroup of $\mathrm{Aut}(V^*)$. Conditions (vii) and (viii) now imply that $\langle x\rangle$ is a Sylow p-subgroup of $N^* = N_{\mathrm{Aut}(V^*)}(\langle\omega^*\rangle)$. Thus $\langle x\rangle$ and $\langle\delta^*\rangle$, and hence also $\langle\omega^*, x\rangle$ and $\langle\omega^*, \delta^*\rangle$, are conjugate in N^*. Thus

$$0^{p'}(G)/\widehat{P} \cong V^* \rtimes \langle\omega^*, x\rangle \cong V^* \rtimes \langle\omega^*, \delta^*\rangle = K^*.$$

It follows that

$$\begin{aligned} 0^{p'}(G) \cong \widetilde{G} &= \{(gA_iB_i, g\widehat{P}) | g \in 0^{p'}(G)\} \\ &\le (0^{p'}(G)/A_iB_i) \times (0^{p'}(G)/\widehat{P}) \cong P \times K^*. \end{aligned}$$

Since $\widetilde{G}$ contains all p-elements of $(0^{p'}(G)/A_iB_i) \times (0^{p'}(G)/\widehat{P})$, it is subnormal in the latter group. Thus $0^{p'}(G)$ is isomorphic to a subnormal subgroup

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of $P \times K^*$. Since P is a p-group, the remark preceding this construction shows that $P \in \mathbf{Fit}(K^*)$. We conclude that $0^{p'}(G)/D_i \in \mathbf{Fit}(K^*)$.

Since $\bigcap_{i=1}^{f} D_i = 1$, we see that $0^{p'}(G)$ is isomorphic to

$$\{(gD_1, \ldots, gD_f) | g \in 0^{p'}(G)\}$$

which, as above, is subnormal in $(0^{p'}(G)/D_1) \times \cdots \times (0^{p'}(G)/D_f)$. Thus $0^{p'}(G)$ is isomorphic to a subnormal subgroup of an element of $\mathbf{Fit}(K^*)$, and so $G \in \mathbf{Fit}(K^*)$ and $\mathbf{X} = \mathbf{Fit}(K^*)$.

4. Concluding remarks

One major difference between the Fitting classes $\mathbf{F}_1$, say, of Construction 2.2 and $\mathbf{F}_2$ of Construction 3.1 is that $\mathbf{S}_r\mathbf{S}_p \subseteq \mathbf{F}_1$, while $\mathbf{S}_r\mathbf{S}_p \cap \mathbf{F}_2 = \mathbf{N}_{\{p,r\}}$, where $\mathbf{S}_r\mathbf{S}_p$ is the class of all (finite) extensions of r-groups by p-groups and $\mathbf{N}_{\{p,r\}}$ is the class of all nilpotent $\{p, r\}$-groups, for the given values of p and r. Thus the extra conditions (vii) and (viii) of Section 3 yield more precise information about the $\{p, r\}$-groups in $\mathbf{Fit}(K^*)$.

To show that the latter is not always the case, we let p, q, r, α, m and t satisfy (i), ..., (vi) of Section1, and add in the new condition

(vii)$'$ $p \,|\, |\mathrm{GL}(m, F)|$

(in contrast to (viii) of Section 3). We can take $p = 5$, $q = 11$, $r = 3$, $\alpha = 1$, $t = 5$ and $m = 4$, to see that such numbers *do* exist. We let $\widehat{K} = \widehat{V} \rtimes \langle\hat{\omega}, \hat{\delta}\rangle$ be constructed as in Section 1, relative to the above numbers.

PROPOSITION 4.1. $\mathbf{Fit}(\widehat{K}) \cap \mathbf{S}_r\mathbf{S}_p \not\subseteq \mathbf{N}_{\{p,r\}}$ *(that is, there are non-nilpotent $\{p, r\}$-groups in* $\mathbf{Fit}(\widehat{K})$*)*.

PROOF. We let s and β be defined as in Section 1. Thus, by (vii)$'$, $s \neq 0$ and $\beta \geq 1$. We let $\hat{h}_1$ be defined analogously to h_1 of Section 1. Thus $\hat{h}_1$ centralises $\hat{\omega}$ in $\mathrm{Aut}(\widehat{V})$ and $\hat{\delta}$ normalises $\langle\hat{h}_1\rangle$. We form the semidirect product $\widehat{G} = \widehat{V} \rtimes \langle\hat{\omega}, \hat{\delta}, \hat{h}_1\rangle$. Since $0^{p'}(\widehat{G}) = \widehat{V} \rtimes \langle\hat{\omega}\rangle \leq \widehat{K}$, we see that $\widehat{K}$ is subnormal in $\widehat{G}$. Since Fitting classes are known to be closed with regard to subnormal products, we see that $(\widehat{K})^{\widehat{G}} \in \mathbf{Fit}(\widehat{K})$ (where $(\widehat{K})^{\widehat{G}}$ is the normal closure of $\widehat{K}$ in $\widehat{G}$).

We define $\widehat{W}_1$ analogously to W_1 in Section 1. Considering the restrictions to $\widehat{W}_1$, we see by Lemma 1.3, that $\widehat{W}_1$ is irreducible under $\langle\hat{\omega}\hat{h}_1\rangle$. As in [3, III.3], say, we may identify $\widehat{W}_1$ with the additive group of $F_{r^{st}}$, the

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field with r^{st} elements. We also identify $\hat{\omega}\hat{h}_1$ with an element of order $p^\beta q^\alpha$ in the multiplicative group of $F_{r^{st}}$ and $\hat{\delta}$ with the Galois automorphism of order p. Letting n be such that $t = np$, and $F_{r^{sn}}$ be the subfield of order r^{sn} which comprises those elements fixed by the action of $\hat{\delta}$, we see by, say [3, III.4], that $\hat{h}_1 \notin F_{r^{sn}}$. Thus $\hat{\delta}$ does *not* centralize $\hat{h}_1$.

We let $\hat{c} = [\hat{\delta}, \hat{h}_1]$. Then $1 \neq \hat{c} \in (\widehat{K})^{\widehat{G}} \cap \langle \hat{h}_1 \rangle$. Since $\langle \hat{c} \rangle$ is characteristic in $\langle \hat{h}_1 \rangle$, which is normal in $\langle \hat{\omega}, \hat{\delta}, \hat{h}_1 \rangle$, we have that $\widehat{V} \rtimes \langle \hat{c} \rangle \trianglelefteq (\widehat{K})^{\widehat{G}} \in$ $\mathbf{Fit}(\widehat{K})$. Thus $\widehat{V} \rtimes \langle \hat{c} \rangle$ is a non-nilpotent group which is an element of $\mathbf{S}_r\mathbf{S}_p \cap$ $\mathbf{Fit}(\widehat{K})$.

We finally indicate why condition (vi) (that is $1 \leq m < p$) of Section 1 is necessary in order to construct Fitting classes like those in Construction 2.2. We let G be a subgroup of $\mathrm{GL}(e, K)$, where K is any field, and define the tensor-products

$$I_e \otimes G = \left\{ \underbrace{\begin{pmatrix} g & & 0 \\ & \ddots & \\ 0 & & g \end{pmatrix}}_{e} \;\middle|\; g \in G \right\},$$

and

$$G \otimes I_e = \left\{ \begin{pmatrix} g_{11}I_e & \cdots & g_{1e}I_e \\ \vdots & & \vdots \\ g_{e1}I_e & \cdots & g_{ee}I_e \end{pmatrix} \;\middle|\; \begin{pmatrix} g_{11} & \cdots & g_{1e} \\ \vdots & & \vdots \\ g_{e1} & \cdots & g_{ee} \end{pmatrix} \in G \right\},$$

(where I_e is the $e \times e$ identity matrix). Thus both tensor-products are groups of $e^2 \times e^2$ matrices which are isomorphic to G.

Proposition 4.2. (i) *$I_e \otimes G$ and $G \otimes I_e$ commute elementwise in* $\mathrm{GL}(e^2, K)$, (ii) *$I_e \otimes G$ and $G \otimes I_e$ are conjugate in* $\mathrm{GL}(e^2, K)$.

Proof. (i) is easy to verify directly. For (ii), we write $\widetilde{G} = I_e \otimes G$ and let $\widetilde{G}$ act operator-isomorphically, with its "natural" action, on the vector-spaces $U_1, \dots, U_e$, all of dimension e over K, where $u_{i1}, \dots, u_{ie}$ is a basis of U_i $(i = 1, \dots, e)$ which induces the given representation of G. We let $U = U_1 \oplus \cdots \oplus U_e$, that is, U is the direct sum of e operator-isomorphic $\widetilde{G}$-submodules. We choose a new basis of U by letting $w_{ij} = u_{ji}$ $i, j = 1, \dots, e$.

We can check that the action of $\widetilde{G}$ with respect to the w_{ij} is represented by $G \otimes I_e$. Thus $\widetilde{G}$ and $G \otimes I_e$ are equivalent representations and hence conjugate in $\mathrm{GL}(e^2, K)$.

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We let $\langle \omega_1, \delta_1 \rangle$ be as in Section 1 and, letting F_{r^n} be the subfield of order r^n in F (as in Section 1, where $t = np$), we see that F_{r^n} is centralised by the action of δ_1. We can thus consider $\langle \omega_1, \delta_1 \rangle$ to be represented over F_{r^n}, that is, $\langle \omega_1, \delta_1 \rangle \leq \mathrm{GL}(p, F_{r^n})$.

We let
$$\widetilde{G} = I_p \otimes \langle \omega_1, \delta_1 \rangle \leq \mathrm{GL}(p^2, F_{r^n}),$$
$$\widehat{G} = \langle \omega_1, \delta_1 \rangle \otimes I_p \leq \mathrm{GL}(p^2, F_{r^n})$$
and we let $\langle \tilde{\omega} \rangle = I_p \otimes \langle \omega_1 \rangle$. We identify $\mathrm{GL}(p^2, F_{r^n})$ with the "natural" subgroup of $\mathrm{GL}(p^2 n, r) = \mathrm{GL}(pt, r)$. By Proposition 4.2 $\widetilde{G}$ and $\widehat{G}$ are conjugate and centralize each other in $\mathrm{GL}(pt, r)$. Since $Z(\langle \omega_1, \delta_1 \rangle) = 1$, we have that $\langle \widetilde{G}, \widehat{G} \rangle = \widetilde{G} \times \widehat{G}$. Now let $\widetilde{V}$ be the direct sum $\widetilde{V} = U_1 \oplus \cdots \oplus U_p$, where $U_i \cong U$ (as in Section 1), for $i = 1, \ldots, p$, and let $\widetilde{K} = \widetilde{V} \rtimes \widetilde{G}$. Thus $\widetilde{K}$ is constructed analogously to K of Section 1, only with $m = p$.

From the above considerations $\widetilde{K}$ and $\widetilde{V} \rtimes \widehat{G}$ are isomorphic and also $\widetilde{V} \rtimes (\widetilde{G} \times \widehat{G})$ is the normal product of $\widetilde{K}$ and $\widetilde{V} \rtimes \widehat{G}$. So $\widetilde{V} \rtimes (\widetilde{G} \times \widehat{G}) \in \mathbf{Fit}(\widetilde{K})$. By considering condition (v) of Construction 2.2, we see that a Fitting class construction directly analogous to that of Construction 2.2, with $\widetilde{V} \rtimes \langle \tilde{\omega} \rangle$ in place of $V \rtimes \langle \omega \rangle$, is not possible.

References

[1] B. Huppert, *Endliche Gruppen I*, (Berlin-Heidelberg-New York 1967).
[2] B. McCann, 'A Fitting class construction' (to appear).
[3] B. McCann, 'Examples of normal products of finite groups and an application to Fitting classes' (submitted).
[4] B. McCann, 'Examples of minimal Fitting classes of finite groups', *Arch. Math.* **49** (1987), 179–186.
[5] R. A. Bryce, *The Fitting Class Generated by a Finite Soluble Group*, (Research Report No. 34-1988, A.N.U. Mathematics Research Report Series).
[6] R. A. Bryce, J. Cossey and E. Ormerod, 'Fitting classes after dark', *Proceedings of the Singapore Group Theory Conference 1987*, pp. 293–321 (de Gruyter, Berlin-New York, 1989).
[7] R. S. Dark, 'Some examples in the theory of injectors of finite soluble groups', *Math. Z.* **27** (1972), 145–156.

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