ADDITIVE FUNCTIONALS ON L_p SPACES

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1. Introduction. In **(1)** a representation theorem was proved for a class of additive functionals defined on the continuous real-valued functions with domain $S = [0, 1]$. The theorem was extended to the case where S is an arbitrary compact metric space in **(3)**. Our present purpose is to consider the corresponding class of additive functionals defined on L_p spaces, $p > 0$. In **(4)** Martin and Mizel have considered functionals defined on the class of bounded measurable functions which, however, satisfy a certain "stochastic" condition which we do not require.

In general, the class of linear functionals appears as a subclass of the class of additive functionals. However it has been shown by M. M. Day **(2)** that if the underlying measure space is non-atomic, then the class of non-trivial linear functionals defined on L_p is empty for $1 > p > 0$. It follows that an additive functional defined on L_p, $1 > p > 0$, is not linear.

In §2 we state our preliminary definitions. In §3 we obtain a general representation for an additive functional defined on L_p, $p > 0$, which reduces to the standard representation theorem for linear functionals when $p \geqslant 1$. The representation utilizes the concept of an additive transformation, which appears as a natural generalization of a linear transformation. In §4 we consider the adjoint of an additive transformation mapping L_p into L_p, $p \geqslant 1$. We recall that the adjoint of a linear transformation mapping L_p into L_p, $p \geqslant 1$, can be interpreted as a linear transformation mapping L_q into L_q, $q = p/(p - 1)$. In §4 we show that the adjoint of an additive transformation mapping L_p into L_p may be interpreted as a class of linear transformations mapping L_q into L_1.

Our proofs utilize methods in **(1)** and in the standard proof for the representation of linear functionals on L_p spaces, $p \geqslant 1$.

2. Preliminaries. In general, we may consider a linear space N whose elements are real-valued functions defined on an underlying space S. For each $f \in N$ there is defined a number $||f|| \geqslant 0$ which may be regarded as a generalized norm. We consider a corresponding space N' and say a mapping T of N into N' is an *additive transformation* if T satisfies the following three requirements:

(1) *Continuity.* For each $\epsilon > 0$ and $b > 0$, there exists $\delta = \delta(b, \epsilon)$ such that $||f|| \leqslant b$, $||g|| \leqslant b$, and $||f - g|| \leqslant \delta$ imply $||T(f) - T(g)|| \leqslant \epsilon$.

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(2) *Boundedness*. For each $b > 0$, there exists $B = B(b)$ such that $||f|| \leqslant b$ implies $||T(f)|| \leqslant B$.

(3) *Additivity*. If f and g satisfy $f(s)g(s) = 0$, $s \in S$, then

$$T(f + g) = T(f) + T(g).$$

Briefly, (1) implies uniform continuity on bounded sets, (2) implies that bounded sets are mapped into bounded sets, and (3) implies that T is additive on functions with disjoint support. When N' is the set of real numbers (with $||T(f)|| = |T(f)|$) we refer to T as an additive functional, which we denote by ϕ.

In particular, we shall be concerned with the case when $(S, \mathfrak{B}, \mu)$ is a finite measure space and $N = L_p = L_p(S, \mathfrak{B}, \mu)$, $p > 0$, with $||f||_p = \{\int_S |f|^p d\mu\}^{1/p}$. If $1 > p > 0$, then $||f||_p$ does not satisfy the triangle inequality and consequently it is not a norm. However, it does satisfy the inequality

$$||f + g||_p \leqslant 2^q[||f||_p + ||g||_p],$$

where $q = (1 - p)/p$; hence L_p is a linear space, $p > 0$.

3. Representation of additive functionals. In this section, we consider $p > 0$ and $L_p = L_p(S, \mathfrak{B}, \mu)$, where $\mu(S) < \infty$. Our representation theorem may be stated as follows.

THEOREM 1. *ϕ is an additive functional in L_p if and only if*

$$\phi(f) = \int_S K(f(s), s)\alpha(s)d\mu, \qquad f \in L_p,$$

where (i) $K(0, s) \equiv 0$, (ii) *$K(x, s)$ is a measurable function of s for each x,* (iii) *$K(x, s)$ is a continuous function of x for $\alpha d\mu$ —* a.a. s, (iv) *for each $b > 0$, there exists $H = H(b)$ such that $|x| \leqslant b$ implies $|K(x, s)| \leqslant H$ for $\alpha d\mu$ —* a.a. s, (v) *if $Tf(s) = K(f(s), s)\alpha(s)$, then T is an additive transformation from L_p into L_1.*

Condition (v) is essentially a compatibility relation between K and α. In general, there will be a class of α's that will satisfy (v) for a given kernel K satisfying (i)–(iv). For example if $K(x, s) = \sin sx$, then we may choose any $\alpha \in L_1$ to satisfy (v).

LEMMA 1. *For each h, $-\infty < h < \infty$, there exists a function $K_h(s)$ which is a measurable function of s and is uniquely defined up to a μ-null set such that*

(1.1) $$K_0(s) = 0, \qquad s \in S,$$

(1.2) $$\phi(h\psi_B) = \int_B K_h(s)d\mu, \qquad B \in \mathfrak{B}.$$

Proof. Let $\mu_h(B) = \phi(h\psi_B)$, $B \in \mathfrak{B}$, where ψ_B denotes the characteristic function of the set B. Conditions (1)–(3) imply that μ_h is a signed measure of finite variation on $\mathfrak{B}$ and μ_h is absolutely continuous with respect to μ. Therefore the Radon–Nikodym theorem implies that there exists a function K_h as above satisfying (1.1) and (1.2).

We note that if ϕ is linear, then $\mu_h(B) = h\mu_1(B)$, $B \in \mathfrak{B}$; hence $K_h(s) = hK_1(s)$, $s \in S$.

LEMMA 2. *There exists a kernel $K(x, s)$ and α satisfying* (i)–(iv) *of Theorem* 1 *such that for each h, $-\infty < h < \infty$, we have*

$$\phi(h\psi_B) = \int_S K(h\psi_B(s), s)\alpha(s)d\mu, \qquad B \in \mathfrak{B}. \tag{2.1}$$

Proof. We utilize the method of proof of **(1**, Lemma 11**)** to first show that $K_h(s)$ is continuous in h for μ – a.a. s. Fix an integer n and for notational convenience let

$$K_l(s) = K_{n+l/2^j}(s), \qquad 1 \leqslant l \leqslant 2^j.$$

Let $\delta > 0$ and set

$$A_0 = \emptyset, \qquad A_l = \{K_l - K_{l-1} \geqslant \delta\} - \bigcup_{i=0}^{l-1} A_i, \quad 1 \leqslant l \leqslant 2^j,$$

$$\text{and } A^j = \bigcup_{l=1}^{2^j} A_l.$$

We shall show that $\lim_{j\to\infty} \mu(A^j) = 0$.

Let

$$y_{j,1} = \sum_{l=1}^{2^j} (n + (l-1)/2^j)\psi_{A_l} \quad \text{and} \quad y_{j,2} = \sum_{l=1}^{2^j} (n + l/2^j)\psi_{A_l}.$$

It follows by our preceding notation and by (1.2) that

$$\phi(y_{j,1}) = \sum_{l=1}^{2^j} \int_{A_l} K_{l-1}(s)d\mu \quad \text{and} \quad \phi(y_{j,2}) = \sum_{l=1}^{2^j} \int_{A_l} K_l(s)d\mu.$$

Therefore by the definition of A_l it follows that $\phi(y_{j,2}) - \phi(y_{j,1}) \geqslant \delta\mu(A^j)$. Since $y_{j,2}(s) - y_{j,1}(s) \leqslant 2^{-j}$, $s \in S$, and $||y_{j,i}|| \leqslant ||(n+1)\psi_s||$, $i = 1, 2$, it follows by Condition (1) that $\lim_j |\phi(y_{j,2}) - \phi(y_{j,1})| = 0$ and hence $\lim_{j\to\infty} \mu(A^j) = 0$. Since $\delta > 0$ was arbitrary, we have

$$\limsup\, [K_l(s) - K_{l-1}(s)] = 0 \qquad \text{for } \mu - \text{a.a. } s.$$

Similarly we show that

$$\liminf\, [K_l(s) - K_{l-1}(s)] = 0 \qquad \text{for } \mu - \text{a.a. } s.$$

It follows that there exists a sequence $\{h_i\}$ dense in $[n, n+1]$ such that

$$\lim_{hi\to hi_0} K_{hi}(s) = K_{hi_0}(s), \qquad \mu - \text{a.a. } s. \tag{2.2}$$

Since

$$(-\infty, \infty) = \bigcup_{-\infty}^{\infty} [n, n+1],$$

it follows that there exists a sequence $\{h_i\}$ dense in $(-\infty, \infty)$ such that (2.2) holds.

If $h = h_i$, we set $K_1(h, s) = K_h(s)$. Otherwise we select $h_i \to h$ and set $K_1(h, s) = \lim_{h_i \to h} K_{h_i}(s)$. Clearly $K_1(h, s)$ is continuous in h for μ − a.a. s. Furthermore an argument similar to the above shows that for each h we have $K_1(h, s) = K_h(s)$ for μ − a.a. s.

Utilizing the method of proof of (**1**, Lemma 12), we can now obtain $K_2(h, s)$ and $\mu_* \sim \mu$ such that

$$\phi(h\psi_B) = \int_B K_2(h, s) d\mu_* \tag{2.3}$$

where $K_2(h, s)$ satisfies conditions (i), (ii), and (iv) of Theorem 1. Moreover utilizing the previous argument we can show that $K_2(h, s)$ can be defined so that for each h, $K_2(h, s)$ is continuous for μ_* − a.a. s. We let α denote the Radon–Nikodym derivative $d\mu_*/d\mu$. Letting $K(h, s) = K_2(h, s)$, we see that $K(h, s)$ satisfies (i)–(iv) of Theorem 1 and (2.1).

Note that if ϕ is linear, then $K(x, s) = x$ and $\alpha(s) = K_1(s)$.

For each $f \in L_p$ we now define $\phi_1(f)$ as

$$\phi_1(f) = \int_S K(f(s), s)\alpha(s) d\mu. \tag{2.4}$$

LEMMA 3. $\phi_1(f) = \phi(f), f \in L_p$.

Proof. Condition (3) and (2.1) imply that (2.4) holds if f is a simple function. Next assume that f is bounded, say $|f(s)| \leqslant b$. We can obtain a sequence of simple functions f_n such that $|f_n| \leqslant b$, $\lim_n f_n(s) = f(s)$, and $\lim_n ||f_n - f||_p = 0$. Condition (1) implies that $\lim_n \phi(f_n) = \phi(f)$ and (iii) implies that

$$\lim_n K(f_n(s), s)\alpha(s) = K(f(s), s)\alpha(s) \qquad \text{for } \mu - \text{a.a. } s.$$

Therefore (iv) and the Lebesgue Bounded Convergence Theorem imply that

$$\lim_n \phi_1(f_n) = \lim_n \int_S K(f_n(s), s)\alpha(s) d\mu = \int K(f(s), s)\alpha(s) d\mu.$$

Since $\phi_1(f_n) = \phi(f_n)$, it follows that $\phi_1(f) = \phi(f)$ for bounded f. Finally consider $f \in L_p$ and let

$$E = \{s: K(f(s), s)\alpha(s) > 0\} \quad \text{and} \quad F = \{s: K(f(s), s)\alpha(s) < 0\}.$$

Let $f_n(s) = f(s)$ if $|f(s)| \leqslant n$ and $f_n(0) = 0$ if $|f(s)| > n$. It follows that $\lim_n ||f_n - f||_p = 0$; hence Condition (1) implies that $\lim_n \phi(f_n) = \phi(f)$. Since f_n is bounded, $\phi_1(f_n) = \phi(f_n)$. Now let

$$A_n = \{s: |f(s)| \leqslant n\}, \qquad E_n = E \cap A_n, \qquad F_n = F \cap A_n,$$
$$f_{n,1} = \psi_{E_n} f_n, \qquad \text{and} \qquad f_{n,2} = \psi_{F_n} f_n.$$

We have $||f_n||_p \leqslant ||f||_p$; hence $||f_{n,i}||_p \leqslant ||f||_p$, $i = 1, 2$. Therefore Condition (2) implies that $|\phi(f_{n,i})| \leqslant B(||f||_p)$, $i = 1, 2$. Hence the following integrals are uniformly bounded in n:

$$\phi(f_{n,i}) = \int_S K(f_{n,i}(s), s)\alpha(s) d\mu, \qquad i = 1, 2.$$

Now we can write

$$\phi(f_{n,1}) = \int_S K(f_{n,1}(s), s)\alpha(s)d\mu = \int_{E_n} K(f(s), s)\alpha(s)d\mu$$

and therefore by the Lebesgue Monotone Convergence Theorem we have

$$\lim_n \phi(f_{n,1}) = \int_E K(f(s), s)\alpha(s)d\mu.$$

Similarly

$$\lim_n \phi(f_{n,2}) = \int_F K(f(s), s)\alpha(s)d\mu.$$

Therefore

$$\phi(f) = \lim_n \phi(f_n) = \lim_n\{\phi(f_{n,1}) + \phi(f_{n,2})\} = \phi_1(f).$$

Proof of Theorem 1. Lemma 3 yields the desired representation for $\phi(f)$, $f \in L_p$. Utilizing Conditions (1) and (2) for ϕ, the validity of (v) follows in a straightforward manner. The converse follows immediately.

4. Adjoint transformations. In this section we define the adjoint transformation T^* of an additive transformation T. We shall then consider a suitable interpretation of T^* when T acts in an L_p space, $p \geqslant 1$. We now assume that N and N' are Banach spaces whose elements are real-valued functions defined on underlying spaces S and S' respectively.

Definition 1. Let T be an additive transformation from N into N' and let λ be a norm-bounded linear functional on N'. We define $T^*\lambda(x) = \lambda(T(x))$, $x \in N$.

LEMMA 4. *Let T and λ be as in Definition* 1. *Then $T^*\lambda$ is an additive functional on N.*

Proof. Immediate.

Lemma 4 implies that in general the adjoint of an additive transformation maps linear functionals into additive functionals. Definition 1 reduces to the usual definition when T is a linear transformation. We shall now restrict our attention to the case $p \geqslant 1$ and $N = N' = L_p$. We consider $q = p/(p-1)$ if $p > 1$ and $q = \infty$ if $p = 1$.

We recall that when T is a linear transformation in L_p, then T^* can be interpreted as a linear transformation in L_q such that

$$\int_S Tf(s)g(s)d\mu = \int_S f(s)T^*g(s)d\mu, \qquad f \in L_p,\ g \in L_q. \tag{4.1}$$

If we write $T_f^*g(s) = f(s)T^*g(s)$ and let $S(f)$ denote the support of f, then we have

$$\int_{S(g)} Tf(s)g(s)d\mu = \int_{S(f)} T_f^*g(s)d\mu. \tag{4.2}$$

We wish to extend (4.2) to additive transformations and we proceed by a series of lemmas.

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LEMMA 5. *Let T be an additive transformation of L_p into L_p and let $g \in L_q$. Then for each h, $-\infty < h < \infty$, there exists a linear transformation T_h^* from L_q into L_1 such that*

$$\int_S T(h\psi_B(s))g(s)d\mu = \int_B T_h^*g(s)d\mu, \qquad B \in \mathfrak{B}. \tag{5.1}$$

Remark. If T is a linear transformation, then $T_h^* = hT_1^*$. However, in general $T_h^* \neq hT_1^*$ when T is an additive transformation.

Proof. If we set $\mu_h(B)$ equal to the left side of (5.1), then μ_h is easily verified to be a signed measure of finite variation on $\mathfrak{B}$ which is absolutely continuous with respect to μ. Therefore by the Radon–Nikodym theorem there exists a measurable function which we denote by $T_h^*\, g$ satisfying (5.1). Given u, $v \in L_q$, we then have

$$\int_B T_h^*(\alpha u + \beta v)d\mu = \int_B (\alpha T_h^*u + \beta T_h^*v)d\mu, \qquad B \in \mathfrak{B}. \tag{5.2}$$

Since B is arbitrary in (5.2), it follows that $T_h^*(\alpha u + \beta v) = \alpha T_h^*u + \beta T_h^*v$. We next show that T_h^* is bounded. Let $g \in L_q$, $E = \{T_h^*g > 0\}$, and $F = \{T_h^*g < 0\}$. By Hölder's inequality and Condition (2) on T we have

$$\left|\int_S T(h\psi_E)(s)g(s)d\mu\right| \leqslant ||T(h\psi_E)||_p||g||_q \leqslant B(|b|)||g||_q, \tag{5.3}$$

$$\left|\int_S T(h\psi_F)(s)g(s)d\mu\right| \leqslant ||T(h\psi_F)||_p||g||_q \leqslant B(|b|)||g||_q, \tag{5.4}$$

where $b = ||h\psi_E||_p$.

It now follows from (5.1), (5.3), and (5.4) that $||T_h^*g||_1 \leqslant 2B(|b|)||g||_q$; hence $||T_h^*|| \leqslant 2B(|b|)$.

Definition 2. Let

$$f = \sum_{i=1}^{n} h_i\, \psi_{B_i}$$

where $h_1, \ldots, h_n$ are the distinct values of f which are taken on the measurable sets $B_1, \ldots, B_n$ respectively, and let $g \in L_q$. We define T_f^*g as

$$T_f^*\, g(s) = \sum_{i=1}^{n} \psi_{B_i}(s)T_{h_i}^*\, g(s), \qquad s \in S.$$

LEMMA 6. *Let f and g be as in Definition 2. Then T_f^* is a linear transformation from L_q into L_1 such that*

$$\int_{S(g)} Tf(s)g(s)d\mu = \int_{S(f)} T_f^*g(s)d\mu.$$

Proof. The linearity follows by Lemma 5. Utilizing Condition (3) on T and a similar decomposition as in the proof of Lemma 5, we obtain $||T_f^*|| \leqslant 2\mathrm{B}(||f||_p)$.

LEMMA 7. *Let $\epsilon > 0$, $b > 0$, and $g \in L_q$. Then there exists $\delta > 0$ such that if u and v are simple functions for which $||u||_p \leqslant b$, $||v||_p \leqslant b$, and $||u - v||_p \leqslant \delta$, then $||T_u^*g - T_v^*g||_1 \leqslant \epsilon$.*

Proof. By Condition (1) on T, there exists $\delta > 0$ such that $||u - v||_p \leqslant \delta$ implies $||Tu - Tv||_p \leqslant \epsilon/2||g||_q$. Let $E = \{T_u{}^*g - T_v{}^*g > 0\}$ and

$$F = \{T_u{}^*g - T_v{}^*g < 0\}.$$

If $u_E = \psi_E\, u$ and $v_E = \psi_E\, v$, then $||u_E||_p \leqslant b$, $||v_E||_p \leqslant b$, $||u_E - v_E||_p \leqslant \delta$. We then have

$$\int_E [T_u{}^*g - T_v{}^*g]d\mu = \int_E [T_{u_E}{}^*g - T_{v_E}{}^*g]d\mu = \int_{S(g)} [Tu_E - Tv_E]g d\mu;$$

hence by Hölder's inequality and the preceding estimate we have

$$\int_E [T_u{}^*g - T_v{}^*g]d\mu \leqslant ||Tu_E - Tv_E||_p ||g||_q \leqslant \epsilon/2.$$

An identical consideration of the integral over F yields the desired result.

Lemma 8. *If f_n is a Cauchy sequence of simple functions in L_p, then $T_{f_n}{}^*\, g$ is a Cauchy sequence in L_1, $g \in L_q$.*

Proof. By Lemma 7.

Definition 3. Let $f \in L_p$ and let f_n be a sequence of simple functions in L_p such that $||f_n||_p \leqslant ||f||_p$ and $\lim_n ||f_n - f||_p = 0$. We define $T_f{}^*g$ for $g \in L_q$ as follows:

$$T_f{}^*g(s) = L_1 \lim_n T_{f_n}{}^*g(s).$$

Theorem 2. *Let $f \in L_p$ and $g \in L_q$. Then $T_f{}^*$ in Definition 3 is a linear operator from L_q into L_1 such that*

$$\int_{S(g)} Tf(s)g(s)d\mu = \int_{S(f)} T_f{}^*g(s)d\mu.$$

Proof. Definition 3 implies that $T_f{}^*$ is linear, and

$$||T_{f_n}{}^*|| \leqslant 2B(||f_n||_p) \leqslant 2B(||f||_p)$$

implies that $||T_f{}^*|| \leqslant 2B(||f||_p)$. Now we may assume $S(f_n) = S(f)$ in Definition 3; hence

$$\int_{S(f)} T_{f_n}{}^*g(s)d\mu = \int_{S(g)} Tf_n(s)g(s)d\mu.$$

It now follows by Definition 3 and an application of Hölder's inequality that we have

$$\begin{aligned}\int_{S(g)} Tf(s)g(s)d\mu &= \lim_n \int_{S(g)} Tf_n(s)g(s)d\mu,\\ &= \lim_n \int_{S(f)} T_{f_n}{}^*g(s)d\mu,\\ &= \int_{S(f)} T_f{}^*g(s)d\mu,\end{aligned}$$

which is the desired result.

References

1. R. V. Chacon and N. Friedman, *Additive functionals*, Arch. Rational Mech. Anal., *18* (1965), 230–240.

2. M. M. Day, *The spaces L^p with* $0 < p < 1$, Bull. Amer. Math. Soc., *46* (1940) 816–823.
3. N. Friedman and M. Katz, *A representation theorem for additive functionals*, Arch. Rational Mech. Anal., *21* (1966), 49–57.
4. A. D. Martin and V. J. Mizel, *A representation theorem for certain nonlinear functionals*, Arch. Rational Mech. Anal., *15* (1964), 353–367.

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