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BOUNDS ON BETTI NUMBERS

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Introduction. The Betti numbers $\beta_n(k)$ of the residue class field $k = R/\mathfrak{m}$ of a commutative local ring $(R, \mathfrak{m})$ have been studied for about 20 years, primarily as the coefficients of the Poincaré series of E. Several authors have obtained results about the growth of the sequence $\{\beta_n(k)\}$. For example, Gulliksen [**3**] showed that when R is non-regular, the sequence is non-decreasing. More recently, Avramov [**1**] studied asymptotic properties of $\{\beta_n(k)\}$ and found that under certain conditions the growth is exponential, i.e., there is a natural number p such that for all n, $\beta_{pn}(k) \geqq 2^n$.

In this paper, we examine the sequence $\{\beta_n(M)\}$ for arbitrary finitely generated non-free modules M over any commutative local artin ring R. We establish the following bounds:

(1) $\beta_{n+1}(M)/\beta_n(M) < l(R), \quad n \geqq 2,$

(2) $\beta_2(M)/\beta_1(M) \leqq l(R),$ and

(3) $\beta_n(M)/\beta_{n+1}(M) < l(R)/l\ (\text{socle}\ (R)), \quad n \geqq 1,$

where $l(X)$ is the length of X.

While these results apply to a much wider class of modules, they do not appear to imply either of the two stated earlier. The only other bounds applying to all modules over certain rings are, as far as we know, those we have given in [**4**], [**5**], and [**6**]. In [**4**] and [**6**] we exhibited classes of rings for which $\beta_{n+1}(M)/\beta_n(M) > 1$. Larger classes were given in [**5**] and [**6**] for which the weaker inequality $\beta_{n+1}(M)/\beta_n(M) \geqq 1$ holds. In [**5**] we also gave a different inequality for a certain subclass of these rings [**5**, Theorem 3.8]:

THEOREM. *Let $S = R/\mathfrak{m}^{k+1}$, where $(R, \mathfrak{m})$ is a regular local ring of dimension $d \geqq 2$, and $k \geqq 1$. Let M be any finitely generated non-free S-module. Then for all $n \geqq 1$,*

$$\beta_{n+2}(M)/\beta_n(M) \geqq \sum_{i=1}^{d} \binom{k+i-1}{k}.$$

The following lemma is the key to our main result.

LEMMA 1. *Let $(R, \mathfrak{m})$ be any commutative noetherian local ring, and let X*

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be a module of finite length. Then for any finitely generated module M,

(1) $l(\mathrm{Tor}_{n+1}(M, X)) - l(\mathrm{Tor}_n(M, X)) \geqq \beta_{n+1}(M) - l(X)\beta_n(M)$.

Proof. We proceed by induction on $q = l(X)$. If $q = 1$, then $X \approx R/\mathfrak{m}$, and (1) reduces to equality. Now if $q > 1$, there is a module Y of length $q - 1$ such that the sequence

$$0 \rightarrow R/\mathfrak{m} \rightarrow X \rightarrow Y \rightarrow 0$$

is exact. By tensoring it with M, we obtain, from the long exact homology sequence, the following 5-term exact sequence:

(a) $$\mathrm{Tor}_{n+1}(M, X) \rightarrow \mathrm{Tor}_{n+1}(M, Y) \rightarrow \mathrm{Tor}_n(M, R/\mathfrak{m}\mathrm{E}) \rightarrow \mathrm{Tor}_n(M, X) \rightarrow \mathrm{Tor}_n(M, Y).$$

Now if $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E$ is an exact sequence of modules of finite length, then, by the additivity of length,

$$l(A) - l(B) + l(C) - l(D) + l(E) \geqq 0,$$

or equivalently,

$$l(A) - l(D) \geqq l(B) - l(E) - l(C).$$

Applying this to (a), we have:

$$l(\mathrm{Tor}_{n+1}(M, X)) - l(\mathrm{Tor}_n(M, X)) \geqq l(\mathrm{Tor}_{n+1}(M, Y)) - l(\mathrm{Tor}_n(M, Y)) - \beta_n(M)$$

Now, by induction,

$$l(\mathrm{Tor}_{n+1}(M, Y)) - l(\mathrm{Tor}_n(M, Y)) \geqq \beta_{n+1}(M) - l(Y)\beta_n(M) = \beta_{n+1}(M) - (q - 1)\beta_n(M).$$

The combination of these two inequalities yields (1).

When R is, in addition, artin, we may replace the X in the lemma with R. Since $\mathrm{Tor}_n(-, R) = 0$, the left side of (1) vanishes. This proves:

LEMMA 2. *Let (R, m) be any commutative local artin ring, and M any finitely generated R-module. Then for all $n \geqq 1$,*

$$\beta_{n+1}(M) \leqq l(R)\beta_n(M).$$

This brings us to our main result:

THEOREM 3. *Let R be artin. Then for any non-free M,*

(1) $\beta_{n+1}(M)/\beta_n(M) < l(R)$, *for $n \geqq 2$, and*

(2) $\beta_n(M)/\beta_{n+1}(M) < l(R)/l(\mathrm{socle}(R))$, *for $n \geqq 1$.*

Proof. Let I = socle (R). From the exact sequence $0 \to I \to R \to R/I \to 0$ we obtain, for $n \geqq 1$:

$$\mathrm{Tor}_{n+1}(M, R/I) \approx \mathrm{Tor}_n(M, I) \approx \oplus \sum \mathrm{Tor}_n(M, R/\mathfrak{m}),$$

where the direct sum term refers to $l(I)$ copies. Hence

$$l(\mathrm{Tor}_{n+1}(M, R/I)) = l(I)\beta_n(M), \quad \text{for } n \geqq 1.$$

Thus, by Lemma 1, we have: for $n \geqq 2$,

$$l(I)(\beta_n(M) - \beta_{n-1}(M)) \geqq \beta_{n+1}(M) - l(R/I)\beta_n(M).$$

Therefore:

$$l(I)\beta_{n-1}(M) + \beta_{n+1}(M) \leqq [l(I) + l(R/I)]\,\beta_n(M) = l(R)\beta_n(M).$$

Since M is not free, $\beta_i(M) > 0$ for all $i \geqq 0$. Hence, for $n \geqq 2$,

(1) $\beta_{n+1}(M)/\beta_n(M) < l(R)$ and

(2) $\beta_{n-1}(M)/\beta_n(M) < l(R)/l(I)$.

This theorem can be extended to Macauley rings:

COROLLARY 4. *Let* $(R, \mathfrak{m})$ *be any local Macauley ring, and* M *any finitely generated non-free* R*-module. Let* d = dim R. *Then*

(1) $\beta_{n+1}(M)/\beta_n(M) < l(\bar{R})$, $\quad n \geqq d + 2$, *and*

(2) $\beta_n(M)/\beta_{n+1}(M) < l(\bar{R})/l(\text{socle }(\bar{R}))$, $\quad n \geqq d + 1$,

where $\bar{R} = R/(X_1, \ldots, x_d)R$, *and* $x_1, \ldots, x_d$ *is a maximal regular* R*-sequence.*

Proof. Let K_d be the dth syzygy of M. Then $x_1, \ldots, x_d$ is a regular K_d-sequence. Hence $\mathrm{Tor}_i^R(K_d, \bar{R}) = 0$ for all $i \geqq 1$ and so

$$\mathrm{Tor}_i^{\bar{R}}(\bar{K}_d, R/\mathfrak{m}) \approx \mathrm{Tor}_i^R(K_d, R/\mathfrak{m}) \quad \text{for all } i \geqq 1,$$

by [**6**, Proposition 4.1.1], where $\bar{K}_d = K_d \otimes_R \bar{R}$. Thus $\beta_i^R(K_d) = \beta_i^{\bar{R}}(\bar{K}_d)$. Since $\beta_i(K_d) = \beta_{i+d}(M)$, the result now follows by applying Theorem 3 to the artin ring $\bar{R}$.

As an application of Theorem 3 we have:

COROLLARY 5. *Let* $(R, \mathfrak{m})$ *be a local Gorenstein ring of dimension* 0 (*i.e., a local QF ring*). *Let* A *be an* m *by* n *matrix with entries in* $\mathfrak{m}$, *such that* max $\{m/n, n/m\} \geqq l(R)$. *Then either* Ker A *or* Coker A *has a non-zero free summand.*

Proof. Suppose not. The sequence

$$0 \to \mathrm{Ker}\, A \to R^n \xrightarrow{A} R^m \to \mathrm{Coker}\, A \to 0$$

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is exact. Since R is a 0-dimensional Gorenstein ring, Coker A is reflexive, and hence embeddable in R^p for some p. Choose the minimal such p. If Coker $A \not\subset \mathfrak{m}R^p$, then Coker A has a free summand. Assume, therefore, that Coker $A \subset \mathfrak{m}R^p$. Similarly, by assuming that Ker A has no free summand, we may suppose that Ker $A \subset \mathfrak{m}R^n$. Let

$$B = \text{Coker } (\text{Coker } A \subset \mathfrak{m}R^p);$$

we may assume that B is imbedded in R^q, say. If $B \not\subset \mathfrak{m}R^q$, then B has a free summand, say $B \approx R \oplus B'$. But from the exactness of

$$0 \to \text{Coker } A \to R^p \to R \oplus B' \to 0,$$

it follows that

$$0 \to \text{Coker } A \to R^{p-1} \to B' \to 0$$

is also exact, thus contradicting the minimality of p. Thus $B \subset \mathfrak{m}R^q$ and so

$$R^n \xrightarrow{A} R^m \to R^p \to C \to 0$$

is the beginning of a minimal free resolution of $C = \text{Coker } (B \subset R^q)$. Hence $m = \beta_2(C)$ and $n = \beta_3(C)$. But then our hypothesis that $\max \{m/n, n/m\} \geqq l(R)$ contradicts Theorem 3. Hence at least one of Ker A and Coker A has a free summand.

We conclude this paper with a remark.

Remark. If $l(X) < \infty$, then by Lemma 1, with $M = R/\mathfrak{m}$, we have

$$\beta_{n+1}(X) - \beta_n(X) \geqq \beta_{n+1}(R/\mathfrak{m}) - l(X)\beta_n(R/\mathfrak{m}).$$

Thus if $l(X) < \liminf \{\beta_{n+1}(R/\mathfrak{m})/\beta_n(R/\mathfrak{m})\}$, then $\{\beta_n(X)\}$ is eventually strictly increasing.

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