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RESEARCH ARTICLE

Improved bounds for the solutions of renewal equations

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Abstract

Sequences of non-decreasing (non-increasing) lower (upper) bounds for the renewal-type equation as well as for the renewal function which are improvements of the famous corresponding bounds of Marshal [(1973). Linear bounds on the renewal function. *SIAM Journal on Applied Mathematics* 24(2): 245–250] are given. Also, sequences such bounds converging to the ordinary renewal function are obtained for several reliability classes of the lifetime distributions of the inter-arrival times, which are refinements of all of the existing known corresponding bounds. For the first time, a lower bound for the renewal function with DMRL lifetimes is given. Finally, sequences of such improved bounds are given for the ordinary renewal density as well as for the right-tail of the distribution of the forward recurrence time.

1. Introduction

Consider the renewal-type equation

$$Z(t) = r(t) + \int_0^t Z(t-y)\, dF(y), \quad t \geq 0, \tag{1.1}$$

where $Z(t)$ is the unknown function, $r(t)$ is a real-valued measurable function which is bounded on finite intervals with $r(t) = 0$ for $t < 0$ and F is a distribution function (df) of a non-negative random variable with $F(0) = 0$.

Let $\{X_1, X_2, \ldots\}$ be a sequence of independent and identically distributed random variables having common df F. Let also X represents a generic random variable of X_i' s. In the rest of the paper, we assume that the random variable X has at least the first two moments to be finite, that is, $\mu = E(X) < \infty$ and $\mu_2 = E(X^2) < \infty$.

The Stieltjes-type convolution of functions $g : [0, \infty) \to \mathbb{R}$ and F will be denoted by $g * F$ and is defined as $(g * F)(t) = \int_0^t g(t-y)\, dF(y)$. Also let $F^{*n}(x) = 1 - \bar{F}^{*n}(x),\ n \geq 1,\ x \geq 0$, the n-fold convolution of the df F with itself, with $F^{*1}(x) = F(x)$, $F^{*0}(x) = 0$, for $x < 0$ and $F^{*0}(x) = 1$, for $x \geq 1$.

Define for $t > 0$, the renewal process $N(t) = \sup\{n \in N : S_n \leq t, \}$, if $X_1 \leq t$, to be the number of renewals in $[0, t]$ with $N(t) = 0$, if $X_1 > t$, where S_n are the partial sum $S_n = X_1 + \cdots + X_n, n \geq 1$, with $Pr(S_n \leq t) = F^{*n}(t)$. The renewal function which is of primary interest in renewal theory is defined as $M(t) = E[N(t)]$ and is given by $M(t) = \sum_{n=1}^{\infty} F^{*n}(t)$. It is well-known that $M(t)$ is the solution of the renewal-type equation

$$M(t) = F(t) + \int_0^t M(t-y)\, dF(y) = F(t) + \int_0^t F(t-y)\, dM(y), \tag{1.2}$$

that is, $M(t)$ satisfies an equation of the form (1.1) with $r(t) = F(t)$. Also many authors consider as a renewal function, the function $U(t) = E[\nu(t)]$, where $\nu(t) = \inf\{n \in N : S_n \geq t\} = N(t) + 1, t \geq 0$ is the level t first-passage time. Hence, $U(t)$ is given by $U(t) = \sum_{n=0}^{\infty} F^{*n}(t) = 1 + M(t)$, implying that $U(t)$ is the solution of the following renewal equation

$$U(t) = 1 + \int_0^t U(t-y)\, dF(y). \tag{1.3}$$

Therefore, $U(t)$ also satisfies an equation of the form (1.1) with $r(t) = 1$.

Renewal-type equations which are Volterra-type integral equations are frequently encountered in several applications (such as in reliability analysis, queueing theory, renewal theory and ruin theory) when regenerative arguments are used in modeling. Especially, the renewal function $M(t)$ and/or $U(t)$ has a wide variety of applications in several areas, such as in queueing theory and its applications, including the design and performance evaluation of service systems as well as computer and telecommunication networks, maintenance policy analysis and optimization, product warranty policy analysis, supply chain planning, production and inventory control, inventory policy analysis, inventory system optimization, spare part demand forecasting, reliability modeling and availability analysis, and reliability modeling for a system that is subject to shocks (see, e.g., [15,16] and the references therein).

For the most of the distributions of the inter-arrival times, since renewal equations on the one hand usually do not have analytical solutions and, on the other hand, it is very difficult and complicated to obtain such solutions, as stated by Ran *et al.* [23], the development of alternate approaches (such as bounds, numerical and approximation methods) usually have to be used. Realizing the importance of the renewal function, researchers have developed several simple and accurate approximations to the renewal function. For more details, we refer to Xie *et al.* [29], Kambo *et al.* [17] and the references therein. The other approach, that is, to obtain bounds, has also a great practical importance. For example, as stated by Mitra and Basu [21], if F is the life distribution of a device and repairs or replacements are instantaneous, $N(t)$ would represent the number of failures/repairs in $[0, t)$ under *a perfect repair (replacement on failure)* strategy. Hence, the bounds can be very useful since they provide, in a sense, benchmark figures for the number of failures/repairs in the interval $[0, t)$. Bounds for the renewal function are obtained by several researchers. Some references are Barlow and Proschan [1,2], Lorden [18], Stone [25], Erikson [12], Marshall [20], Deley [11], Waldmann [26], Brown [6,7], Xie [28], Bhattacharjee [4,5], Ran *et al.* [23], Politis and Koutras [22] and Losidis and Politis [19]. For a survey of some bounds for the renewal function, see Beichelt and Fatti [3].

If the df F is absolutely continuous having a probability density function (pdf) f, then there exists the renewal density $u(t) = dU(t)/dt$ which of course is equal to the renewal density $m(t) = dM(t)/dt$. Since $u(t) = \sum_{n=1}^{\infty} f^{*n}(t)$, where $f^{*n}(t) = dF^{*n}(t)/dt$ is the n-fold convolution of the pdf f, then $u(t)$ is the solution of

$$u(t) = f(t) + \int_0^t u(t-y)\, dF(y), \tag{1.4}$$

and thus, $u(t) = m(t)$ satisfies the renewal-type equation (1.1) with $r(t) = f(t)$.

Bounds for the renewal density are obtained, for example, by Xie [28] and Losidis and Politis [19].

The forward recurrence time is a very important concept in renewal processes. It is the time between any given time t and the next epoch of the renewal process under consideration, denoted by the random variable $\gamma(t)$. It is also called residual lifetime, residual waiting time or excess lifetime at time $t \geq 0$ and is given by $\gamma(t) = S_{N(t)+1} - t$. If $V_y(t)$ is the df of $\gamma(t)$, that is, $V_y(t) = \Pr[\gamma(t) \leq y]$, then $V_y(t)$ satisfies the renewal-type equation

$$V_y(t) = F(t+y) - F(t) + \int_0^t V_y(t-x)\, dF(x). \tag{1.5}$$

Therefore, $V_y(t)$ is of the form of (1.1) with $r(t) = F(t + y) - F(t)$. It is well known that if X_i has an exponential distribution with rate $\lambda > 0$, then by memoryless property of the exponential distribution, it holds $\gamma(t) \sim \text{Exp}(\lambda)$. However, for the general renewal process, the distribution of $\gamma(t)$ is complicated and depends on time t. Therefore, it is very useful to find bounds and asymptotic results for the distribution of $\gamma(t)$. Some upper bounds for the tail $\bar{V}_y(t)$ are obtained by Lorden [18, Thm. 4] and Chang [9, Prop. 4.1 and 4.2] who improved the Lorden's upper bound. Also, simple two-sided bounds for the tail $\bar{V}_y(t)$ are obtained by Chen [10, Thm. 1] under the assumption that $\gamma(t)$ is stochastically decreasing and/or increasing in $t \geq 0$.

The paper is organized as follows: In Section 2, we give sequences of increasing (decreasing) lower (upper) bounds, of $Z(t)$. Some of these are given in terms of the renewal function $M(t)$. In Section 3, by applying the results of Section 2, we give increasing (decreasing) lower (upper) bounds for the renewal function $U(t)$, which are improvements of the corresponding Marshall's [20] and Waldmann's [26] bounds. In Section 4, we give several lower and upper bounds based on reliability properties (bounded mean residual life time, NWUE, NBUE, bounded failure rate, IMRL, DMRL, DFR, IFR reliability classes) of the distribution function F of the inter-arrival times, which are either new or improvements of corresponding existing bounds. Particularly, a lower bound for the renewal function with DMRL lifetimes is given for first time. In Section 5, we give two-sided bounds for the renewal density, whereas in Section 6, we give sequences of increasing (decreasing) lower (upper) bounds for the right-tail of the forward recurrence time. Several numerical examples are also given to illustrate the effectiveness of our new bounds.

2. Some bounds for the renewal-type function $Z(t)$

2.1. Bounds for $Z(t)$ in terms of $M(t)$

Using Laplace transforms, it is well-known that we can obtain the solution $Z(t)$ of (1.1) in terms of the renewal function $M(t)$ and/or $U(t)$, since it holds

$$Z(t) = r(t) + \int_0^t r(t-y)\, dM(y) = r(t) + \int_0^t r(t-y)\, dU(y). \tag{2.1}$$

Since the solution of $Z(t)$ ig given from the above equation, bounds for the renewal function $M(t)$ and/or $U(t)$ are also useful to obtain bounds for $Z(t)$. So, we shall give some bounds for $Z(t)$ in terms of $M(t)$. In order to do this, we need the following proposition (for the proof, see Appendix). By convention, $\sum_a^b(\cdot) = 0$, if $b < a$.

Proposition 2.1. *If $Z(t)$ satisfies (1.1) and $M(t)$ satisfies (1.2), then for every $n = 1, 2, 3, \ldots$*

$$(i) \quad Z(t) = \sum_{m=1}^{n} (r * F^{*(m-1)})(t) + \int_0^t Z(t-y)\, dF^{*n}(y) \tag{2.2}$$

$$(ii) \quad M(t) = \sum_{m=1}^{n} F^{*m}(t) + \int_0^t M(t-y)\, dF^{*n}(y). \tag{2.3}$$

Now, using the above proposition, we get a lower and an upper bound for $Z(t)$ by comparing $r(t)$ and $F(t)$. Thus, we have the following (for the proof, see Appendix).

Theorem 2.2. *Let $w(t) = r(t) - F(t)$. If $w(t) \geq (\leq)\, 0$, then for every $n = 1, 2, 3, \ldots$*

$$Z(t) \geq (\leq) M(t) + \sum_{m=1}^{n} (w * F^{*(m-1)})(t). \tag{2.4}$$

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Note that the bound in (2.4) is exact at $t = 0$, since it is equal to $r(0) = Z(0)$. Also, both the lower and the upper bound in (2.4) are getting tighter as n increases since $w(t) \geq 0$ for the lower bound and $w(t) \leq 0$ for the upper bound.

Further bounds for $Z(t)$ which are given at the next Theorem 2.3 can be obtained by examining except the sign, an additional property for the function $w(t) = r(t) - F(t)$ concerning its monotonicity.

Theorem 2.3. *(i) If $w(t) = r(t) - F(t)$ is differentiable with $w'(t) \leq (\geq) 0$, then for every $n = 0, 1, 2, \ldots$, it holds*

$$Z(t) \geq (\leq) M(t) + w(t)[1 + M(t)] + \varepsilon_n(t), \quad t \geq 0, \tag{2.5}$$

where $\varepsilon_n(t) = \sum_{m=1}^{n} \{(w * F^{*m})(t) - w(t)F^{*m}(t)\} \geq (\leq) 0$.

(ii) If $w'(t) \leq (\geq) 0$ and $w(t) \geq (\leq) 0$, then for every $n = 0, 1, 2, \ldots$, the bound (2.5) is a refinement of the bound (2.4).

The bound in (2.5) is exact at $t = 0$. Since $\varepsilon_n(t) \geq 0$ ($\varepsilon_n(t) \geq 0$) for the lower (upper) bound, increasing n in (2.5) yields a tighter and tighter bound. Therefore, the bound in (2.5) for every $n = 1, 2, 3, \ldots$ is a refinement of the bound given in (A.5).

The bounds of Theorems 2.2 and 2.3 are very useful to obtain bounds for delayed renewal processes by comparing the df of the first renewal time in the delayed renewal process with the df $F(t)$ of the inter-arrival times of the ordinary renewal process.

2.2. *A sequence of two-sided bounds for $Z(t)$*

In this subsection, we shall give general two-sided bounds for $Z(t)$, in the sense that they do not depend on the renewal function $M(t)$. Thus, we have the following.

Theorem 2.4. *Let*

$$\sigma_U(t) = \sup_{\substack{0 \leq z \leq t \\ \bar{F}(z) > 0}} \left\{ \frac{r(z)}{\bar{F}(z)} \right\}, \quad \sigma_L(t) = \inf_{\substack{0 \leq z \leq t \\ \bar{F}(z) > 0}} \left\{ \frac{r(z)}{\bar{F}(z)} \right\}, \tag{2.6}$$

and

$$\psi_L(t) = r(t) - \sigma_L(t)\bar{F}(t) \geq 0, \quad \psi_U(t) = \sigma_U(t)\bar{F}(t) - r(t) \geq 0. \tag{2.7}$$

Then, for every $n = 1, 2, 3, \ldots$, it holds:

$$(i) \quad Z(t) \leq r(t) + (\sigma_U * F)(t) - \sum_{m=1}^{n} (\psi_U * F^{*m})(t) \tag{2.8}$$

$$\leq \sigma_U(t) - \sum_{m=1}^{n} (\psi_U * F^{*(m-1)})(t). \tag{2.9}$$

$$(ii) \quad Z(t) \geq r(t) + (\sigma_L * F)(t) + \sum_{m=1}^{n} (\psi_L * F^{*m})(t) \tag{2.10}$$

$$\geq \sigma_L(t) + \sum_{m=1}^{n} (\psi_L * F^{*(m-1)})(t). \tag{2.11}$$

Proof. (i) At first, we shall prove (2.9) by mathematical induction on $n = 1, 2, 3, \ldots$ Since $\sigma_U(t)$ is a non-decreasing function, then for $0 \leq y \leq t$, it holds $\sigma_U(t - y) \leq \sigma_U(t)$, and since $r(t) \leq \sigma_U(t)\bar{F}(t)$,

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from (2.1) we obtain

$$\begin{aligned} Z(t) &= r(t) + \int_0^t r(t-y)\,dU(y) \\ &\le r(t) + \int_0^t \sigma_U(t-y)\bar{F}(t-y)\,dU(y) \\ &\le r(t) + \sigma_U(t)\int_0^t \bar{F}(t-y)\,dU(y). \end{aligned} \tag{2.12}$$

Integrating by parts, from (1.3) we get

$$\begin{aligned} U(t) &= 1 - \int_0^t U(t-y)\bar{F}'(y) \\ &= 1 - \left\{U(0)\bar{F}(t) - U(t)\bar{F}(0) + \int_0^t U'(t-y)\bar{F}(y)\,dy\right\} \\ &= F(t) + U(t) - \int_0^t \bar{F}(t-y)\,dU(y), \end{aligned}$$

and thus, it holds

$$int_0^t \bar{F}(t-y)\,dU(y) = F(t). \tag{2.13}$$

Then, (2.12) yields

$$Z(t) \le r(t) + \sigma_U(t)F(t) = \sigma_U(t) - \psi_U(t),$$

which proves that the upper bound in (2.9) holds for $n = 1$. If we assume that the upper bound in (2.9) holds true for some $n = 1, 2, 3, \ldots$, then by inserting the upper bound in (2.9) into the integral of the right-hand side of (1.1), we get

$$\begin{aligned} Z(t) &\le r(t) + \int_0^t \left\{\sigma_U(t-y) - \sum_{m=1}^{n}(\psi_U * F^{*(m-1)})(t-y)\right\} dF(y) \\ &\le r(t) + \sigma_U(t)\int_0^t dF(y) - \sum_{m=1}^{n}(\psi_U * F^{*m})(t) \\ &= r(t) + \sigma_U(t) - \sigma_U(t)\bar{F}(t) - \sum_{m=2}^{n+1}(\psi_U * F^{*(m-1)})(t) \\ &= r\sigma_U(t) - \sum_{m=1}^{n+1}(\psi_U * F^{*(m-1)})(t). \end{aligned}$$

Therefore, the upper bound in (2.9) holds also true for $n + 1$, and hence the upper bound in (2.9) holds for all $n = 1, 2, 3, \ldots$

By inserting the upper bound in (2.9) into the integral of the right-hand side of (1.1) we immediately obtain (2.8). Let $B_1(t)$ and $B_2(t)$ denote the upper bounds in (2.9) and (2.8), respectively. In order to show that the bound in (2.8) is a refinement of the upper bound given in (2.9), it suffices to show that

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$B_2(t) \le B_1(t)$. Indeed, since $\sigma_U(t)$ is a non-decreasing function in $t \ge 0$, it holds

$$\begin{aligned} B_2(t) &= r(t) + \int_0^t \sigma_U(t-y)\, dF(y) - \sum_{m=1}^{n-1} (\psi_U * F^{*m})(t) - (\psi_U * F^{*n})(t) \\ &\le r(t) + \sigma_U(t)F(t) - \sum_{m=1}^{n-1} (\psi_U * F^{*m})(t) \\ &= \sigma_U(t) - \psi_U(t) - \sum_{m=1}^{n-1} (\psi_U * F^{*m})(t) \\ &= \sigma_U(t) - \sum_{m=1}^{n} (\psi_U * F^{*(m-1)})(t) \\ &= B_1(t). \end{aligned}$$

(ii) The lower bound in (2.11) follows in a similar way, by observing that $r(t) \ge \sigma_L(t)\bar{F}(t)$ and $\sigma_L(t)$ is a non-increasing function in t. The rest of the proof is similar as in (i). Note, that all the upper and the lower bounds in (2.8)–(2.11) are getting tighter as n increases since $\psi_L(t) \ge 0$ for the lower bound and $\psi_U(t) \le 0$ for the upper bound.

Also, all the bounds are exact at $x = 0$. Indeed, for $x = 0$, the upper bound in (2.9) is equal to $r(0)$, and the upper bound in (2.8) is equal to

$$\sigma_U(0) - \psi(0) = \sigma_U(0) - \{\sigma_U(0)\bar{F}(0) - r(0)\} = r(0) = Z(0).$$

Similarly, the lower bounds in (2.10) and (2.11) are also equal to $r(0)$. □

Remark 1. (i) Note, that Theorem 2.4 is still holds if we replace $\sigma_U(t)$ and $\sigma_L(t)$ with the weaker functions σ_u and σ_l, where

$$\sigma_u = \sup_{\substack{z \ge 0 \\ \bar{F}(z) > 0}} \left\{ \frac{r(z)}{\bar{F}(z)} \right\}, \quad \sigma_l = \inf_{\substack{z \ge 0 \\ \bar{F}(z) > 0}} \left\{ \frac{r(z)}{\bar{F}(z)} \right\}.$$

Then, (2.8) and (2.10) are reduced to

$$R_n(\sigma_\ell, t) \le Z(t) \le R_n(\sigma_u, t), \quad n = 1, 2, 3 \ldots$$

where

$$R_n(\sigma, t) = \sum_{m=0}^{m} (r \star F^{*m}) + \sigma F(t) - \sigma \sum_{m=1}^{n} (\bar{F} \star F^{*m})(t). \tag{2.14}$$

It can be easily proved, for example, by mathematical induction, that

$$\bar{F}^{*n}(t) = \sum_{m=0}^{n-1} (\bar{F} \star F^{*m})(t), \quad n = 1, 2, 3 \ldots$$

Also from Resnick [24, Sect. 3.5], it follows that the general solution to (1.1) is

$$\begin{aligned} Z(t) &= r(t) + \sum_{m=0}^{\infty} \int_0^t r(t-y)\, dF^{*m}(y) \\ &= \sum_{m=0}^{\infty} (r \star F^{*m})(t). \end{aligned}$$

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Then,

$$\begin{aligned}\lim_{n\to\infty} R_n(\sigma,t) &= \sum_{m=0}^{\infty}(r\star F^{*m})(t)+\sigma F(t)-\sigma\left[\sum_{m=0}^{\infty}(\bar{F}\star F^{*m})(t)-\bar{F}(t)\right]\\ &= Z(t)+\sigma F(t)-\sigma\left[\lim_{n\to\infty}\bar{F}^{*n}-\bar{F}(t)\right]\\ &= Z(t),\end{aligned}$$

since $\lim_{n\to\infty}\bar{F}^{*n}=1$.

Therefore, both the upper and lower bound $R_n(\sigma_u,t)$ and $R_n(\sigma_\ell,t)$ converge to $Z(t)$.

(ii) Let $I_n(\psi_U(t))$ be the upper bound in (2.8) and $I_n(\psi_L(t))$ (2.10). For simplicity, let us examine the case $n=1$. Since, $\sigma_U(t)\le\sigma_u$ for any $t\ge 0$, we get that

$$\begin{aligned}I_1(\psi_U(t)) &= r(t)+(\sigma_U\star F)(t)-(\psi_U\star F)(t)\\ &= r(t)+(\sigma_U\star F)(t)-(\sigma_U\bar{F}\star F)(t)+(r\star F)(t)\\ &= r(t)+(\sigma_U F\star F)(t)+(r\star F)(t)\\ &\le r(t)+\sigma_u(F\star F)(t)+(r\star F)(t)\\ &= r(t)+\sigma_u F^{*2}(t)+(r\star F)(t).\end{aligned}$$

Also, it is

$$\begin{aligned}R_1(\sigma_u,t) &= r(t)+(r\star F)(t)+\sigma_u F(t)-\sigma_u(\bar{F}\star F)(t)\\ &= r(t)+(r\star F)(t)+\sigma_u F(t)-\sigma_u(F(t)-F^{*2}(t))\\ &= r(t)+(r\star F)(t)+\sigma_u F^{*2}(t),\end{aligned}$$

and hence, it holds that $I_1(\psi_U(t))\le R_1(\sigma_u,t)$. Similarly, since $\sigma_L(t)\ge\sigma_\ell$ for any $t\ge 0$, we can prove that $I_1(\psi_L(t))\ge R_1(\sigma_\ell,t)$. Therefore, the two-sided bound for $Z(t)$ given by (2.8) and (2.10) is tighter than that given by (2.14) for $n=1$.

3. Improvements of Marshall's and Waldmann's bounds for the renewal function

The linear bounds of Marshal [20] are recognized to be the "best" linear bounds for the renewal function. Let,

$$\alpha_l=\inf_{t\in A}\left\{\frac{\bar{F}_e(t)}{\bar{F}(t)}\right\},\quad \alpha_u=\sup_{t\in A}\left\{\frac{\bar{F}_e(t)}{\bar{F}(t)}\right\}, \tag{3.1}$$

where $A=\{t\ge 0;\ \bar{F}(t)>0\}$. Then, Marshall [20] proved the following two-sided linear bound for the renewal function $U(t)$,

$$\frac{t}{\mu}+\alpha_l\le U(t)\le\frac{t}{\mu}+\alpha_u, \tag{3.2}$$

as well as that

$$\frac{t}{\mu}+\bar{F}_e(t)+\alpha_l F(t)\le U(t)\le\frac{t}{\mu}+\bar{F}_e(t)+\alpha_u F(t), \tag{3.3}$$

where $F_e(t)=1-\bar{F}_e(t)=(\int_0^t\bar{F}(y)\,dy)/\mu$ is the equilibrium df of F.

Note that since $\bar{F}_e(t)\le\alpha_u\bar{F}(t)$ and $\bar{F}_e(t)\ge\alpha_l\bar{F}(t)$ the upper (lower) bound in (3.3) is a refinement of the upper (lower) bound given in (3.2).

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Let $A_t = \{0 \le z \le t;\ \bar{F}(z) > 0\}$ and

$$\alpha_U(t) = \sup_{z \in A_t} \{\bar{F}_e(z)/\bar{F}(z)\}, \quad \alpha_L(t) = \inf_{z \in A_t} \{\bar{F}_e(z)/\bar{F}(z)\}. \tag{3.4}$$

Waldmann [26, Cor. 2] by applying a monotonicity argument to obtain upper and lower bounds for $Z(t)$, proved the following two-sided bound for $U(t)$,

$$\frac{t}{\mu} + \bar{F}_e(t) + \alpha_L(t)F(t) \le U(t) \le \frac{t}{\mu} + \bar{F}_e(t) + \alpha_U(t)F(t). \tag{3.5}$$

Of course, Waldmann's two-sided bound is an improvement of the aforementioned Marshall' s two-sided bounds, since $\alpha_U(t) \le \alpha_u$ and $\alpha_L(t) \ge \alpha_l$.

In the following Proposition 3.1, we shall give a two-sided bound which is tighter than that given in (3.5) and thus it is tighter than that given in (3.3) and (3.4). The proof is motivated by Feller [14] and is much simpler than that given by Waldmann [26].

Proposition 3.1. *If $\alpha_U(t)$ and $\alpha_L(t)$ are given by (3.4), then it holds the following two-sided bound for $U(t)$*

$$\frac{t}{\mu} + \bar{F}_e(t) + (\alpha_L * F)(t) \le U(t) \le \frac{t}{\mu} + \bar{F}_e(t) + (\alpha_U * F)(t). \tag{3.6}$$

Proof. Let $Z(t) = t/\mu + \alpha_U(t) - 1$. Since $\alpha_U(t)$ is a non-decreasing function, then for $0 \le y \le t$, it holds $\alpha_U(t-y) \le \alpha_U(t)$, and thus from (1.1), we get

$$\begin{aligned}
\frac{t}{\mu} + \alpha_U(t) - 1 &= r(t) + \int_0^t \left(\frac{t-y}{\mu} + \alpha_U(t-y) - 1\right) dF(y) \\
&\le r(t) + \int_0^t \frac{t-y}{\mu}\, dF(y) + [\alpha_U(t) - 1] \int_0^t dF(y) \\
&= r(t) + \frac{t}{\mu} - F_e(t) + [\alpha_U(t) - 1]F(t),
\end{aligned}$$

implying that

$$r(t) \ge [\alpha_U(t) - 1]\bar{F}(t) + F_e(t).$$

Since, $\alpha_U(t) \ge \bar{F}_e(t)/\bar{F}(t)$, from the above relation, we get that $r(t) \ge F(t)$. Inserting this bound for $r(t)$ into the right-hand side of (2.1) and taking into account (1.2), we get

$$Z(t) \ge F(t) + \int_0^t F(t-y)\, dM(y),$$

that is, it holds $M(t) \le Z(t)$, or equivalently

$$U(t) \le \frac{t}{\mu} + \alpha_U(t). \tag{3.7}$$

Now, inserting the upper bound in (3.7) into the right-hand side of (1.3), we obtain

$$\begin{aligned}
U(t) &\le 1 + \int_0^t \left(\frac{t-y}{\mu} + \alpha_U(t-y)\right) dF(y) \\
&= 1 + \frac{t}{\mu} - F_e(t) + (\alpha_U * F)(t),
\end{aligned}$$

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which yields the upper bound in (3.6). The proof for the lower bound is similar, if we take $Z(t) = t/\mu + \alpha_L(t) - 1$ and using that $\alpha_L(t)$ is a non-increasing function, we obtain

$$U(t) \geq \frac{t}{\mu} + \alpha_L(t). \tag{3.8}$$

Inserting the lower bound in (3.8) into the right-hand side of (1.3), we obtain the lower bound in (3.6). □

The upper bound in (3.7) and the lower bound in (3.8) are improvements of the linear two-sided bound in (3.3) obtained by [20]. Also, since

$$(\alpha_U * F)(t) \leq \alpha_U(t)F(t), \quad (\alpha_L * F)(t) \geq \alpha_L(t)F(t),$$

it follows that the two-sided bound in (3.6) is a refinement of the two-sided bound in (3.5) obtained by Marshall [20] and thus is also a refinement of the two-sided bound in (3.6) obtained by [20].

Example 1. Consider a distribution for the r.v. given in Marshall [20], having df

$$F(t) = \begin{cases} \frac{5}{7}t, & 0 \leq t < 1, \\ 1 - \frac{2}{7}e^{-(t-1)}, & t \geq 1, \end{cases}$$

with $\mu = E(X) = 13/14$ and $\mu_2 = E(X^2) = 5/3$.

Marshall [20] obtained the following general two-sided linear bounds given by (3.2)

$$1.076t - 0.86 \leq U(t) \leq 1.076t + 1.08, \quad t \geq 0.$$

Let $L_M(t)(U_M(t))$ denotes the lower (upper) bound for $U(t)$ obtained by Marshall [20] and given by (3.3), $L_W(t)(U_W(t))$ denotes the lower (upper) bound obtained by Waldmann [26] and given by (3.4) and $L_{CT}(t)(U_{CT}(t))$ denotes our new lower (upper) bound given by (3.6). Obviously, the bound $L_M(t)(U_M(t))$ is a refirement of the aforementioned Marshall's [20] lower (upper) bound.

Note that a general lower bound for $U(t)$ was given by Barlow and Proschan [1] who poved that

$$U(t) \geq \frac{t}{\mu F_e(t)} =: L_{\text{BP}}(t), \quad t \geq 0, \tag{3.9}$$

and a famous upper bound was obtained by Lorden [18] who proved that

$$U(t) \leq \frac{t}{\mu} + \frac{\mu_2}{\mu^2} =: U_L(t), \quad t \geq 0.$$

For the above distribution, we shall compare numerically the aforementioned lower bounds $L_{\text{BP}}(t), L_M(t), L_W(t)$ and $L_{CT}(t)$ as well as the upper bounds $U_L(t), U_M(t), U_W(t)$ and $U_{CT}(t)$.

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It turns out that the after some routine calculations, these bounds are

$$L_M(t) = \begin{cases} 0.384615t^2 + 0.614286t + 1, & 0 \le t < 1, \\ 1.07692t + 0.168474e^{-t} + 0.86, & t \ge 1, \end{cases}$$

$$U_M(t) = \begin{cases} 0.384615t^2 + 0.771429t + 1, & 0 \le t < 1, \\ 1.07692t - 0.0023897e^{-t} + 1.08, & t \ge 1, \end{cases}$$

$$L_W(t) = \begin{cases} \dfrac{91 - 35t^2}{91 - 65t}, & 0 \le t < 0.6, \\ 0.384615(t^2 + 1.59993t + 2.6), & 0.6 \le t < 1, \\ 1.07692t + 0.167309e^{-t} + 0.8615, & t \ge 1, \end{cases}$$

$$U_W(t) = \begin{cases} 0.384615t^2 + 0.714286t + 1, & 0 \le t < \dfrac{33}{35}, \\ 0.384615t^2 + \dfrac{5(5t^2 - 14t + 13)t}{13(7 - 5t)} + 1, & \dfrac{33}{35} \le t < 1, \\ 1.07692t + 1.07692, & t \ge 1, \end{cases}$$

$$L_{CT}(t) = \begin{cases} 0.192308t^2 + 0.538462t - 0.246154\log(7 - 5t) + 1.47899, & 0 \le t \le 0.6, \\ 0.384615t^2 + 0.61538t + 1.02237, & 0.6 < t \le 1, \\ 0.246154\log(12. - 5.t) + 0.192308t^2 + 0.61538t - 0.293419e^{-t} & \\ \quad +1.08536e^{-t}\Gamma(0, 12/5 - t) + 0.797222, & 1 < t < 1.6, \\ 1.07692t + 0.196084e^{-1.t} + 0.861537, & t \ge 1.6, \end{cases}$$

$$U_{CT}(t) = \begin{cases} 0.384615t^2 + 0.71422839011t + 1, & 0 \le t < \dfrac{33}{35}, \\ 0.192308t^2 + 0.538462t - 0.246154\log(7 - 5t) + 1.54022, & \dfrac{33}{35} \le t < 1, \\ 1.13186t + 0.0597444e^{-t} + 0.94652, & 1 \le t < \dfrac{68}{35}, \\ 0.19231t^2 + 0.769233t - 0.178646e^{-t} + 0.24615\log(12 - 5t) & \\ \quad +1.08536e^{-t}\Gamma(0, 12/5 - t) + 0.66015, & \dfrac{68}{35} \le t < 2, \\ 1.07692t - 0.0983675e^{-t} + 1.07692, & t \ge 2, \end{cases}$$

$$L_{\mathrm{BP}}(t) = \begin{cases} \dfrac{14}{14 - 5t}, & 0 \le t \le 1, \\ \dfrac{14t}{13 - 4e^{-(t-1)}}, & t > 1, \end{cases}$$

$$U_L(t) = \frac{14}{13}t + \frac{5}{3}\left(\frac{14}{13}\right)^2.$$

Table 1 presents the values of these bounds for various values of $0 \le t \le 50$. We observe that the general lower bound given by Barlow and Proschan [1] is worse than all other bounds $L_M(t), L_W(t)$ and L_{CT}. Also, the general upper bound $U_L(t)$ given by Lorden [18] is worse than all upper bounds $U_M(t), U_W(t)$ and $U_{CT}(t)$. We also observe that our new lower (upper) bound $L_{CT}(t)(U_{CT}(t))$ improve the corresponding lower (upper) bound $L_M(t)$ and $L_W(t)$ ($U_M(t)$ and $U_W(t)$), respectively, for any $t > 0$ and especially for small values of t, whereas for larger values of t, the bounds are approximately equal.

Example 2. In this example, we consider an interarrival random variable X given in Example 7.16 in Beichelt and Fatti [3], having df $F(t) = (1 - e^{-t})^2$, $t \ge 0$, with $\mu = \frac{3}{2}$. If $\mu_e(t)$ denotes the failure rate of the random variable having df $F_e(t)$, then

$$\mu_e(t) = \frac{2(2 - e^{-t})}{4 - e^{-t}}, \quad t \ge 0.$$

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Table 1. *Numerical values for Marshall's, Waldmann's and CT lower and upper bounds as well as Lorden's upper bound and Barlow and Proschan's lower bound.*

t	$L_{BP}(t)$	$L_M(t)$	$L_W(t)$	$L_{CT}(t)$	$U_L(t)$	$U_M(t)$	$U_W(t)$	$U_{CT}(t)$
0.0	0.000000	1.000000	1.000000	1.000000	1.932939	1.000000	1.000000	1.000000
0.1	1.037037	1.065275	1.072781	1.074011	2.040631	1.080989	1.075275	1.075274
0.2	1.076923	1.138242	1.148718	1.153329	2.148323	1.169670	1.158242	1.158241
0.3	1.120000	1.218901	1.228671	1.238209	2.256016	1.266044	1.248901	1.248900
0.4	1.166667	1.307253	1.313846	1.328978	2.363708	1.370110	1.347253	1.347252
0.5	1.217391	1.403297	1.405983	1.426067	2.471400	1.481868	1.453297	1.453296
0.6	1.272727	1.507033	1.507676	1.530059	2.579093	1.601319	1.567033	1.567032
0.7	1.333333	1.618462	1.619212	1.641598	2.686785	1.728462	1.688462	1.688461
0.8	1.400000	1.737582	1.738440	1.760828	2.794477	1.863297	1.817582	1.817581
0.9	1.473684	1.864396	1.865360	1.887750	2.902170	2.005824	1.954396	1.954394
1.0	1.555556	1.998901	1.999973	2.022365	3.009862	2.156044	2.153846	2.100362
1.5	1.986026	2.512976	2.514216	2.520759	3.548323	2.694851	2.692308	2.657645
2.0	2.428767	3.036647	3.037989	3.041920	4.086785	3.233523	3.230769	3.217454
2.5	2.890775	3.566137	3.567541	3.569940	4.625247	3.772112	3.769231	3.761153
3.0	3.371149	4.099157	4.100599	4.102069	5.163708	4.310650	4.307692	4.302792
3.5	3.866897	4.634318	4.635783	4.636689	5.702170	4.849159	4.846154	4.843180
4.0	4.374709	5.170778	5.172257	5.172821	6.240631	5.387649	5.384615	5.382811
4.5	4.891604	5.708025	5.709512	5.709869	6.779093	5.926127	5.923077	5.921981
5.0	5.415133	6.245751	6.247243	6.247474	7.317554	6.464599	6.461538	6.460873
10.0	10.769640	11.629238	11.630738	11.630777	12.702170	11.849231	11.846154	11.846146
20.0	21.538462	22.398462	22.399962	22.399999	23.471400	22.618462	22.615385	22.615382
50.0	53.846154	54.706154	54.707654	54.707691	55.779093	54.926154	54.923077	54.923074

Since $\mu_e(t)$ is strictly increasing function in $t \geq 0$, then using that

$$\frac{\bar{F}_e(x)}{\bar{F}(x)} = \frac{1}{\mu \mu_e(x)},$$

we find

$$\alpha_u = \sup_{\substack{t \geq 0 \\ \bar{F}(t)>0}} \left\{ \frac{\bar{F}_e(t)}{\bar{F}(t)} \right\} = \sup_{t \geq 0} \left\{ \frac{4 - e^{-t}}{3(2 - e^{-t})} \right\} = 1,$$

$$\alpha_U(t) = \sup_{\substack{0 \leq x \leq t \\ \bar{F}(x)>0}} \left\{ \frac{\bar{F}_e(x)}{\bar{F}(x)} \right\} = \sup_{0 \leq x \leq t} \left\{ \frac{4 - e^{-x}}{3(2 - e^{-x})} \right\} = 1,$$

$$\alpha_\ell = \inf_{\substack{t \geq 0 \\ \bar{F}(x)>0}} \left\{ \frac{\bar{F}_e(x)}{\bar{F}(x)} \right\} = \inf_{t \geq 0} \left\{ \frac{4 - e^{-t}}{3(2 - e^{-t})} \right\} = \frac{2}{3},$$

and

$$\alpha_L(t) = \inf_{\substack{0 \leq x \leq t \\ \bar{F}(x)>0}} \left\{ \frac{\bar{F}_e(x)}{\bar{F}(x)} \right\} = \inf_{0 \leq x \leq t} \left\{ \frac{4 - e^{-x}}{3(2 - e^{-x})} \right\} = \frac{4 - e^{-t}}{3(2 - e^{-t})}.$$

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Therefore, the upper bounds $U_M(t), U_W(t)$ and $U_{CT}(t)$ given in (3.3), (3.5) and (3.6), respectively, are the same, and thus, we obtain

$$U(t) \le \tfrac{2}{3}t + \tfrac{2}{3}(2 - \tfrac{1}{2}e^{-t})e^{-t} + (1 - e^{-t})^2, \quad t \ge 0.$$

Also, since $a_L(t) \ge a_\ell$ for any $t \ge 0$, it follows that the lower bound $L_{CT}(t)$ given in (3.6) is a refinement of the lower bound $L_M(t)$ given in (3.3) obtained by Marshall [20] and of course is refinement of (3.2). As in the previous Example 1, let us denote by $L_{\mathrm{BP}}(t), L_M(t), L_W(t)$ and $L_{CT}(t)$ the lower bounds given in (3.9), (3.3), (3.5) and (3.6), respectively. These bounds, for any $t \ge 0$, are given by

$$\begin{aligned}
L_{\mathrm{BP}}(t) &= \frac{2t}{3 - (4 - e^{-t})e^{-t}},\\
L_M(t) &= \frac{2t}{3} + \frac{4 - e^{-t} - e^{-t}(1 - e^{-t})^2}{3(2 - e^{-t})},\\
L_W(t) &= \frac{2t}{3} + \frac{4 - e^{-t}}{3(2 - e^{-t})},\\
L_{CT}(t) &= \frac{2t}{3} + \frac{(4 - e^{-t})(1 - (1 - e^{-t})^2)}{3(2 - e^{-t})},\\
&\quad + \frac{1}{6}e^{-2t}(-10e^t + 4e^{2t} + (2e^t - 1)\log(2e^t - 1) + 6).
\end{aligned}$$

In Table 2, we give the numerical values of these bounds for several values of $0 \le t \le 50$. As in the Example 1, we observe that the lower bounds $L_{\mathrm{BP}}(t)$ obtained by Barlow and Proschan [1] is worse than all lower bounds $L_M(t), L_W(t)$ and $L_{CT}(t)$. Also for the values $0 \le t \le 10$, the lower bound $L_W(t)$ is better than the bound $L_M(t)$, while our new lower bound $L_{CT}(t)$ is a refinement of the lower bounds $U_M(t)$ and $U_W(t)$. Note that for $t > 10$, the bounds $U_M(t), U_W(t)$ and $U_{CT}(t)$ are approximately equal.

Improvements of the bounds for the renewal function $U(t)$ given in Proposition 3.1 are obtained in Corollary 3.2, with a different (but even simple) proof than that given in Proposition 3.1.

Using Theorem 2.4, one can directly obtain a sequence of general two-sided bounds for the renewal function $U(t)$. Thus, we get the following main result of this section.

Corollary 3.2. *If $\alpha_U(t)$ and $\alpha_L(t)$ are given by (3.4), and*

$$\xi_L(t) = \bar{F}_e(t) - \alpha_L(t)\bar{F}(t) \ge 0, \quad \xi_U(t) = \alpha_U(t)\bar{F}(t) - \bar{F}_e(t) \ge 0, \tag{3.10}$$

then, for every $n = 1, 2, 3, \ldots$, it holds

$$\begin{aligned}
&\frac{t}{\mu} + \bar{F}_e(t) + (\alpha_L * F)(t) + \sum_{m=1}^{n}(\xi_L * F^{*m})(t) \le U(t)\\
&\quad \le \frac{t}{\mu} + \bar{F}_e(t) + (\alpha_U * F)(t) - \sum_{m=1}^{n}(\xi_U * F^{*m})(t).
\end{aligned}\tag{3.11}$$

The upper (lower) bound is monotone non-increasing (non-decreasing) in $n \ge 1$.

Proof. Since for $r(t) = \bar{F}_e(t)$, then $Z(t)$ satisfying (1.1) is $Z(t) = U(t) - t/\mu$, $\sigma_U(t), \sigma_L(t)$ defined in relation (2.6), are reduced to $\alpha_U(t), \alpha_L(t)$ given by (3.4), and $\psi_U(t), \psi_L(t)$ defined in (2.7) are reduced to $\xi_U(t)$ and $\xi_L(t)$ given by (3.11) respectively. Applying now Theorem 2.4 with $r(t) = \bar{F}_e(t)$, the result follows. Since $\xi_U(t) \ge 0$ and $\xi_L(t) \ge 0$, it follows that the upper (lower) bound in (3.11) is monotone non-increasing (non-decreasing) in $n \ge 1$. □

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Table 2. *Numerical values for Barlow and Proschan's, Marshall's, Waldmann's and CT lower bounds.*

t	L_{BP}	$L_M(t)$	$L_W(t)$	$L_{CT}(t)$
0.0	0.00000	1.00000	1.00000	1.00000
0.1	1.00310	1.00624	1.00874	1.00907
0.2	1.01164	1.02344	1.03103	1.03306
0.3	1.02470	1.04960	1.06278	1.06807
0.4	1.04160	1.08311	1.10137	1.11119
0.5	1.06185	1.12263	1.14509	1.16025
0.6	1.08504	1.16706	1.19273	1.21360
0.7	1.11089	1.21553	1.24343	1.27004
0.8	1.13913	1.26730	1.29659	1.32867
0.9	1.16958	1.32177	1.35172	1.38882
1.0	1.20205	1.37845	1.40847	1.45002
1.5	1.39065	1.68326	1.70852	1.76259
2.0	1.61487	2.00611	2.02419	2.07742
2.5	1.86679	2.33558	2.34760	2.39315
3.0	2.14031	2.66749	2.67518	2.71108
3.5	2.43045	3.00030	3.00511	3.03196
4.0	2.73311	3.33345	3.33641	3.35579
4.5	3.04498	3.66671	3.66853	3.68215
5.0	3.36350	4.00002	4.00113	4.01052
10.0	6.66707	7.33333	7.33334	7.33348
20.0	13.33333	14.00000	14.00000	14.00000
50.0	33.33333	34.00000	34.00000	34.00000

Note that the lower and the upper bound in (3.11) are exact at $t = 0$ and for every $n = 1, 2, 3, \ldots$

Remark 2. Corollary 3.2 still holds if we replace $\alpha_U(t)$ and $\alpha_L(t)$ with the weaker functions α_u and α_l defined by (3.1). Then, from Corollary 3.2 and for every $n = 0, 1, 2, \ldots$, we get that

$$\frac{t}{\mu} + \bar{F}_e(t) + \alpha_l F(t) + \sum_{m=1}^{n-1} (\xi_l * F^{*m})(t) \le U(t) \le \frac{t}{\mu} + \bar{F}_e(t) + \alpha_u F(t) - \sum_{m=1}^{n-1} (\xi_u * F^{*m})(t), \tag{3.12}$$

where $\xi_l(t) = \bar{F}_e(t) - \alpha_l \bar{F}(t) \ge 0$, $\xi_u(t) = \alpha_u \bar{F}(t) - \bar{F}_e(t) \ge 0$. Substituting $\xi_l(t)$ and $\xi_u(t)$ in the above relation and using that $\sum_{m=1}^{n-1} (\bar{F} * F^{*m})(t) = F(t) - F^{*n}(t)$, we get the following two-sided bound,

$$J_n(a_\ell, t) \le U(t) \le J_n(a_u, t), \quad \text{for every } n = 0, 1, 2, \ldots. \tag{3.13}$$

where

$$J_n(a, t) = \frac{t}{\mu} + \bar{F}_e(t) + aF^{*n}(t) + \sum_{m=1}^{n-1} (\bar{F}_e * F^{*m})(t).$$

Since $M(t) = U(t) + 1$, Marshall [20, Thm. 1] we obtain the following two-sided bounds.

$$I_n(b_\ell, t) \le U(t) \le I_n(b_u, t), \quad \text{for every } n = 0, 1, 2, \ldots.$$

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with

$$I_n(b,t) = 1 + \frac{t}{\mu} + bF^{*n}(t) - \sum_{m=1}^{n}(F_e * F^{*(m-1)})(t) + \sum_{m=1}^{n} F^{*m}(t)$$

and

$$b_l = \inf_{t\in A}\left\{\frac{F(t) - F_e(t)}{\bar{F}(t)}\right\}, \quad b_u = \sup_{t\in A}\left\{\frac{F(t) - F_e(t)}{\bar{F}(t)}\right\}.$$

Since it can be easily shown that $I_n(a-1,t) = J_n(a,t)$ and since $b_l = a_l - 1,\ b_u = a_u - 1$, it follows that the lower (upper) bounds in (3.13) is exactly the aforementioned lower (upper) bounds obtained by Marshall [20] for every $n = 0, 1, 2, \ldots$

Marshall [20] using mathematical induction in n, proved that both the upper and the lower bounds $I_n(b_l,t)$ and $I_n(b_u,t)$ converge to $U(t)$. This follows immediately from (3.13), since

$$\sum_{m=1}^{\infty}(\bar{F}_e * F^{*m})(t) = U(t) - \frac{t}{\mu} - \bar{F}_e(t) \quad \text{and} \quad \lim_{n\to\infty} F^{*n}(t) = 0,$$

implying that $\lim_{n\to\infty} J_n(\alpha,t) = U(t)$ for every real α. Thus, it holds that

$$\lim_{n\to\infty} J_n(\alpha_l,t) = \lim_{n\to\infty} J_n(\alpha_u,t) = U(t).$$

It deserves mentioning that a disadvantage of the upper bound obtained by Marshall [20] is that α_u could be infinite and thus no linear upper bound exist in such a case. For example, this could be happen if the df F has an increasing mean residual lifetime. Of course, there is no problem even in this case with the function $\alpha_U(t)$.

Example 3. In this example, we consider that the r.v. X follows the two-parametric Pareto distribution having df

$$F(x) = 1 - \left(\frac{\lambda}{\lambda+t}\right)^{\alpha}, \quad t \geq 0,\ \lambda > 0,\ a > 1.$$

Obviously, the distribution has an increasing mean residual lifetime. Since

$$\bar{F}_e(x) = \left(\frac{\lambda}{\lambda+t}\right)^{\alpha-1}, \quad t \geq 0.$$

It follows that

$$a_U(t) = \sup_{\substack{0\leq x\leq t\\ \bar{F}(x)>0}}\left\{\frac{\bar{F}_e(x)}{\bar{F}(x)}\right\} = \sup_{0\leq x\leq t}\left\{1 + \frac{x}{\lambda}\right\} = 1 + \frac{t}{\lambda},$$

$$a_u = \sup_{\substack{t\geq 0\\ \bar{F}(t)>0}} \frac{\bar{F}_e(t)}{\bar{F}(t)} = \sup_{t\geq 0}\left\{1 + \frac{t}{\lambda}\right\} = \infty,$$

$$a_L(t) = \inf_{\substack{0\leq x\leq t\\ \bar{F}(x)>0}} \frac{\bar{F}_e(x)}{\bar{F}(x)} = \inf_{0\leq x\leq t}\left\{1 + \frac{x}{\lambda}\right\} = 1,$$

and

$$a_\ell = \inf_{\substack{t\geq 0\\ \bar{F}(t)>0}} \frac{\bar{F}_e(t)}{\bar{F}(t)} = \inf_{t\geq 0}\left\{1 + \frac{t}{\lambda}\right\} = 1.$$

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Therefore, since, $a_\ell = a_L(t) = 1,\ t \geq 0$, the lower bounds given in (3.3), (3.5) and (3.6) are the same. Comparing the upper bounds we observe that Marshall's [20] upper bound given in (3.2) and (3.3) becomes infinity while our new upper bound in (3.6) is finite. Of course, the same holds true for the corresponding Waldmann's [26] upper bound given in (3.5).

Now, under a monotonicity condition for the function $\xi_U(t)$ ($\xi_L(t)$), we shall give another upper (lower) bounds for $U(t)$ which are tighter than the corresponding bounds obtained by Waldmann [26] and given in Proposition 3.1. Thus, we have the following

Theorem 3.3. *Let $\alpha_U(t)$, $\alpha_L(t)$ are given by (3.4), and $\xi_U(t)$, $\xi_L(t)$ are given by (3.10).*

(i) If $\xi'_U(t) \leq 0$, then

$$U(t) \leq \frac{\frac{t}{\mu} + \alpha_U(t)\bar{F}(t) + (\alpha_U * F)(t)}{1 + \xi_U(t)} \tag{3.14}$$

$$\leq \frac{\frac{t}{\mu} + \alpha_U(t)}{1 + \xi_U(t)}. \tag{3.15}$$

(ii) If $\xi'_L(t) \leq 0$, then

$$U(t) \geq \frac{\frac{t}{\mu} + \alpha_L(t)\bar{F}(t) + (\alpha_L * F)(t)}{1 - \xi_L(t)} \tag{3.16}$$

$$\geq \frac{\frac{t}{\mu} + \alpha_L(t)}{1 - \xi_L(t)} \tag{3.17}$$

Proof. (i) Since $\alpha_U(t)$ and $-\xi_U(t)$ are non-decreasing functions in $t \geq 0$, for $0 \leq y \leq t$, it follows that $\alpha_U(t-y) \leq \alpha_U(t)$ and $-\xi_U(t-y) \leq -\xi_U(t)$, and hence from the general solution for $U(t)$, we have that

$$\begin{aligned}
U(t) &= \frac{t}{\mu} + \bar{F}_e(t) + \sum_{n=1}^{\infty} \int_0^t \bar{F}_e(t-y)\, dF^{*n}(y) \\
&= \frac{t}{\mu} + \bar{F}_e(t) + \sum_{n=1}^{\infty} \int_0^t [\alpha_U(t-y)\bar{F}(t-y) - \xi_U(t-y)] dF^{*n}(y) \\
&\leq \frac{t}{\mu} + \bar{F}_e(t) + \alpha_U(t) \sum_{n=1}^{\infty} \int_0^t \bar{F}(t-y)\, dF^{*n}(y) - \xi_U(t) \sum_{n=1}^{\infty} \int_0^t dF^{*n}(y), \\
&= \frac{t}{\mu} + \bar{F}_e(t) + \alpha_U(t)F(t) - \xi_U(t)[U(t) - 1] \\
&= \frac{t}{\mu} + \alpha_U(t) - \xi_U(t)U(t),
\end{aligned}$$

from which (3.14) follows directly.

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Now, by inserting the upper bound of the above relation into the right-hand side of (1.3), we obtain

$$\begin{aligned} U(t) &\le 1 + \int_0^t \left\{ \frac{t-y}{\mu} + \alpha_U(t-y) - \xi_U(t-y)U(t-y) \right\} dF(y) \\ &\le 1 + \frac{t}{\mu} - F_e(t) + (\alpha_U * F)(t) - \xi_U(t) \int_0^t U(t-y)\, dF(y) \\ &= \frac{t}{\mu} + \alpha_U(t)\bar{F}(t) - \xi_U(t) + (\alpha_U * F)(t) - \xi_U(t)[U(t) - 1], \end{aligned}$$

which gives the upper bound in (3.11).

Since, $\alpha_U(t)\bar{F}(t) + (\alpha_U * F)(t) \le \alpha_U(t)\bar{F}(t) + \alpha_U(t)F(t) = \alpha_U(t)$, it follows that the bound in (3.14) is tighter than that given in (3.15).

In order to show that the bound in (3.14) is a refinement of the upper bound given in (3.6), it suffices to show that

$$\xi_U(t) \left\{ \frac{t}{\mu} + \bar{F}_e(t) + (\alpha_U * F)(t) - 1 \right\} \ge 0.$$

This inequality holds true because of the upper bound in (3.6) and the fact that $\xi_U(t) \ge 0$, $U(t) \ge U(0) = 1$ for $t \ge 0$. Similarly, in order to show that the bound in (3.15) is a refinement of the upper bound given in (3.5), it suffices to show that

$$\xi_U(t) \left\{ \frac{t}{\mu} + \bar{F}_e(t) + \alpha_U(t)F)(t) - 1 \right\} \ge 0.$$

This inequality holds true because of the upper bound in (3.5) and the fact that $\xi_U(t) \ge 0$, $U(t) \ge U(0) = 1$ for $t \ge 0$.

(ii) The proof is similar as in (i). □

The bound in (3.14) is a refinement of the upper bounds given in (3.6) and (3.5), and the bound in (3.15) is a refinement of the upper bound given in (3.5). Also, the bound in (3.16) is a refinement of the lower bounds given in (3.6) and 3.5), and the bound in (3.17) is a refinement of the lower bound given in (3.5).

4. Bounds based on reliability properties of the inter-arrival times

In this section, we obtain bounds for the renewal function $U(t)$ based on some reliability properties of the df F.

4.1. The class of distributions with bounded mean residual lifetime and the NBUE (NWUE) class

Consider the residual lifetime random variable $T_t = X - t/X > t$ for $X > t$, and T_t is undefined otherwise. Then, the right-tail of T_t is

$$\Pr(T_t > y) = \frac{\bar{F}(t+y)}{\bar{F}(t)}, \quad y \ge 0,$$

and the expected value of T_t called the mean residual lifetime of the random variable X is given by

$$r_F(t) = E(T_t) = \int_0^\infty \Pr(T_t > y)\, dy,$$

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that is, it holds

$$r_F(t) = \frac{\int_t^\infty \bar{F}(y)\,dy}{\bar{F}(t)}.$$

Now, using Corollary 3.2, we obtain the following (for the proof, see Appendix)

Corollary 4.1. *(i) If for some $0 < r_1 < \infty$, it holds $r_F(t) \geq r_1$, then for every $n = 1, 2, 3, \ldots$,*

$$U(t) \geq \frac{t}{\mu} + \bar{F}_e(t) + \frac{r_1 F(t)}{\mu} + \sum_{m=1}^{n} (\xi_{1,L} * F^{*m})(t), \tag{4.1}$$

where $\xi_{1,L}(t) = \bar{F}_e(t) - \frac{r_1}{\mu}\bar{F}(t) \geq 0$.
The lower bound is monotone non-decreasing in $n \geq 1$ for any fixed t and converges to $U(t)$.

(ii) If for some $0 < r_2 < \infty$, it holds $r_F(t) \leq r_2$, then for every $n = 1, 2, 3, \ldots$,

$$U(t) \leq \frac{t}{\mu} + \bar{F}_e(t) + \frac{r_2 F(t)}{\mu} - \sum_{m=1}^{n} (\xi_{1,U} * F^{*m})(t), \tag{4.2}$$

where $\xi_{1,U}(t) = (r_2/\mu)\bar{F}(t) - \bar{F}_e(t) \geq 0$.
The upper bound is monotone non-increasing in $n \geq 1$ for any fixed t and converges to $U(t)$.

If for some r_1, r_2 such that $0 < r_1 < r_2 < \infty$ it holds $r_1 \leq r_F(t) \leq r_2$, then from (4.1) and (4.2), we obtain the following two-sided bound

$$\frac{t}{\mu} + \bar{F}_e(t) + \frac{r_1 F(t)}{\mu} + \sum_{m=1}^{n} (\xi_{1,L} * F^{*m})(t) \leq U(t) \leq \frac{t}{\mu} + \bar{F}_e(t) + \frac{r_2 F(t)}{\mu} - \sum_{m=1}^{n} (\xi_{1,U} * F^{*m})(t).$$

Also, using Theorem 3.3, we obtain the following (for the proof, see Appendix)

Corollary 4.2. *(i) If for some $0 < r_1 < \infty$, it holds $\mu_F(t) \leq 1/r_1$, then*

$$U(t) \geq \frac{t + r_1}{r_1 \bar{F}(t) + \mu F_e(t)}. \tag{4.3}$$

(ii) If for some $0 < r_2 < \infty$, it holds $\mu_F(t) \geq 1/r_2$, then

$$U(t) \leq \frac{t + r_2}{r_2 \bar{F}(t) + \mu F_e(t)}. \tag{4.4}$$

Note that, in this case, the upper (lower) bounds in (3.14) and (3.15) (in (3.16) and (3.17)) are the same, since $a_U(t)(a_L(t))$ is independent of t.

An important class of non-parametric distributions is the new better (worse) than used in expectation or NBUE (NWUE) class which is a subclass of the class of distributions with bounded mean residual lifetime from above (below). The df F is NBUE (NWUE) if the mean residual lifetime $r_F(t)$ satisfies $r_F(t) \leq (\geq) r(0) = E(X)$, that is, if $r_F(t) \leq (\geq)\mu$. Clearly, from $r_F(t) = \mu \bar{F}_e(t)/\bar{F}(t)$, NBUE (NWUE) is equivalent to $\bar{F}_e(t) \leq (\geq) \bar{F}(t)$.

Barlow and Proschan [2, p. 171] proved that if the df F is NBUE (NWUE), then

$$U(t) \leq (\geq) 1 + \frac{t}{\mu}, \quad t \geq 0. \tag{4.5}$$

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Since by Jensen' s inequality it always holds $\mu_2 \geq \mu^2$, it follows that for the class of NBUE distributions, the upper bound of Barlow and Proschan in (4.5) is a refinement of Lorden's [18] upper bounds $U_L(t)$ given in Example 1.

Using now the previous Corollary 4.1, we get improved bounds than that given in (4.5). Thus, by setting $r_1 = r_2 = E(X) = \mu$ in Corollary 4.1, we obtain the following (for the proof, see Appendix)

Corollary 4.3. *If the df F is NWUE, then for every $n = 1, 2, 3, \ldots$,*

$$U(t) \geq \frac{t}{\mu} + \bar{F}_e(t) + F(t) + \sum_{m=1}^{n} (\xi_{2,L} * F^{*m})(t), \tag{4.6}$$

where $\xi_{2,L}(t) = \bar{F}_e(t) - \bar{F}(t) \geq 0$.

The lower bound is monotone non-decreasing in $n \geq 1$ for any fixed t and converges to $U(t)$.

(ii) If the df F is NBUE, then for every $n = 1, 2, 3, \ldots$,

$$U(t) \leq \frac{t}{\mu} + \bar{F}_e(t) + F(t) - \sum_{m=1}^{n} (\xi_{2,U} * F^{*m})(t), \tag{4.7}$$

where $\xi_{2,U}(t) = \bar{F}(t) - \bar{F}_e(t) \geq 0$. The upper bound is monotone non-increasing in $n \geq 1$ for any fixed t and converges to $U(t)$.

Note that (4.6) and (4.7) can also be obtained by using Theorem 2.2. Indeed, if $r(t) = F_e(t)$, then $Z(t) = t/\mu$ and $w(t) = F_e(t) - F(t) = \overline{F}(t) - \overline{F}_e(t)$. Since $w(t) \geq (\leq) 0$ if the df is NBUE(NWUE), then (4.6) and (4.7) follow directly from (2.4). Finally, in this subsection, we shall give another lower (upper) bound for $U(t)$ if the df F is NBUE (NWUE), which is the best one of all the aforementioned bounds. Thus, we have the following

Theorem 4.4. *(i) If the df F is NWUE, then for every $n = 1, 2, 3, \ldots$,*

$$U(t) \geq \frac{1 + t/\mu + \varepsilon_{n,L}(t)}{\bar{F}(t) + F_e(t)}, \tag{4.8}$$

for all t such that $\mu_F(t) \leq 1/\mu$, where

$$\varepsilon_{n,L}(t) = [F_e(t) - F(t)] \sum_{m=1}^{n} F^{*m}(t) - \sum_{m=1}^{n} (F_e * F^{*m})(t) + \sum_{m=1}^{n} F^{*(m+1)}(t) \geq 0.$$

The lower bound is monotone non-decreasing in $n \geq 1$ for any fixed t, converges to $U(t)$ and is a refinement of the bound in (4.6) for every $n = 1, 2, 3, \ldots$.

(ii) If the df F is NBUE, then for every $n = 1, 2, 3, \ldots$,

$$U(t) \leq \frac{1 + t/\mu - \varepsilon_{n,U}(t)}{\bar{F}(t) + F_e(t)}, \tag{4.9}$$

for all t such that $\mu_F(t) \geq 1/\mu$, where

$$\varepsilon_{n,U}(t) = \sum_{m=1}^{n} (F_e * F^{*m})(t) - \sum_{m=1}^{n} F^{*(m+1)}(t) - [F_e(t) - F(t)] \sum_{m=1}^{n} F^{*m}(t) \geq 0.$$

The upper bound is monotone non-increasing in $n \geq 1$ for any fixed t, converges to $U(t)$ and is a refinement of the bound in (4.7) for every $n = 1, 2, 3, \ldots$.

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Proof. (i) Let $w(t) = F_e(t) - F(t)$. Since the df $F$is NWUE, then $w(t) \leq 0$. Also, it holds

$$w'(t) = \bar{F}(t)\left(\frac{1}{\mu} - \mu_F(t)\right) \geq 0.$$

Consider now, the equilibrium renewal function satisfying (1.1) with $r(t) = F_e(t)$ and thus $Z(t) = t/\mu$. Then, since $w(t) \leq 0$ and $w'(t) \geq 0$, from Theorem 2.3 we obtain for every $n = 1, 2, 3, \ldots$

$$M(t) \geq \frac{1}{1 + w(t)}\left(\frac{t}{\mu} - w(t) - \varepsilon_n(t)\right), \tag{4.10}$$

where $-\varepsilon_n(t) = \sum_{m=1}^{n}\{(w * F^{*m})(t) - w(t)F^{*m}(t)\} \geq 0$ reduces to $\varepsilon_{n,L}(t) \geq 0$. Hence, (4.8) follows directly from (4.10).

Let $L_n(t)$ be the lower bound in (4.8). Since $\varepsilon_{n,L}(t) \geq 0$ for all $t \geq 0$, it follows that $L_n(t)$ is monotone non-decreasing in $n \geq 1$ for any fixed t. Also, since

$$\begin{aligned}
\varepsilon_{n,L}(t) &= [F_e(t) - F(t)]\sum_{m=1}^{n} F^{*m}(t) - \sum_{m=0}^{n}(F_e * F^{*m})(t) + F_e(t) + \sum_{m=1}^{n+1} F^{*m}(t) - F(t)\\
&= [\bar{F}(t) + F_e(t)]\sum_{m=1}^{n} F^{*m}(t) - \sum_{m=0}^{n}(F_e * F^{*m})(t) + F_e(t) + F^{*(n+1)}(t) - F(t)
\end{aligned}$$

we get

$$\lim_{n\to\infty} \varepsilon_{n,L}(t) = [\bar{F}(t) + F_e(t)]M(t) - \frac{t}{\mu} + F_e(t) - F(t)$$

implying that $\lim_{n\to\infty} L_n(t) = 1 + M(t) = U(t)$, that is, $L_n(t)$ converges to $U(t)$.

Since as stated previously, the bound in (4.6) is also obtained using relation (2.4) in Theorem 2.2, it follows from Theorem 2.3(ii), that the bound $L_n(t)$ is a refinement of the bound given in (4.6) for every $n = 1, 2, 3, \ldots$

(ii) The proof is similar as in (i) by reversing the inequalities. □

Since $\varepsilon_{n,U}(t) \geq 0$ and $\varepsilon_{n,L}(t) \geq 0$ for every $n \geq 0$ and for all $t \geq 0$, from Theorem 4.4 we obtain the following weaker but simpler lower (upper) bound for $U(t)$, namely

$$U(t) \geq (\leq)\frac{1 + t/\mu}{\bar{F}(t) + F_e(t)}, \tag{4.11}$$

for all t such that $\mu_F(t) \leq (\geq)1/\mu$, when the df F is NWUE (NBUE).

Obviously, the bound in (4.11) is also a refinement of the corresponding bound given by (4.5) obtained by Barlow and Proschan [2] for all t such that $\mu_F(t) \leq (\geq)1/\mu$. Note that the lower (upper) bound in (4.11) is also obtained from Corollary 4.2 with $r_1 = \mu$ ($r_2 = \mu$) if df F is NWUE (NBUE).

Example 4. (i) Let the r.v. X follows the two-parametric Pareto distribution with $\bar{F}(t) = \lambda^a/(\lambda + t)^a$, $a > 1$, $\lambda > 0$, $t \geq 0$. Then, the df F is NWUE and the condition $\mu_F(t) \leq \frac{1}{\mu}$ is equivalent to $t \geq \mu$. Therefore, for this distribution, the lower bound in (4.8) and (4.11) holds for all $t \geq \mu$.

Let us denote by $L_{\mathrm{BP}}(t)$ the lower bound obtained by Barlow and Prochan [2] given in (4.5), by $L_1(t)$, $t \geq 0$ the lower bound given in Corollary 4.3, by $L_2(t)$, $t \geq \mu$ the lower bound given by (4.11), $L_3(t)$, $t \geq \mu$ the lower bound given in Theorem 4.4 for $n = 1$ and by $L_M(t), t \geq 0$ the Marshall's [20] general lower bound given by (3.3). Note that since $a_L(t) = a_\ell = 1$, $t \geq 0$ (see Example 3) this bound is also equal to the lower bound obtained by Waldmann [26] and given in (3.5) as well as to the lower bound given in Proposition 3.1.

Table 3. *Numerical values for $L_{\mathrm{BP}}(t)$, $L_M(t)$, $L_1(t)$, $L_2(t)$ and $L_3(t)$ lower bounds.*

t	$L_{\mathrm{BP}}(t)$	$L_M(t)$	$L_1(t)$	$L_2(t)$	$L_3(t)$
1.	2.	2.25000	2.41667	2.66667	3.36505
2.	3.	3.22222	3.42689	3.85714	4.83261
3.	4.	4.18750	4.37924	4.92308	5.98737
4.	5.	5.16000	5.33046	5.95238	7.04434
5.	6.	6.13889	6.28931	6.96774	8.06612
6.	7.	7.12245	7.25583	7.97674	9.07369
7.	8.	8.10938	8.22861	8.98246	10.0751
8.	9.	9.09877	9.20625	9.98630	11.0737
9.	10.	10.0900	10.1877	10.9890	12.0711
10.	11.	11.0826	11.172	11.9910	13.0681
11.	12.	12.0764	12.1587	12.9925	14.0649
12.	13.	13.0710	13.1473	13.9936	15.0618
13.	14.	14.0663	14.1373	14.9945	16.0588
14.	15.	15.0622	15.1286	15.9953	17.056
15.	16.	16.0586	16.1209	16.9959	18.0534
16.	17.	17.0554	17.1141	17.9963	19.0510
17.	18.	18.0525	18.1080	18.9967	20.0487
18.	19.	19.0499	19.1025	19.9971	21.0466
19.	20.	20.0475	20.0975	20.9974	22.0447
20.	21.	21.0454	21.093	21.9976	23.0429
50.	51.	51.0192	51.0389	51.9996	53.019
100.	101.	101.010	101.020	102.000	103.010

Now, consider that $a = 2, \lambda = 1$, implying that $\mu = 1$ and let us compare numerically the aforementioned bounds for $1 \le t \le 100$. After some routine calculations, we find that

$$
\begin{aligned}
L_{\mathrm{BP}}(t) &= 1 + t, \quad t \ge 0,\\
L_M(t) &= t + \frac{1}{1+t} + 1 - \left(\frac{1}{1+t}\right)^2 = 1 + t + \frac{t}{(1+t)^2}, \quad t \ge 0,\\
L_1(t) &= \frac{t^6 + 9t^5 + 35t^4 + 76t^3 + 86t^2 + 46t + 8}{(t+1)^2(t+2)^3} + \frac{4(-1+t^2)\log(1+t)}{(1+t)(t+2)^4}, \quad t \ge 0,\\
L_2(t) &= \frac{(1+t)^3}{1 + t(1+t)}, \quad t \ge 1,\\
L_3(t) &= \frac{1}{t^2 + t + 1}\left((t+1)^2(t+2) - 1 - \frac{t^2(t+2)}{(t+1)^2} + \frac{t^4 + 7t^3 + 8t^2 + 2t}{(t+2)^3},\right.\\
&\quad \left. - \frac{4(t+1)^2(t+5)\log(1+t)}{(t+2)^4}\right), \quad t \ge 1,
\end{aligned}
$$

In Table 3, we present the values of the above lower bounds for various values of $1 \le t \le 100$.

We observe that for any $1 \le t \le 100$, it holds

$$L_3(t) > L_2(t) > L_1(t) > L_M(t) > L_{\mathrm{BP}}(t)$$

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and thus the lower bound $L_3(t)$ given in Theorem 4.4 is the best of all bounds $L_1(t), L_2(t), L_M(t)$ and $L_{\mathrm{BP}}(t)$, while the lower bound $L_{\mathrm{BP}}(t)$ is the worst. Also the simplest bound $L_2(t)$ is a refinement of the bounds $L_1(t), L_M(t)$ and $L_{\mathrm{BP}}(t)$.

(ii) Also, if the r.v. X follows the Weibull distribution with $\bar{F}(t) = e^{-ct^2}$, $c > 0$, $t \geq 0$, then the df F is NBUE and the condition $\mu_F(t) \geq 1/\mu$ is equivalent to $t \geq 1/2\sqrt{c\pi}$. Thus, the upper bounds in (4.9) and (4.11) hold for all $t \geq 1/2\sqrt{c\pi}$.

4.2. The class of distributions with bounded failure rate

Now, consider the class of absolutely continuous distribution functions F with failure rate $\mu_F(t) = f(t)/\bar{F}(t)$ bounded from below and/or above. Namely, suppose that for some $m_1, m_2 \in (0, \infty)$, it holds that $\mu_F(t) \geq m_1$ or $\mu_F(t) \leq m_2$ and/or $m_1 \leq \mu_F(t) \leq m_2$ for $m_2 > m_1$ applying Corollary 4.5 we obtain the following (for the proof, see Appendix).

Corollary 4.5. *Suppose that the df F is absolutely continuous with bounded failure rate $\mu_F(t) < \infty$.*

(i) If $\mu_F(t) \geq m_1$ for some $0 < m_1 < \infty$, then for every $n = 1, 2, 3, \ldots$

$$1 + m_1 t \leq U(t) \leq \frac{t}{\mu} + \bar{F}_e(t) + \frac{F(t)}{m_1\mu} - \sum_{m=1}^{n} (\xi_{3,U} * F^{*m})(t),$$

where $\xi_{3,U}(t) = \bar{F}(t)/m_1\mu - \bar{F}_e(t) \geq 0$.

The upper bound is monotone non-increasing in $n \geq 1$ for any fixed t and converges to $U(t)$.

(ii) If $\mu_F(t) \leq m_2$ for some $0 < m_2 < \infty$, then for every $n = 1, 2, 3, \ldots$

$$\frac{t}{\mu} + \bar{F}_e(t) + \frac{F(t)}{m_2\mu} + \sum_{m=1}^{n} (\zeta_{1,L} * F^{*m})(t) \leq U(t) \leq 1 + m_2 t,$$

where $\zeta_{1,L}(t) = \bar{F}_e(t) - \bar{F}(t)/m_2\mu \geq 0$.

The lower bound is monotone non-decreasing in $n \geq 1$ for any fixed t and converges to $U(t)$.

(iii) If $m_1 \leq \mu_F(t) \leq m_2$, for some m_1, m_2 such that $0 < m_1 < m_2 < \infty$, then for every $n = 1, 2, 3, \ldots$

$$A_L(t) \leq U(t) \leq A_U(t),$$

where

$$A_L(t) = \max\left\{1 + m_1 t,\ \frac{t}{\mu} + \bar{F}_e(t) + \frac{F(t)}{m_1\mu} + \sum_{m=1}^{n} (\xi_{3,L} * F^{*m})(t)\right\},$$

and

$$A_U(t) = \min\left\{1 + m_2 t,\ \frac{t}{\mu} + \bar{F}_e(t) + \frac{F(t)}{m_1\mu} - \sum_{m=1}^{n} (\xi_{3,U} * F^{*m})(t)\right\}.$$

The class of absolutely continuous distributions with bounded failure rate is a large one, in the sense that includes many families of distributions. The NBUFR (NWUFR) class is the special case with $m_1 = \mu_F(0)$ ($m_2 = \mu_F(0)$) and this also includes IFR, IFRA, NBU and NBU (2) (DFR, DFRA, NWU and NWU (2)) classes (e.g., [13]). Note that $\mu_F(0)$ always exists for an absolutely continuous df F.

Therefore, by setting $m_1 = f(0)$, from Corollary 4.5, we get a two-sided bound when the df F is IFR (i.e., the failure rate is a non-decreasing function), while for $m_2 = f(0)$, we get a two-sided bound when the df F is DFR (i.e., the failure rate is a non-increasing function).

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4.3. The IMRL (DMRL) class

A two-sided bound for the renewal function $U(t)$ if the df F has increasing mean residual lifetime (IMRL) was given by Brown [6, Thm. 2(i) and (iv)] who proved that

$$\frac{t}{\mu} + \bar{F}_e(t) + \frac{\mu_2}{2\mu^2}F_2(t) \le U(t) \le \frac{t}{\mu} + \frac{\mu_2}{2\mu^2}, \tag{4.12}$$

provided that $\mu_2 = E(X^2) < \infty$, where $F_2(t) = (2\mu/\mu_2)\int_0^t \bar{F}_e(y)\,dy$ is the equilibrium df of the random variable having df F_e. Using (4.12), we easily obtain the following

Corollary 4.6. *If the df F is IMRL, then for every $n = 0, 1, 2, \ldots$*

$$U(t) \le \frac{t}{\mu} + \sum_{m=0}^{n-1}(\bar{F}_e * F^{*m})(t) + \frac{\mu_2}{2\mu^2}F^{*n}(t), \tag{4.13}$$

and

$$U(t) \ge \frac{t}{\mu} + \sum_{m=0}^{n}(\bar{F}_e * F^{*m})(t) + \frac{\mu_2}{2\mu^2}(F_2 * F^{*n})(t). \tag{4.14}$$

Both the upper and lower bound converge to $U(t)$ as $\to \infty$.

When the df F has decreasing mean residual lifetime (DMRL), to the best of our knowledge there does not exist in the literature lower bounds for the renewal function $U(t)$. Of course, since the df F is also NBUE, one can use the upper bounds for the NBUE case. In the following corollary, we give two-sided bounds when the df F is DMRL (IMRL).

Corollary 4.7. *If the df F is DMRL (IMRL), then*

$$U(t) \ge (\le)\frac{t}{\mu} + \bar{F}_e(t) + \frac{1}{\mu}(r_F * F)(t), \tag{4.15}$$

Proof. Since the functions $\alpha_L(t)$ and $\alpha_U(t)$ can be rewritten as

$$\alpha_L(t) = \frac{1}{\mu}\inf_{\substack{0\le z\le t\\ \bar{F}(z)>0}}\{r_F(z)\} \quad \text{and} \quad \alpha_{\mathrm{U}}(\mathrm{t}) = \frac{1}{\mu}\sup_{\substack{0\le z\le t\\ \bar{F}(z)>0}}\{\mathrm{r_F(z)}\},$$

it follows that $\alpha_L(t) = r_F(t)/\mu$, if the df F is DMRL and $\alpha_U(t) = \frac{r_F(t)}{\mu}$ if the df is IMRL. Therefore, if the df F is DMRL, then

$$\xi_L(t) = \bar{F}_e(t) - \alpha_L(t)\bar{F}(t) = 0,$$

and

$$\xi_U(t) = \alpha_U(t)\bar{F}(t) - \bar{F}_e(t) = 0,$$

if the df F is IMRL. Hence, the result follows immediately from (3.11). □

We note that an upper (lower) bound for $U(t)$ when the df F is DMRL (IMRL) is given by (4.9) (4.8) for all t such that $\mu_F(t) \ge (\le) 1/\mu$, since F is also NBUE (NWUE).

Since $(r_F * F)(t) \ge (\le) r_F(t)F(t)$, if the df F is DMRL (IMRL), from Corollary 4.7, we get a weaker but simpler lower (upper) bound, given by

$$U(t) \ge (\le)\frac{t}{\mu} + \bar{F}_e(t) + \frac{1}{\mu}r_F(t)F(t),$$

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or equivalently

$$U(t) \geq (\leq) \frac{t}{\mu} + \frac{\bar{F}_e(t)}{\bar{F}(t)}. \tag{4.16}$$

Since the DMRL (IMRL) implies the NBUE (NWUE) property, and thus $\bar{F}_e(t) \leq (\geq) \bar{F}(t)$, it follows that the upper (lower) bound in (4.16) is a refinement of the corresponding upper (lower) bound in (4.5) obtained by Barlow and Proschan [2] when the df F is NBUE (NWUE).

4.4. The DFR (IFR) class

Supposing that the df F is absolutely continuous having pdf $f(t) = F'(t)$, then the failure rate of X is defined by $\mu_F(t) = f(t)/\bar{F}(t)$. In many situations of practical interest, the failure rate is a strictly monotone non-increasing function in t, and this is associated with the situation where the distribution of X has a thick right tail. The df F is said to be a decreasing failure rate or DFR, if $\bar{F}(y+t)/\bar{F}(t)$ is a non-decreasing function in t for fixed $y \geq 0$, that is, if $\bar{F}(t)$ is log-convex. From the definition of the failure rate function, it is evident that if the df F is absolutely continuous, then the DFR property is equivalent to $\mu_F(t)$ non-increasing in t. In this situation, the random variable X has a thick tail.

Before we proceed, we need the following Lemma ([8, Lemma 3]).

Lemma 4.8. *If $g_1(t)$ and $g_2(t)$ are differentiable functions with $g_1'(t)g_2'(t) \leq (\geq) 0$ on $[a, b]$, then*

$$\int_a^b g_1(t)g_2(t)\, dt \leq (\geq) \frac{1}{b-a} \int_a^b g_1(t)\, dt \int_a^b g_2(t)\, dt.$$

Let the df F is DFR. Then, it is well-known that the renewal density $u(t)$ is a decreasing function in $t \geq 0$ (see, e.g., [6]). Let $r(t) = \bar{F}(t)$. Then $Z(t) = 1$, and since $r'(t)u'(t) \geq 0$, from Lemma 4.8, we obtain

$$\bar{F}(t) + \frac{U(t) - 1}{t} \int_0^t \bar{F}(y)\, dy \geq 1,$$

yielding the following lower bound

$$U(t) \geq 1 + \frac{tF(t)}{\mu F_e(t)}. \tag{4.17}$$

Let us compare the bound given in (4.17) with some lower bounds stated previously.

- If the df F is DFR, then F is also NWUE, implying that $F(t) \geq F_e(t)$ and hence the bound in (4.17) is a refinement of the lower bound $1 + t/\mu$ obtained by Barlow and Proschan [2].
- From $\mu F_e(t) = \int_0^t \bar{F}(y)\, dy \geq t\bar{F}(t)$, it can be easily verified that the bound in (4.17) is also a refinement of Erickson' s lower bound $t\mu F_e(t)$ [12].
- Since F is also NWUE, from Corollary 4.3 it holds

$$U(t) \geq \frac{t}{\mu} + \bar{F}_e(t) + F(t), \tag{4.18}$$

which is a refinement of the aforementioned lower bound of Barlow and Proschan [2].
Since $\mu F_e(t) = \int_0^t \bar{F}(y)\, dy \leq \int_0^t dy = t$, it follows that the bound in (4.17) is tighter than that given in (4.18).
- However, from $\mu F_e(t) \geq t\bar{F}(t)$, it can be easily checked that the lower bound in (4.11) is a refinement of the bound given in (4.17), and thus, this is also true for the lower bound given in (4.8) of Theorem 4.4.

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Since the df F is DFR, is also IMRL, all the bounds stated previously for the IMRL and NWUE case are applicable when the df F is DFR. More precisely, all the bounds given in (4.6), (4.8), (4.11), (4.13), (4.14), (4.15) and (4.16) are applicable.

If the df F is IFR, Brown [7, Thm. 2.11] proved that

$$U(t) \geq \frac{t}{\mu} + \frac{\sigma^2}{\mu^2}, \quad \text{where } \sigma^2 = \mathrm{Var}(X).$$

By inserting this bound into the right-hand side of (1.3) and using mathematical induction in $n \geq 0$, we can easily obtain the following.

Corollary 4.9. *If the df F is IFR, then for every $n = 0, 1, 2, \ldots$*

$$U(t) \geq \frac{t}{\mu} + \sum_{m=0}^{n-1} (\bar{F}_e * F^{*m})(t) + \frac{\sigma^2}{\mu^2} F^{*n}(t).$$

The bound converges to $U(t)$ as $n \to \infty$.

Also, if the df F is IFR, then F is also DMRL and NBUE, implying that all the bounds given in (4.7), (4.9), (4.11), (4.15) and (4.16) are applicable when the df F is IFR.

5. Bounds for the renewal density

In this section, we assume that the df F is absolutely continuous and the failure rate $\mu_F(t) = f(t)/\bar{F}(t)$ exists. At first, we shall give an upper bound for the renewal density $u(t)$ in terms of the renewal function $M(t)$, using Lemma 4.8. Thus, we have the following.

Corollary 5.1. *(i) If the df F is DFR, then*

$$u(t) \leq f(t) + \frac{1}{t} F(t) M(t), \quad t > 0.$$

(ii) If the pdf $f(t)$ is increasing in $t \in [0, t_0]$, for $0 < t_0 \leq \infty$, then

$$u(t) \leq f(t) + \frac{1}{t} F(t) M(t), \quad 0 < t \leq t_0.$$

Proof. (i) It is well known, that if the pdf F is DFR, then the $u(t)$ is a decreasing function in $t \geq 0$. Also since $d\mu_F(t)/dt \leq 0$, it holds that $f'(t) \leq 0$ implying that $f'(t)u'(t) \geq 0$. Hence, the direct application of Lemma 4.8 to (1.4) yields the the required bound.

(ii) Since the pdf $f(t)$ is increasing in $t \in [0, t_0]$, it follows that $u(t)$ is also increasing in the same interval [28, Thm. 4.8]. Now, since $f'(t)u'(t) \geq 0$ as in (i) we get the following required bound. □

Now, we shall give a general two-sided bound for $u(t)$ by applying Theorem 2.4. Since for $r(t) = f(t)$, the function $Z(t)$ satisfying (1.1) is reduced to $u(t)$ satisfying (1.4), the direct application of Theorem 2.4 gives us the following sequence of general two-sided bounds for $u(t)$

Corollary 5.2. *Let*

$$\sigma_{1,U}(t) = \sup_{\substack{0 \leq z \leq x \\ \bar{F}(z) > 0}} \{\mu_F(z) > 0\}, \quad \sigma_{1,L}(t) = \inf_{\substack{0 \leq z \leq x \\ \bar{F}(z)}} \{\mu_F(z)\}, \tag{5.1}$$

and $\psi_{1,U}(t) = \bar{F}(t)[\sigma_{1,U}(t) - \mu_F(t)] \geq 0$, $\psi_{1,L}(t) = \bar{F}(t)[\mu_F(t) - \sigma_{1,L}(t)] \geq 0$.

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Then, for every $n = 1, 2, 3, \ldots$ *and* $t \geq 0$, *holds:*

$$(i) \quad u(t) \leq f(t) + (\sigma_{1,U} * F)(t) - \sum_{m=1}^{n} (\psi_{1,U} * F^{*m})(t) \tag{5.2}$$

$$\leq f(t) + \sigma_{1,U}(t)F(t) - \sum_{m=1}^{n-1} (\psi_{1,U} * F^{*m})(t). \tag{5.3}$$

$$(ii) \quad u(t) \geq f(t) + (\sigma_{1,L} * F)(t) + \sum_{m=1}^{n} (\psi_{1,L} * F^{*m})(t) \tag{5.4}$$

$$\geq f(t) + \sigma_{1,L}(t)F(t) + \sum_{m=1}^{n-1} (\psi_{1,L} * F^{*m})(t). \tag{5.5}$$

Proof. (i) If we apply Theorem 2.4 to the renewal density $u(t)$ satisfying (1.4), with $r(t) = f(t)$, then $\sigma_U(t)$, $\sigma_L(t)$, defined by (2.6), $\psi_U(t)$ and $\psi_L(t)$ defined by (2.7) are reduced to $\sigma_{1,U}(t)$, $\sigma_{1,L}(t)$, $\psi_{1,U}(t)$ and $\psi_{1,L}(t)$, respectively, and thus (5.2) follows directly from (2.8). The right-hand side of (2.9) is reduced to

$$\sigma_{1,U}(t) - \sum_{m=1}^{n} (\psi_{1,U} * F^{*(m-1)})(t) = \sigma_{1,U}(t) - \psi_{1,U}(t) - \sum_{m=2}^{n} (\psi_{1,U} * F^{*(m-1)})(t)$$

$$= f(t) + \sigma_{1,U}(t)F(t) - \sum_{m=1}^{n-1} (\psi_{1,U} * F^{*m})(t),$$

giving the upper bound in (5.3).

(ii) Follows in a similar way as in (i). □

Xie [28, Cor. 4.3] proved that

$$\sigma_{1,L}(t) \leq u(t) \leq \sigma_{1,U}(t), \quad t \geq 0. \tag{5.6}$$

Since for all $t \geq 0$, $\sigma_{1,U}(t) \geq \mu_F(t)$ and $\sigma_{1,L}(t) \leq \mu_F(t)$, it follows that

$$f(t) + \sigma_{1,U}(t)F(t) \leq \sigma_{1,U}(t) \quad \text{and} \quad f(t) + \sigma_{1,L}(t)F(t) \geq \sigma_{1,L}(t),$$

implying that the two-sided bounds given in Corollary 5.2 for every $n = 1, 2, 3, \ldots$, are better than that of Xie' s two-sided bound given in (5.6).

An immediate consequence of Corollary 5.2 are the following corollaries.

Corollary 5.3. *(i) If the df F is IFR, then for all* $t \geq 0$

$$u(t) \leq f(t) + (\mu_F * F)(t). \tag{5.7}$$

(ii) If the df F is DFR, then for all $t \geq 0$

$$u(t) \geq f(t) + (\mu_F * F)(t). \tag{5.8}$$

Proof. (i) If the df F is IFR, then $\sigma_{1,U}(t)$ and $\psi_{1,U}(t)$ defined by Corollary 5.2 are equal to $\sigma_{1,U}(t) = \mu_F(t)$ and $\psi_{1,U}(t) = 0$. Then, (5.7) follows from (5.2).

(ii) If the df F is DFR, then $\sigma_{1,L}(t)$ defined by (5.1) and $\psi_{1,L}(t)$ defined by (5.2) are equal to $\sigma_{1,L}(t) = \mu_F(t)$ and $\psi_{1,L}(t) = 0$. Then, (5.8) follows from (5.4). □

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Xie [28, Cor. 4.4] proved that if the df F is IFR(DFR), then

$$u(t) \leq (\geq)\mu_F(t).$$

Since

$$f(t) + (\mu_F * F)(t) \leq f(t) + \mu_F(t)F(t) = \mu_F(t),$$

if the df F is IFR, and

$$f(t) + (\mu_F * F)(t) \geq f(t) + \mu_F(t)F(t) = \mu_F(t),$$

if the df F is DFR, it follows that the bounds in Corollary 5.3 are tighter than the corresponding bounds given by Xie [28].

Combining (5.8) and Corollary 5.1 (i), we immediately obtain the following

Corollary 5.4. *If the df F is DFR, then*

$$U(t) \geq 1 + \frac{t(\mu_F * F)(t)}{F(t)}.$$

In the following corollary (for the proof, see Appendix), we give bounds for the renewal density function by considering the class of absolutely continuous distribution functions F with bounded failure rate.

Corollary 5.5. *(i) If for some $m_1 \in (0,\infty)$, it holds $\mu_F(t) \geq m_1$, $t \geq 0$, then for every $n = 1, 2, 3, \ldots$*

$$u(t) \geq f(t) + m_1 F(t) + \sum_{m=1}^{n-1} (\hat{\psi}_{1,L} * F^{*m})(t),$$

where $\hat{\psi}_{1,L}(t) = \bar{F}(t)[\mu_F(t) - m_1] \geq 0$.

The lower bound is monotone non-decreasing in $n \geq 1$ for any fixed t and converges to $u(t)$.

(ii) If for some $m_2 \in (0,\infty)$, it holds $\mu_F(t) \leq m_2$, $t \geq 0$, then for every $n = 1, 2, 3 \ldots$

$$u(t) \leq f(t) + m_2 F(t) + \sum_{m=1}^{n-1} (\hat{\psi}_{1,U} * F^{*m})(t),$$

where $\hat{\psi}_{1,U}(t) = \bar{F}(t)[m_2 - \mu_F(t)] \geq 0$.

The upper bound is monotone non-increasing in $n \geq 1$ for any fixed t and converges to $u(t)$.

(iii) If for some $m_1, m_2 \in (0,\infty)$ with $m_1 < m_2$, it holds $m_1 \leq \mu_F(t) \leq m_2$, $t \geq 0$, then for every $n = 1, 2, 3, \ldots$

$$f(t) + m_1 F(t) + \sum_{m=1}^{n-1} (\hat{\psi}_{1,L} * F^{*m})(t) \leq u(t) \leq f(t) + m_1 F(t) - \sum_{m=1}^{n-1} (\hat{\psi}_{1,U} * F^{*m})(t).$$

Let

$$\sigma_1 = \inf_{z\geq 0}\{\mu_F(z)\} \quad \text{and} \quad \sigma_2 = \sup_{z\geq 0}\{\mu_F(z)\}. \tag{5.9}$$

Then, Xie [28, Cor. 4.7], using the two-sided bound for the renewal density $u(t)$ given in (5.6), obtained the following two-sided linear bound for the renewal function $H(t)$

$$\sigma_1 t \leq M(t) \leq \sigma_2 t,$$

and conjectured that this two-sided bound may be improved. An improvement is given in the next section.

Corollary 5.6. *If σ_1 and σ_2 are given by (5.9), then it holds the following general two-sided bound*

$$\sigma_1 t + [F(t) - \sigma_1 \mu F_e(t)] \leq M(t) \leq \sigma_2 t - [\sigma_2 \mu F_e(t) - F(t)].$$

Proof. Since $\sigma_{1,U}(t) \leq \sigma_2$ and $\sigma_{1,L}(t) \geq \sigma_1$, where $\sigma_{1,U}(t)$ and $\sigma_{1,L}(t)$ are given in (5.9), then from Corollary 5.2 and for $n = 1$, we get the following two-sided bound

$$f(t) + \sigma_1 F(t) \leq u(t) \leq f(t) + \sigma_2 F(t).$$

Therefore, it holds

$$F(t) + \sigma_1 \int_0^t F(y)\, dy \leq H(t) \leq F(t) + \sigma_2 \int_0^t F(y)\, dy,$$

and since $\int_0^t F(y)\, dy = t - \mu F_e(t)$, the result follows. □

Since, $f(t) \geq \sigma_1 \bar{F}(t)$ and $f(t) \leq \sigma_2 \bar{F}(t)$, it follows that

$$F(t) \geq \sigma_1 \int_0^t \bar{F}(y)\, dy = \sigma_1 \mu F_e(t) \quad \text{and} \quad F(t) \leq \sigma_2 \int_0^t \bar{F}(y)\, dy = \sigma_2 \mu F_e(t),$$

implying that the two-sided bound in Corollary 5.6 is tighter than the above mentioned Xie' s two-sided bound.

6. Some bounds for the distribution of the forward recurrence time

In this section, we shall give some bounds for the right-tail of the excess lifetime $\gamma(t)$. Let $\bar{V}_y(t) = \Pr[\gamma(t) > y]$. Since the solution of (1.5) is given by

$$V_y(t) = F(t+y) - \int_0^t \bar{F}(t+y-x)\, dM(x), \tag{6.1}$$

and $\bar{V}_y(t) = 1 - V_y(t)$, from (1.5) and (6.1), it follows that the right-tail $\bar{V}_y(t)$ satisfies the renewal-type equation

$$\bar{V}_y(t) = \bar{F}(y+t) + \int_0^t \bar{V}_y(t-x)\, dF(x), \tag{6.2}$$

with solution (in terms of the renewal function $M(t)$)

$$\bar{V}_y(t) = \bar{F}(y+t) + \int_0^t \bar{F}(y+t-x)\, dM(x). \tag{6.3}$$

Let $(\bar{F}(y+\cdot) * F)(t) = \int_0^t \bar{F}(y+t-x)\, dF(x)$. Note that the general solution of (6.2) is given by

$$\bar{V}_y(t) = \sum_{m=0}^{\infty} (\bar{F}(y+\cdot) * F^{*m})(t) = \sum_{m=0}^{\infty} \int_0^t \bar{F}(y+t-x)\, dF^{*m}(x). \tag{6.4}$$

Since $\bar{V}_y(t)$ is of the form (1.1) with $r(t) = \bar{F}(y+t)$, then from Theorem 2.4 we immediately obtain the following sequence of general two-sided bounds for $\bar{V}_y(t)$.

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Corollary 6.1. *Let*

$$\beta_U(t) = \sup_{\substack{0\le z\le t\\ \bar{F}(z)>0}} \left\{\frac{\bar{F}(y+z)}{\bar{F}(z)}\right\} \quad and \quad \beta_L(t) = \inf_{\substack{0\le z\le t\\ \bar{F}(z)>0}} \left\{\frac{\bar{F}(y+z)}{\bar{F}(z)}\right\}. \tag{6.5}$$

Then, for every $n = 1, 2, 3, \ldots$, it holds the following general two-sided bounds

$$\bar{V}_y(t) \le \bar{F}(y+t) + (\beta_U * F)(t) - \sum_{m=1}^{n} (g_{y,U} * F^{*m})(t) \tag{6.6}$$

$$\le \beta_U(t) - \sum_{m=1}^{n} (g_{y,U} * F^{*(m-1)})(t), \tag{6.7}$$

and

$$\bar{V}_y(t) \ge \bar{F}(y+t) + (\beta_L * F)(t) + \sum_{m=1}^{n} (g_{y,L} * F^{*m})(t) \tag{6.8}$$

$$\ge \beta_U(t) + \sum_{m=1}^{n} (g_{y,L} * F^{*(m-1)})(t), \tag{6.9}$$

where

$$g_{y,L}(t) = \bar{F}(y+t) - \beta_L(t)\bar{F}(t) \ge 0, \quad g_{y,U}(t) = \beta_U(t)\bar{F}(t) - \bar{F}(y+t) \ge 0. \tag{6.10}$$

Now, using Corollary 6.1, we can easily get bounds for $\bar{V}_y(t)$ based on some reliability properties of the df F. Thus, we have

Corollary 6.2. *(i) If the df F is IFR, then for every $n = 1, 2, \ldots$*

$$\frac{\bar{F}(y+t)}{\bar{F}(t)} \le \bar{V}_y(t) \le \bar{F}(y+t) + \bar{F}(y)F(t) - \sum_{m=1}^{n} (g_{1,y} * F^{*m})(t) \tag{6.11}$$

$$\le \bar{F}(y) - \sum_{m=1}^{n} (g_{1,y} * F^{*(m-1)})(t), \tag{6.12}$$

where $g_{1,y}(t) = \bar{F}(y)\bar{F}(t) - \bar{F}(y+t) \ge 0$.
The upper bounds are monotone non-increasing in $n \ge 1$ for any fixed t and converge to $\bar{V}_y(t)$.

(ii) If the F is DFR, then for every $n = 1, 2, 3, \ldots$

$$\frac{\bar{F}(y+t)}{\bar{F}(t)} \ge \bar{V}_y(t) \ge \bar{F}(y+t) + \bar{F}(y)F(t) + \sum_{m=1}^{n} (g_{2,y} * F^{*m})(t) \tag{6.13}$$

$$\ge \bar{F}(y) + \sum_{m=1}^{n} (g_{2,y} * F^{*(m-1)})(t), \tag{6.14}$$

where $g_{2,y}(t) = \bar{F}(y+t) - \bar{F}(y)\bar{F}(t) \ge 0$.
The lower bounds are monotone non-decreasing in $n \ge 1$ for any fixed t and converge to $\bar{V}_y(t)$.

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Proof. (i) Since the df F is IFR, then $\bar{F}(y+z)/\bar{F}(z)$ is a non-increasing function in z for fixed $y \geq 0$. Then, from (6.5), it follows that $\beta_U(t) = \bar{F}(y)$, $\beta_L(t) = \bar{F}(y+t)/\bar{F}(t)$, implying from (6.10), that $g_{y,L}(t) = 0$ and $g_{y,U}(t) = \bar{F}(y)\bar{F}(t) - \bar{F}(y+t) = g_{1,y}(t)$, which is non-negative for all y, $t \geq 0$, since the df F is also NBU. Therefore, from (6.6) and (6.7), we immediately obtain (6.11) and (6.12), respectively.

Let $U_{n,y}(t)$ and $\hat{U}_{n,y}(t)$ be the upper bounds in (6.11) and (6.12), respectively. Since $g_{1,y}(t)$ is a non-negative function for all $t \geq 0$, it follows that $U_{n,y}(t)$ and $\hat{U}_{n,y}(t)$ are monotone non-increasing in $n \geq 1$ for any fixed $t \geq 0$. By definition of $g_{1,y}(t)$, the upper bound $U_{n,y}(t)$ can be rewritten as

$$\begin{aligned} U_{n,y}(t) &= \bar{F}(y+t) + \bar{F}(y)F(t) - \bar{F}(y)\sum_{m=1}^{n}(\bar{F} * F^{*m})(t) + \sum_{m=1}^{n}(\bar{F}(y+\cdot) * F^{*m})(t) \\ &= \bar{F}(y) + \bar{F}(y)F(t) - \bar{F}(y)[\bar{F}^{*(n+1)}(t) - \bar{F}(t)] + \sum_{m=0}^{n}(\bar{F}(y+\cdot) * F^{*m})(t) \\ &= \bar{F}(y) - \bar{F}(y)\bar{F}^{*(n+1)}(t) + \sum_{m=0}^{n}(\bar{F}(y+\cdot) * F^{*m})(t), \end{aligned}$$

and thus from (6.4) and using that $\lim_{n\to\infty} \bar{F}^{*(n+1)}(t) = 1$, we get that $\lim_{n\to\infty} U_{n,y}(t) = \bar{V}_y(t)$, implying that $U_{n,y}(t)$ converges to $\bar{V}_y(t)$ as $n \longrightarrow \infty$.

Also,

$$\begin{aligned} \hat{U}_{n,y}(t) &= \bar{F}(y) - \bar{F}(y)\sum_{m=1}^{n}(\bar{F} * F^{*(m-1)})(t) + \sum_{m=1}^{n}(\bar{F}(y+\cdot) * F^{*(m-1)})(t) \\ &= \bar{F}(y) - \bar{F}(y)\bar{F}^{*n}(t) + \sum_{m=0}^{n-1}(\bar{F}(y+\cdot) * F^{*m})(t), \end{aligned}$$

from which it follows that $\lim_{n\to\infty} U_{n,y}(t) = \bar{V}_y(t)$. Therefore, $U_{n,y}(t)$ converges to $\bar{V}_y(t)$ as $n \longrightarrow \infty$.

(ii) When the df F is IFR, then $\beta_L(t) = \bar{F}(y)$, $\beta_U(t) = \bar{F}(y+t)/\bar{F}(t)$, and the bounds in (6.13) and (6.14) follows directly from (6.8) and (6.9), respectively. The rest of the proof is exactly the same as in (i). □

When the df F is DFR, we can also obtain and some other upper bounds for $\bar{V}_y(t)$ in terms of the renewal function $M(t)$.

Corollary 6.3. *If the df F is DFR, then for every* $n = 1, 2, 3, \ldots$

$$\bar{V}_y(t) \leq F(t) + \bar{F}(y+t) - [\bar{F}(t) + F(y+t)]M(t) - \varepsilon_n(t) \tag{6.15}$$

where

$$\begin{aligned} \varepsilon_n(t) = \sum_{m=1}^{n} F^{*(m+1)}(t) - \sum_{m=1}^{n}\int_0^t \bar{F}(y+t-x)\,dF^{*m}(x) \\ - \sum_{m=1}^{n}[F(y+t) - F(t)]F^{*m}(t) \geq 0 \end{aligned}$$

and

$$\bar{V}_y(t) \leq \bar{F}(y+t) + \frac{\mu}{t}[\bar{F}_e(y) - \bar{F}_e(y+t)]M(t). \tag{6.16}$$

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Proof. Let $w(t) = F(y+t) - F(t) \geq 0$. Then, $w'(t) = f(t+y) - f(t) \leq 0$, since the pdf f is a non-increasing function. Then, for $r(t) = F(y+t)$, from Theorem 2.3, we obtain

$$V_y(t) \geq H(t) + [F(y+t) - F(t)][1 + M(t)] + \varepsilon_n(t),$$

with $\varepsilon_n(t) = \sum_{m=1}^{n} \{(w * F^{*m})(t) - w(t)F^{*m}(t)\} \geq 0$, which gives (6.15).

Since the function $\bar{F}(y+t-x)$ is non-decreasing in x, for $0 \leq x \leq t$, and the renewal density $u(t)$ is a non-increasing function in $t \geq 0$, from (6.3) and Lemma 4.8, we obtain

$$\begin{aligned}
\bar{V}_y(t) &= \bar{F}(y+t) + \int_0^t \bar{F}(y+t-x)u(x)\,dx \\
&\leq \bar{F}(y+t) + \frac{1}{t}\int_0^t \bar{F}(y+x)\,dx \int_0^t u(x)\,dx \\
&= \bar{F}(y+t) + \frac{1}{t}\left\{\int_0^{y+t} \bar{F}(x)\,dx - \int_0^y \bar{F}(x)\,dx\right\} M(t),
\end{aligned}$$

and thus (6.16) follows. □

Corollary 6.4. *(i) If for some $m_1 \in (0,\infty)$, it holds $\mu_F(t) \geq m_1$, $t \geq 0$, then for every $n = 1, 2, 3, \ldots$*

$$\bar{V}_y(t) \leq \bar{F}(y+t) + e^{-m_1 y}F(t) - \sum_{m=1}^{n}(g_{3,y} * F^{*m})(t) \tag{6.17}$$

$$\leq e^{-m_1 y} - \sum_{m=1}^{n}(g_{3,y} * F^{*(m-1)})(t), \tag{6.18}$$

where $g_{3,y}(t) = \bar{F}(t)e^{-m_1 y} - \bar{F}(t+y) \geq 0$

The upper bounds are monotone non-increasing in $n \geq 1$ for any fixed t and converge to $\bar{V}_y(t)$.

(ii) If for some $m_2 \in (0,\infty)$, it holds $\mu_F(t) \leq m_2$, $t \geq 0$, then for every $n = 1, 2, 3, \ldots$

$$\bar{V}_y(t) \geq \bar{F}(y+t) + e^{-m_1 y}F(t) + \sum_{m=1}^{n}(g_{4,y} * F^{*m})(t) \tag{6.19}$$

$$\geq e^{-m_2 y} + \sum_{m=1}^{n}(g_{4,y} * F^{*(m-1)})(t), \tag{6.20}$$

where $g_{4,y}(t) = \bar{F}(t+y) - \bar{F}(t)e^{-m_2 y} \geq 0$.

The lower bounds are monotone non-decreasing in $n \geq 1$ for any fixed t and converge to $\bar{V}_y(t)$.

Proof. (i) Since

$$\frac{\bar{F}(y+z)}{\bar{F}(z)} = \exp\left\{-\int_z^{z+y} \mu_F(x)\,dx\right\},$$

then

$$\frac{\bar{F}(y+z)}{\bar{F}(z)} \leq \exp\left\{-\int_z^{z+y} m_1 dx\right\} = e^{-m_1 y},$$

and thus, from (6.5) and (6.10), we get $\beta_U(t) = e^{-m_1 y}$ and $g_{y,U}(t) = \bar{F}(t)e^{-m_1 y} - \bar{F}(y+t) = g_{3,y}(t)$. Now, the bounds in (6.17) and (6.18) follow immediately from (6.6) and (6.7), respectively.

Let $A_{n,y}(t)$ and $\hat{A}_{n,y}(t)$ be the upper bounds in (6.17) and (6.18), respectively. Since $g_{3,y}(t) \geq 0$ is a non-negative function for all $t \geq 0$, it follows that $A_{n,y}(t)$ and $\hat{A}_{n,y}(t)$ are monotone non-increasing in $n \geq 1$ for any fixed $t \geq 0$. By definition of $g_{3,y}(t)$, the upper bound $A_{n,y}(t)$ can be rewritten as

$$\begin{aligned}A_{n,y}(t) &= \bar{F}(y+t) + e^{-m_1 y}F(t) - e^{-m_1 y}\sum_{m=1}^{n}(\bar{F} * F^{*m})(t) \\ &\quad + \sum_{m=1}^{n}(\bar{F}(y+\cdot) * F^{*m})(t) \\ &= e^{-m_1 y}F(t) - e^{-m_1 y}[\bar{F}^{*(n+1)}(t) - \bar{F}(t)] + \sum_{m=0}^{n}(\bar{F}(y+\cdot) * F^{*m})(t) \\ &= e^{-m_1 y} - e^{-m_1 y}\bar{F}^{*(n+1)}(t) + \sum_{m=0}^{n}(\bar{F}(y+\cdot) * F^{*m})(t).\end{aligned}$$

Using that $\lim_{n\to\infty} \bar{F}^{*(n+1)}(t) = 1$ and relation (6.4), it follows that $\lim_{n\to\infty} A_{n,y}(t) = \bar{V}_y(t)$, that is, $A_{n,y}(t)$ converges to $\bar{V}_y(t)$.

Also, it holds that

$$\begin{aligned}\hat{A}_{n,y}(t) &= e^{-m_1 y} - e^{-m_1 y}\sum_{m=1}^{n}(\bar{F} * F^{*(m-1)})(t) + \sum_{m=1}^{n}(\bar{F}(y+\cdot) * F^{*(m-1)})(t) \\ &= e^{-m_1 y} - e^{-m_1 y}\bar{F}^{*n}(t) + \sum_{m=0}^{n-1}(\bar{F}(y+\cdot) * F^{*m})(t),\end{aligned}$$

from which we get that $\lim_{n\to\infty} \hat{A}_{n,y}(t) = \bar{V}_y(t)$, implying also that the bound $\hat{A}_{n,y}(t)$ converges to $\bar{V}_y(t)$.

(ii) As in (i), we find $\beta_L(t) = e^{-m_2 y}$ and $g_{y,L}(t) = \bar{F}(y+t) - \bar{F}(t)e^{-m_2 y} = g_{4,y}(t)$. Therefore, the bounds in (6.19) and (6.20) follow directly from (6.8) and (6.9), respectively. The rest of the proof is similar as in (i). □

If it holds $m_1 \leq \mu_F(t) \leq m_2$, for some $m_1, m_2 \in (0, \infty)$ with $m_1 < m_2$, then from Corollary 6.4 we obtain a two-sided bound for $\bar{V}_y(t)$.

7. Conclusion

In this paper, we derive sequences of non-decreasing (non-increasing) general lower (upper) bounds for the solution of a proper renewal equation, for the renewal function, the renewal density and the right-tail probability of the forward recurrence time of a renewal process. Some of these bounds are given for first time, whereas all the proposed bounds are refinements of the corresponding existing ones. These bounds can be applied for any distribution F of the inter-arrival times.

Also, by considering first-order reliability classes (such as IFR, DFR, IMRL, DMRL, NWUE and NBUE) we provide a series of such new bounds for the renewal function, the renewal density and the right-tail probability of the forward recurrence time of a renewal process. In order to obtain such bounds, one has to compare the tails $\bar{F}(t)$ and $\bar{F}_e(t)$. This is achieved, by using the functions $a_U(t)$ and $a_L(t)$ defined by (3.4), since these functions are defined through the tails $\bar{F}(t)$ and $\bar{F}_e(t)$. This approach cannot be applied directly to higher-order reliability classes for the df F, since in this case we have to compare higher-order equilibrium distributions of F. Hence, one has to define other functions than $a_U(t)$ and $a_L(t)$. This is an interesting open problem. Similar conclusions also hold with the use of functions $\beta_U(t)$ and $\beta_L(t)$ (defined by (5.1) in order to obtain bounds for the right-tail probability of the forward recurrence time).

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Appendix

Proof of Proposition 2.1. (i) We shall prove (2.2) by mathematical induction on $n = 1, 2, 3, \ldots$. Obviously, (2.2) it holds true for $n = 1$, because of (1.1). Let $\hat{Z}(s) = \int_0^\infty e^{-st} Z(t)\, dt$, $\hat{r}(s) = \int_0^\infty e^{-st} r(t)\, dt$ and $\hat{f}^{*m}(s) = \int_0^\infty e^{-st}\, dF^{*m}(t)$ denote the Laplace transforms of the functions $Z(t)$, $r(t)$ and $f^{*m}(t)$, respectively. Assuming that (2.2) holds for some $n \geq 1$, then taking Laplace transforms on both sides of (2.2) and (1.1), we obtain

$$\hat{Z}(s) = \sum_{m=1}^{n} \hat{r}(s)\hat{f}^{*(m-1)}(s) + \hat{Z}(z)\hat{f}^{*n}(s),$$

and

$$\hat{Z}(s) = \hat{r}(s) + \hat{Z}(s)\hat{f}(z),$$

which yield

$$\begin{aligned}\hat{Z}(s) &= \sum_{m=1}^{n} \hat{r}(s)\hat{f}^{*(m-1)}(s) + [\hat{r}(s) + \hat{Z}(z)\hat{f}(s)]\hat{f}^{*n}(z) \\ &= \sum_{m=1}^{n+1} \hat{r}(s)\hat{f}^{*(m-1)}(s) + \hat{Z}(z)f^{*(n+1)}(s)\end{aligned}$$

implying that (2.2) is also holds for $n+1$. Therefore, (2.2) holds true for all $n = 1, 2, 3, \ldots$

(ii) The relation (2.3) follows directly from (2.2) with $r(t) = F(t)$, since then $Z(t) = M(t)$. □

Proof of Theorem 2.2. Consider the function $h(t) = Z(t) - M(t)$. Then, from (2.2) and (2.3), it follows that $h(t)$ satisfies the renewal-type equation

$$h(t) = \bar{r}_n(t) + \int_0^t h(t-y)\,dF^*(y), \tag{A.1}$$

with $\bar{r}_n(t) = \sum_{m=1}^{n}(r * F^{*(m-1)}(t) - \sum_{m=1}^{n} F^{*m}(t)$.

Let M_n be such that

$$M_n(t) = F^{*n}(t) + \int_0^t M_n(t-y)\,dF^{*n}(y), \quad n = 1, 2, 3, \ldots \tag{A.2}$$

with $M_1(t) = M(t)$.

Since (A.1) is of the form (1.1), then using that

$$Z(t) = r(t) + \int_0^t r(t-y)\,dM(y) = r(t) + \int_0^t r(t-y)\,dU(y)$$

and taking into account the relation (A.2), we obtain

$$h(t) = \bar{r}_n(t) + \int_0^t \bar{r}_n(t-y)\,dM_n(t). \tag{A.3}$$

By observing that

$$\begin{aligned}\bar{r}_n(t) &= \sum_{m=1}^{n}(r * F^{*(m-1)}(t) - \sum_{m=1}^{n} F^{*m}(t) \\ &= \sum_{m=1}^{n}(r * F^{*(m-1)}(t) - \sum_{m=1}^{n}(F * F^{*(m-1)})(t) \\ &= \sum_{m=1}^{n}(w * F^{*(m-1)}(t),\end{aligned}$$

from (A.3), it follows that for every $n = 1, 2, 3, \ldots$, it holds

$$Z(t) = M(t) + \sum_{m=1}^{n} (w * F^{*(m-1)}(t) + \sum_{m=1}^{n} \int_0^t (w * F^{*(m-1)})(t-y)\, dM_n(y). \tag{A.4}$$

If $w(t) \geq (\leq)\, 0$, then (2.4) it follows immediately from (A.4), since the third term in (A.4) is positive (negative). □

Proof of Theorem 2.3. From (A.4) and for $n = 1$, we obtain

$$Z(t) = M(t) + w(t) + \int_0^t w(t-y)\, dM(y).$$

Since $w'(t) \leq (\geq) 0$, then for $0 \leq y \leq t$ it holds $w(t-y) \geq (\leq) w(t)$, and thus from the above relation we get that

$$Z(t) \geq (\leq) M(t) + w(t) + w(t) \int_0^t dM(t),$$

that is,

$$Z(t) \geq (\leq) M(t) + w(t) + w(t)M(t). \tag{A.5}$$

Therefore, (2.5) holds for $n = 0$.

For every $n = 1, 2, 3, \ldots$, from (A.5) and using Proposition 2.1, we obtain

$$(Z * F^{*n})(t) \geq (\leq)(M * F^{*n})(t) + (w * F^{*n})(t) + (wM * F^{*n})(t),$$

or

$$\begin{aligned} Z(t) \geq (\leq) & \sum_{m=1}^{n} (r * F^{*(m-1)})(t) + M(t) - \sum_{m=1}^{n} F^{*m}(t) + (w * F^{*n})(t) \\ & + (wM * F^{*n})(t). \end{aligned} \tag{A.6}$$

Since $r(t) = w(t) + F(t)$ and $w'(t) \leq (\geq) 0$, then

$$\sum_{m=1}^{n} (r * F^{*(m-1)})(t) = \sum_{m=1}^{n} (w * F^{*(m-1)})(t) + \sum_{m=1}^{n} F^{*m}(t),$$

and

$$\begin{aligned} (wM * F^{*n})(t) &= \int_0^t w(t-y)M(t-y)\, dF^{*n}(y) \\ &\geq (\leq) w(t) \int_0^t M(t-y)\, dF^{*n}(y) \\ &= w(t)M(t) - w(t) \sum_{m=1}^{n} F^{*n}(t), \end{aligned}$$

where the last relation follows from (2.3). Using the last two relations, (A.6) yields

$$Z(t) \geq (\leq) M(t) + w(t)M(t) + (w * F^{*n})(t) + \sum_{m=1}^{n} (w * F^{*(m-1)})(t) - w(t) \sum_{m=1}^{n} F^{*m}(t), \tag{A.7}$$

and since

$$\begin{aligned}\sum_{m=1}^{n}(w * F^{*(m-1)})(t) + (w * F^{*n})(t) - w(t)\sum_{m=1}^{n} F^{*m}(t) \\ = w(t) + \sum_{m=2}^{n+1}(w * F^{*(m-1)})(t) - w(t)\sum_{m=1}^{n} F^{*m}(t) \\ = w(t) + \varepsilon_n(t),\end{aligned}$$

from (A.7), we immediately obtain that the bound (2.5) is also holds for every $n \geq 1$. Therefore, (2.5) holds true for every $n \geq 0$. Obviously, if $w'(t) \leq (\geq)0$, then

$$(w * F^{*m})(m) \geq (\leq)w(t)F^{*m}(t),$$

implying that $\varepsilon_n(t) \geq (\leq)0$.

(ii) Let $L_{n,1}(t)$ and $L_{n,2}(t)$ be the bounds in (2.4) and (2.5), respectively.

Since for every $n = 1, 2, 3, \ldots$, $\sum_{m=1}^{n} F^{*m}(t) \leq \sum_{m=1}^{\infty} F^{*m}(t) = M(t)$, it follows that if $w(t) \geq (\leq)0$, then

$$w(t)\sum_{m=1}^{n} F^{*m}(t) \leq (\geq)w(t)M(t),$$

yielding that

$$L_{n,2}(t) \geq (\leq)M(t) + w(t) + \sum_{m=1}^{n}(w * F^{*m})(t) = M(t) + \sum_{m=0}^{n}(w * F^{*m})(t) = L_{n+1}(t),$$

for every $n = 1, 2, 3, \ldots$

Obviously, from (A.5) (or equivalently, from (2.5) for $n = 0$) and from (2.4) for $n = 1$, we get

$$L_{0,2}(t) = M(t) + w(t) + w(t)M(t) \geq M(t) + w(t) = L_{1,1}(t).$$

Therefore, it holds that

$$L_{n,2}(t) \geq L_{n+1,1}(t), \quad \text{for every } n = 0, 1, 2, \ldots$$

and thus applying Theorem 2.2 for $n + 1$ and Theorem 2.3 for n, then for the corresponding bounds we conclude that the bound (2.5) is tighter than that of (2.4). □

Proof of Corollary 4.1. Since $r_F(t) = \mu\bar{F}_e(t)/\bar{F}(t)$, from (3.4) it follows that

$$\alpha_U(t) = \frac{1}{\mu}\sup_{\substack{0\leq z\leq t\\ \bar{F}(z)}}\{r_F(z)\} \quad \text{and} \quad \alpha_L(t) = \frac{1}{\mu}\inf_{\substack{0\leq z\leq t\\ \bar{F}(z)}}\{r_F(z)\}. \tag{A.8}$$

(i) From $r_F(t) \geq r_1$, it follows that $\alpha_L(t) = r_1/\mu$ and thus $\xi_L(t)$ given by (3.10) becomes equal to $\xi_{1,L}(t)$. Then, the lower bound follows immediately from (3.11). Let $L_{1,n}(t)$ denotes the lower bound. Since $\xi_{1,L}(t) \geq 0$ for all $t \geq 0$, it follows that $L_{1,n}(t)$ is monotone non-decreasing in $n \geq 1$ for any fixed t. By definition of $\xi_{1,L}(t)$ and since

$$\bar{F}^{*n}(t) = \bar{F}(t) + \sum_{m=1}^{n-1}(\bar{F} * F^{*m})(t),$$

the lower bound $L_{1,n}(t)$ can be rewritten as

$$L_{1,n}(t) = \frac{t}{\mu} + \frac{r_1}{\mu}F(t) + \sum_{m=0}^{n}(\bar{F}_e * F^{*m})(t) - \frac{r_1}{\mu}\sum_{m=1}^{n}(\bar{F} * F^{*m})(t)$$
$$= \frac{t}{\mu} + \frac{r_1}{\mu}F(t) + \sum_{m=0}^{n}F^{*m}(t) - \sum_{m=0}^{n}(F_e * F^{*m})(t) - \frac{r_1}{\mu}\bar{F}^{*(n+1)}(t)$$
$$+ \frac{r_1}{\mu}\bar{F}(t).$$

Since,

$$\sum_{n=0}^{\infty}(F_e * F^{*m})(t) = \frac{t}{\mu}, \qquad \lim_{n\to\infty}\bar{F}^{*(n+1)}(t) = 1,$$

it follows that

$$\lim_{n\to\infty} L_{1,n}(t) = \sum_{m=0}^{\infty} F^{*m}(t),$$

and hence the bound $L_{1,n}(t)$ converges to $U(t)$.

(ii) From $r_F(t) \le r_2$, then $\alpha_U(t) = r_2/\mu$ and $\xi_U(t)$ given by (3.10) is reduced to $\xi_{1,U}(t)$. Now, the required upper bound is immediately obtained by applying (3.11). The rest of the proof is similar as in (i). □

Proof of Corollary 4.2. (i) Since $\mu_F(t) \le 1/r_1$, it follows that $r_F(t) \ge r_1$ (see, e.g., [27, rel. (2.3.18)]) and thus from (A.8) we get $a_L(t) = /\mu$. Therefore, $\xi_L(t)$ defined by (3.10) is reduced to $\xi_L(t) = \bar{F}_e(t) - (r_1/\mu)\bar{F}(t)$, implying that

$$\xi_L'(t) = \frac{1}{\mu}[r_1 f(t) - \bar{F}(t)] = \frac{\bar{F}(t)}{\mu}[r_1\mu_F(t) - 1] \le 0,$$

and since

$$1 - \xi_L(t) = F_e(t) + \frac{r_1}{\mu}\bar{F}(t), \quad (a_L * F)(t) = \frac{r_1}{\mu}F(t),$$

from (3.16) (or equivalently from (3.17)), we obtain

$$U(t) \ge \frac{t/\mu + (r_1/\mu)\bar{F}(t) + (r_1/\mu)F(t)}{(r_1/\mu)\bar{F}(t) + F_e(t)},$$

and thus (4.3) follows directly.

(ii) Similarly, since $\mu_F(t) \ge 1/r_2$, then it holds $r_F(t) \le r_2$ (see again in rel. (2.3.18) of [27]) and thus we find that $a_U(t) = r_2/\mu$ and $\xi_U(t)$ defined by (3.10) becomes $\xi_U(t) = (r_2/\mu)\bar{F}(t) - \bar{F}_e(t)$, yielding that

$$\xi_U'(t) = \frac{1}{\mu}[\bar{F}(t) - r_2 f(t)] = \frac{\bar{F}(t)}{\mu}[1 - r_2\mu_F(t)] \le 0$$

and hence (4.4) follows immediately as above from (3.14) (or equivalently (3.15)). □

Proof of Corollary 4.5. (i) Since

$$\bar{F}(t) = \exp\left\{-\int_0^t \mu_F(y)\,dy\right\} \le \exp\left\{-\int_0^t m_1\,dy\right\} = e^{-m_1 t},$$

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it follows that $X <_{st} Y$, where the random variable Y follows the exponential distribution with parameter m_1 having df $G(t) = 1 - \bar{G}(t)$. Therefore, it holds $\sum_{i=1}^{k} X_i <_{st} \sum_{i=1}^{k} Y_i$, for every $k = 1, 2, 3, \ldots$, where $\{Y_i : i \geq 1\}$ is a sequence of i.i.d. random variables having df G. This implies that $F^{*k}(t) \geq G^{*k}(t)$, for every $k = 1, 2, 3, \ldots$ and hence $U(t) \geq 1+H_G(t)$, where $H_G(t) = \sum_{k=1}^{\infty} G^{*k}(t)$. Since, $G(t) = 1-e^{-m_1 t}$, it follows that $H(t) = m_1 t$, which yields $U(t) \geq 1 + m_1 t$.

Since $\mu_F(t) \geq m_1$, it follows that $r_F(t) \leq 1/m_1$ (see [27, rel. (2.3.18)]), and thus applying Corollary 4.1(ii) with $r_2 = 1/m_1$, we immediately obtain the upper bound.

Let $U_{3,n}(t)$ be the upper bound. Since $\xi_{3,U}(t) \geq 0$ it follows that $U_{3,n}(t)$ is monotone non-decreasing in $n \geq 1$ for any fixed $t \geq 0$. By definition of $\xi_{3,U}(t)$, the bound $U_{3,n}(t)$ can be rewritten as

$$\begin{aligned} U_{3,n}(t) &= \frac{t}{\mu} + \frac{F(t)}{m_1\mu} - \frac{1}{m_1\mu}\sum_{m=1}^{n}(\bar{F} * F^{*m})(t) + \sum_{m=0}^{n}(\bar{F}_e * F^{*m})(t) \\ &= \frac{t}{\mu} + \frac{F(t)}{m_1\mu} - \frac{1}{m_1\mu}[\bar{F}^{*(n+1)}(t) - \bar{F}(t)] + \sum_{m=0}^{n} F^{*m}(t) \\ &\quad - \sum_{m=0}^{n}(F_e * F^{*m})(t), \end{aligned}$$

and since

$$\sum_{n=0}^{\infty}(F_e * F^{*m})(t) = \frac{t}{\mu}, \qquad \lim_{n\to\infty} \bar{F}^{*(n+1)}(t) = 1,$$

$$\lim_{n\to\infty} U_{3,n}(t) = \frac{t}{\mu} + \frac{1}{m_1\mu} - \frac{1}{m_1\mu}\lim_{n\to\infty} \bar{F}^{*(n+1)}(t) + U(t) - \sum_{m=0}^{\infty}(F_e * F^{*m})(t) = U(t),$$

that is, the upper bound $U_{3,n}(t)$ converges to $U(t)$.

(ii) Since $\mu_F(t) \leq m_2$, it follows that $r_F(t) \geq 1/m_2$ ([27, rel. (2.3.18)]) and thus the lower bound follows directly from Corollary 4.1(i) with $r_1 = 1/m_2$. As in (i), by reversing the inequalities we get that $U(t) \leq 1 + m_2 t$. The rest of the proof is similar as in (i). □

Proof of Corollary 4.6. We shall prove (4.13) by mathematical induction in $n \geq 0$. Obviously, from (4.12), it follows that (4.13) holds true for $n = 0$. Supposing that (4.13) holds true for some $n \geq 0$, then by inserting the upper bound in (4.12) into the right-hand side of (1.3), we obtain

$$\begin{aligned} U(t) &\leq 1 + \int_0^t \frac{t-y}{\mu}\, dF(y) + \sum_{m=0}^{n-1}(\bar{F}_e * F^{*(m+1)})(t) + \frac{\mu_2}{2\mu^2}F^{*(n+1)}(t) \\ &\leq 1 + \frac{t}{\mu} - F_e(t) + \sum_{m=1}^{n}(\bar{F}_e * F^{*m})(t) + \frac{\mu_2}{2\mu^2}F^{*(n+1)}(t) \\ &= \frac{t}{\mu} + \sum_{m=0}^{n}(\bar{F}_e * F^{*m})(t) + \frac{\mu_2}{2\mu^2}F^{*(n+1)}(t), \end{aligned}$$

and thus (4.13) is also holds for $n + 1$, implying that (4.13) holds true for all $n = 0, 1, 2, \ldots$

The lower bound in (4.14) follows by a similar way if we insert the lower bound in (4.12) into the right-hand side of (1.3).

Since

$$\sum_{m=0}^{\infty}(\bar{F}_e * F^{*m})(t) = U(t) - \frac{t}{\mu} \quad \text{and} \quad \lim_{n\to\infty} F^{*n}(t) = 0,$$

the upper bound in (4.13) converges to $U(t)$ as $\to \infty$.

Also, since

$$0 \leq (F_2 * F^{*n})(t) \leq \int_0^t dF^{*n}(t) = F^{*n}(t),$$

it follows that $\lim_{n\to\infty}(F_2 * F^{*n})(t) = 0$. Therefore, it also holds that the lower bound in (4.14) converges to $U(t)$ as $\to \infty$. □

Proof of Corollary 5.5. The lower bound in (i) and the upper bound (ii) are derived directly from Corollary 5.2 since $\sigma_{1,L}(t) = m_1$, and $\sigma_{1,U}(t) = m_2$. The two-sided bound in (iii) follows from (i) and (ii).

Since $\hat{\psi}_{1,L}(t) \geq 0$ ($\hat{\psi}_{1,U}(t) \geq 0$) it follows that the lower bound in (i) (upper bound in (ii)) is monotone non-decreasing (monotone non-increasing) in $n \geq 1$ for all $t \geq 0$. Let $u_n(t)$ be the lower bound in (i). By definition of $\hat{\psi}_{1,L}(t)$, the bound $u_n(t)$ can also be rewritten in the form

$$\begin{aligned} u_n(t) &= f(t) + m_1 F(t) + \sum_{m=1}^{n-1}(f * F^{*m})(t) - m_1 \sum_{m=1}^{n-1}(\bar{F} * F^{*m})(t) \\ &= f(t) + m_1 F(t) + \sum_{m=1}^{n-1} f^{*(m+1)}(t) - m_1[\bar{F}^{*n}(t) - \bar{F}(t)] \\ &= m_1 - m_1 \bar{F}^{*n}(t) + \sum_{m=1}^{n} f^{*m}(t). \end{aligned}$$

Since $\lim_{n\to\infty} \bar{F}^{*n}(t) = 1$, $\sum_{m=1}^{\infty} f^{*m}(t) = u(t)$, it follows that $\lim_{n\to\infty} u_n(t) = u(t)$, that is., $u_n(t)$ converges to $u(t)$. Similarly, it can be shown that the lower bound in (ii) also converges to $u(t)$ as $n \to \infty$.

(iii) Follows directly from (i) and (ii). □

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