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CHAIN CONDITIONS FOR MODULAR LATTICES WITH FINITE GROUP ACTIONS

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This paper establishes the following combinatorial result concerning the automorphisms of a modular lattice.

THEOREM. *Let M be a modular lattice and let G be a finite subgroup of the automorphism group of M. If the sublattice, M^G, of (common) fixed points (under G) satisfies any of a large class of chain conditions, then M satisfies the same chain condition. Some chain conditions in this class are the following: the ascending chain condition; the descending chain condition; Krull dimension; the property of having no uncountable chains, no chains order-isomorphic to the rational numbers; etc.*

Unknown to the author, John R. Isbell [**1**] proved the ascending chain version in 1969 in response to a query of Peter M. Neumann. As it turns out his proof was a peculiar one, from which one could obtain very little more than the stated result. The type of proof that one would like would construct a finite family of G-invariant G-lattice polynomials $f_1, f_2, \ldots, f_k$ in one variable, such that whenever $p < q$ in M, at least one of the induced inequalities $f_i(p) \leqq f_i(q)$ $(i = 1, 2, \ldots, k)$ must also be strict. The ascending chain condition result follows immediately: if $p_1 < p_2 < p_3 < \ldots$ is an infinite ascending chain in M, then at least one of the k chains $f_i(p_1) < f_i(p_2) < f_i(p_3) < \ldots$ in M^G will also have infinitely many distinct terms. Such a proof, unlike Isbell's, would also yield such conclusions as that if M^G has no uncountable chains, then neither does M, etc. We establish the existence of such G-invariant G-lattice polynomials, and hence provide a proof of just this sort.

As one can imagine such a theorem has many applications. In Section 2 we content ourselves with applications to lattices of ideals of particular rings and to lattices of submodules of certain modules. It was these applications which motivated us to prove the theorem in the first place.

1. The theorem. In this section we prove the main theorem and investigate some of the various aspects surrounding it.

Before we begin the proof, we need to set some terminology and notation. Let F denote the set of all unary $(G, \wedge, \vee)$-polynomial operations $M \rightarrow M$. Note that F may be described as the lattice of set-maps $M \rightarrow M$, generated by the actions of G, under pointwise $\wedge$ and $\vee$. It is also closed under composition. Thus it is a lattice-ordered semigroup of isotone maps; not in general homo-

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morphisms. An element $f(x)$ in F is called a *G-lattice polynomial*. For example, if S and T are subsets of G, then $f(x) = \vee_S(\wedge_T x^t)^s$ is a G-lattice polynomial where it is understood, for convenience of notation, that the 'inf' is taken over all $t \in T$ and the 'sup' is taken over all $s \in S$. Likewise, $g(x) = \wedge_S(\vee_T x^t)^s$ is a G-lattice polynomial. A *G-invariant* G-lattice polynomial is a G-lattice polynomial which is invariant under the action of G, i.e., it takes its values in M^G. For example, if S is a subset of G, then $\wedge_G x^g$, $\vee_G x^g$, $\wedge_G(\vee_S x^s)^g$, $\vee_G(\wedge_S x^s)^g$ are G-invariant polynomials.

We begin with a technical lemma, but first a definition. If $f(x)$ is a G-lattice polynomial and $p \leqq q$ are elements of M, we say that $f(x)$ *trivializes* the pair (p, q) if $f(p) = f(q)$.

LEMMA 1.1. *Let S be a finite subset of G and g an element of $G \backslash S$. Suppose that $p \leqq q$ are elements of M such that (p, q) is trivialized by both the G-lattice polynomials* (1) $\wedge_{S \cup \{g\}} x^t$ *and* (2) $\vee\{(\wedge_S x^s)^{t^{-1}}: \ t \in S \cup \{g\}\}$. *Then (p, q) is also trivialized by $\wedge_S x^s$.*

Proof. The statements that (1) and (2) trivialize (p, q) say

(1′) $\wedge_{S \cup \{g\}} q^t = \wedge_{S \cup \{g\}} p^t$ and

(2′) $\vee\{(\wedge_S q^s)^{t^{-1}}: \ t \in S \cup \{g\}\} = \vee\{(\wedge_S p^s)^{t^{-1}}: \ t \in S \cup \{g\}\}$.

We transform (2′) as follows. First, we apply g to it. Then, we weaken it to an inequality: we decrease the left-hand join by dropping all but the $t = g$ term, while on the right-hand side we increase every meet except the $t = g$ term, by dropping all terms but the $s = t$ one. The $s = t$ terms are all the same, and so coalesce into one term giving

(2″) $\wedge_S q^s \leqq (\wedge_S p^s) \vee p^g$.

Let us, finally, increase the right-hand side still further by changing p^g to q^g. We now recall that the resulting inequality $a \leqq b$ can be rewritten $a = b \wedge a$. In this case, our "b" is a join, the first term of which is $\leqq a$ (because $p \leqq q$). This allows us to apply modularity, and the result is $\wedge_S q^s = (\wedge_S p^s) \vee (q^g \wedge (\wedge_S q^s))$. Now the second term on the right is just the left-hand side of (1′). This is equal to the right-hand side of (1′), which is equal to or less than the other term, and hence is absorbed thereby, giving the desired equation $\wedge_S q^s = \wedge_S p^s$. Hence (p, q) is trivialized by $\wedge_S x^s$.

Now we assume that G is finite with $|G| = n$ and we choose an arbitrary ordering of its elements, $g_1, g_2, \ldots, g_n$, except we make $g_1 = e$. Fix this ordering. Let I denote the set of all sequences $1 = i(1) < \ldots < i(m) \leqq n$ where $m \geqq 1$. We write a typical member of $I \backslash \{(1)\}$ as $i = (i', i(m))$ where $i' \in I$. We define G-lattice polynomials $f(i; \wedge; x)$ and $f(i; \vee; x)$ inductively:

(4) $f((1); \wedge; x) = f((1); \vee; x) = x$

(5) $f(i', i(m); \wedge; x) = \wedge_{j=1}^{i(m)} f(i'; \vee; x)^{g_j}$

(6) $f(i', i(m); \vee; x) = \vee_{j=1}^{i(m)} f(i'; \wedge; x)^{g_j^{-1}}$.

For a group $G = \{g_1 = e, g_2, g_3\}$ of order 3 the G-lattice polynomials defined above are as follows: $f((1); \wedge; x) = f((1); \vee; x) = x$, $f((1, 2); \wedge; x) = x \wedge x^{g_2}$, $f((1, 2); \vee; x) = x \vee x^{g_2^{-1}}$, $f((1, 3); \wedge; x) = x \wedge x^{g_2} \wedge x^{g_3}$, $f((1, 3); \vee; x) = x \vee x^{g_2^{-1}} \vee x^{g_3^{-1}}$, $f((1, 2, 3); \wedge; x) = (x \vee x^{g_2^{-1}}) \wedge (x \vee x^{g_2^{-1}})^{g_2} \wedge (x \wedge x^{g_2^{-1}})^{g_3}$ and $f((1, 2, 3); \vee; x) = (x \wedge x^{g_2}) \vee (x \wedge x^{g_2})^{g_2^{-1}} \vee (x \wedge x^{g_2})^{g_3^{-1}}$. The last four G-lattice polynomials are G-invariant. In general, the number of G-invariant G-lattice polynomials obtained in this way is $2^{|G|-1}$ and these are the ones that we are interested in. They are $f(i', n; *; x)$ with $* \in \{\wedge, \vee\}$.

Although the minimal sequence $(1) \in I$ cannot be decomposed as $(i', i(m))$, let us formally identify it with the symbol $(1, 1)$, so that we get such a decomposition with $i' = 1$. Then from (4) we can see that (5) and (6) still hold for this sequence. Now (5) and (6) extended (as noted) combine with Lemma 1.2 and its dual to give the following.

COROLLARY 1.2. *Let $i = (i', i(m)) \in I$ with $i(m) < n$, and let $p \leqq q$ in M.*

(1) *If $f(i', i(m) + 1; \wedge; x)$ and $f(i, i(m) + 1; \vee; x)$ both trivialize (p, q), then $f(i; \wedge; x)$ trivializes (p, q).*

(2) *If $f(i', i(m) + 1; \vee; x)$ and $f(i, i(m) + 1; \wedge; x)$ both trivialize (p, q), then $f(i; \vee; x)$ trivializes (p, q).*

Now let us partition I as $I = I_1 \cup I_2 \cup \ldots \cup I_n$, where $I_k = \{i = (i', i(m)) \in I: i(m) = k\}$. Corollary 1.2 guarantees that if for some $m < n$, both $f(i; \wedge; x)$ and $f(i; \vee; x)$ trivialize (p, q) for all $i \in I_{m+1}$, then the same is true for all $i \in I_m$. Hence, by induction, if $f(i; \wedge; x)$ and $f(i; \vee; x)$ trivialize (p, q) for all $i \in I_n$, then the identity polynomials $f((1); \wedge; x) = f((1); \vee; x) = x$ trivialize (p, q), i.e., $p = q$. However, for $i \in I_n$ the polynomials $f(i; \wedge; x)$ and $f(i; \vee; x)$ are G-invariant since the "$\wedge$" and "$\vee$", respectively, are taken over the entire group. Therefore we have proven the following lemma.

LEMMA 1.3. *Suppose that $p \leqq q$ in M are such that each of the $2^{|G|-1}$ G-invariant G-lattice polynomials $f(i; *; x)$ $(i \in I_{|G|}, * \in \{\wedge, \vee\})$ trivializes (p, q). Then $p = q$.*

THEOREM. *If M is a modular lattice and G is a finite group of automorphisms acting on M such that M^G has* ACC, *resp.* DCC, *resp. Krull dimension* [**2**], *then so does M. The same holds for the property of having no uncountable chains, no chains order-isomorphic to the rational numbers, etc.* (See Note 1.)

Proof. The proof of the ACC and DCC follow from Lemma 1.3 as outlined in the introduction. The proof of the Krull dimension likewise follows by using the same technique and a simple induction. The remainder of the proof is clear.

Note 1. In [**3**] Bergman defines a class P of algebraic structures to be *weakly subdirectly prime* if no member of P can be represented as a finite subdirect product of structures, none of which contains a substructure isomorphic to a member of P. Examples of weakly subdirectly prime classes of totally ordered sets are the singleton classes consisting: of any infinite cardinal, of the ordered set of real numbers, of the ordered set of rational numbers, and the class of all totally ordered sets of a given infinite cardinality. Before knowing of the existence of G-invariant G-lattice polynomials, Bergman [**3**] characterized those classes of chains to which such a theorem would apply as those classes which have a chain isomorphic to a member of a weakly subdirectly prime class of totally ordered sets.

Examples. (1) The theorem is false for arbitrary lattices. Take a lattice L which has two infinite ascending chains $p_1 < p_2 < p_3 < \ldots$ and $q_1 < q_2 < q_3 < \ldots$ such that for each i, $p_i \vee q_i = 1$ and $p_i \wedge q_i = 0$. Let $G = \langle g \rangle$ be a group of order 2 acting on L by $p_i^g = q_i$ for each i, $1^g = 1$, and $0^g = 0$. Then $L^G = \{0, 1\}$ has ACC, but L does not.

(2) The theorem is false for infinite groups. Let $R = \oplus\{C_i: \ i \in \mathbf{Z}$ and $C_i = \mathbf{Z}_2\}$. Then the lattice of subgroups of R is a modular lattice. Moreover $G = \mathbf{Z}$ acts on R by a natural shift to the right. Obviously, M^G satisfies ACC but M does not.

(3) In a modular lattice M not every property of M^G is inherited by M, viz., the distributive property. For example, take the lattice M of subgroups of Klein's four group and let a group G of order 2 act on it by interchanging two of the subgroups of order 2 and by fixing all the other subgroups.

Of course, it follows from the theorem that if M^G has finite length, then M has finite length. The next corollary shows moreover that $l(M) \leqq |G|l(M^G)$.

COROLLARY 1.4. *If M is a modular lattice and G is a finite group of automorphisms of M such that the length, $l(M^G)$, is finite, then $l(M) < \infty$ and $l(M) \leqq |G|l(M^G)$.*

Proof. That $l(M) < \infty$ follows from the theorem. Let $h : M \rightarrow \{0, 1, \ldots, l(M)\}$ be the "height" function. Then it is not hard to prove that if $p < q$ in M with $h(q) - h(p) = 1$, then for any $r \in M$, $h(q \vee r) - h(p \vee r) \leqq 2$, and to deduce by induction that $h(f(q)) - h(f(p)) \leqq |G|$ where $f(x) = \vee_G x^g$. Now take elements $0 = p_0 < p_1 < \ldots < p_{l(M)} = 1$ in M with $h(p_i) = i$. By applying $f(x)$, we get a chain of elements in M^G ascending from 0 to 1, with the height changing at each step by at most the $|G|$. Hence there must be at least $l(M)/|G|$ nontrivial steps, so $l(M^G) \geqq l(M)/|G|$.

2. Applications. Let G be a finite group of automorphisms acting on a ring R and let $R^G = \{r \in R: \ r^g = r$ for each g in $G\}$. A left (two-sided) ideal I of R is ***G*-*invariant*** if $I^g = I$ for each g in G. The motivation for this paper comes from the author's work on the study of the relation between the ideal structures of R and R^G. It is clear from Fisher-Osterburg [**7**] that this breaks into two problems: (1) the relation between the lattice for all ideals and the lattice of G-invariant ideals of R, and (2) the relation between the lattice of

G-invariant ideals of R and the lattice of ideals of R^G. We shed some light on the former with this first corollary.

COROLLARY 2.1. *Let R be a ring and let G be a finite group of automorphisms acting on R. If R satisfies the* ACC (DCC) *on G-invariant left (two-sided) ideals, then R satisfies the* ACC (DCC) *on all left (two-sided) ideals.*

Proof. Since the lattice of left (two-sided) ideals of R is modular, the result follows immediately from the theorem.

COROLLARY 2.2. *Let R be a ring and let G be a finite group of anti-automorphisms acting on R. If R satisfies the* ACC (DCC) *on G-invariant ideals, then R satisfies the* ACC (DCC) *on all ideals. In particular, if a ring R with involution "$*$" satisfies the* ACC (DCC) *on $*$-invariant ideals, then R satisfies the* ACC (DCC) *on all ideals.*

Proof. The result follows from the theorem since G acts as a group of automorphisms on the lattice of ideals of R.

COROLLARY 2.3. *Let R be a subring of S. Assume that there exists a finite subset $\{u_1, u_2, \ldots, u_m\}$ of S such that $S = \sum\{Ru_i: \; 1 \leqq i \leqq m\}$, $Ru_i = u_iR$ for $1 \leqq i \leqq m$, and $\{u_1, u_2, \ldots, u_m\}$ forms a group under multiplication.*

(i) *A left S-module is Artinian (Noetherian) if and only if it is Artinian (Noetherian) as a left R-module.*

(ii) *An (S, S)-bimodule is Artinian (Noetherian) if and only if it is Artinian (Noetherian) as an (R, R)-bimodule. In particular, S satisfies* DCC (ACC) *on two-sided ideals if and only if R satisfies* DCC (ACC) *on two-sided ideals.*

Proof. (i) Let A be a left S-module which is Artinian (Noetherian) as an S-module. If M is the lattice of R-submodules of A and $G = \{u_1, u_2, \ldots, u_m\}$, then M^G is the lattice of S-submodules of A. The theorem now applies to yield the result.

(ii) Let B be an (S, S)-bimodule which is Artinian (Noetherian) as an (S, S)-bimodule. If M is the lattice of (R, R)-bisubmodules of B and $G = \{u_1, u_2, \ldots, u_m\}$, then $G \times G$ acts on M by $p^{(g,h)} = gph$ where $p \in M$. Moreover $M^{G \times G}$ is the lattice of (S, S)-bisubmodules of B. Again the theorem applies. The proof of the converse is found in Reiter [**9**].

Remark. Formanek and Jategaonker [**10**] have proven the Noetherian half of Corollary 2.3(i) *without* the assumption that $\{u_1, u_2, \ldots, u_m\}$ forms a group under multiplication; however, they do not have the Artinian half even with the assumption that $\{u_1, \ldots, u_m\}$ forms a group. That much is new. It would be interesting to know if the Artinian half can be proven without the assumption that $\{u_1, \ldots, u_m\}$ forms a group?

Let G be a group of automorphisms acting on a ring R. Then the *skew group ring*, $R * G$, is the free left R-module on the set G with multiplication defined by $(rg)(sh) = rs^{g^{-1}}(gh)$ where $r, s \in R$ and $g, h \in G$

COROLLARY 2.4. *Let G be a finite group of automorphisms acting on a ring R.*

(i) *A left $R*G$-module is Artinian (Noetherian) if and only if it is Artinian (Noetherian) as a left R-module. In particular, if $R*G$ is left Artinian (Noetherian), then R is left Artinian (Noetherian). Moreover, if $|G|$ is invertible in R, then R^G is also left Artinian (Noetherian).*

(ii) *An $(R*G, R*G)$-bimodule is Artinian (Noetherian) if and only if it is Artinian (Noetherian) as an (R, R)-bimodule. In particular, $R*G$ satisfies* DCC (ACC) *on two-sided ideals if and only if R satisfies* DCC (ACC) *on two-sided ideals. Moreover, if $|G|$ is invertible in R, then R^G also satisfies* DCC (ACC) *on two-sided ideals.*

Proof. The first statements in (i) and (ii) follow from Corollary 2.3. If $|G|$ is invertible in R, then $R^G \cong e(R*G)e$ where $e^2 = e$ by [**8**, Corollary 1.4]. The last statements follow from this.

Now we restrict our attention to chains of semiprime ideals in order to answer a question in [**7**].

COROLLARY 2.5. *Let R be a ring and let G be a finite group of automorphisms acting on R. Then R satisfies the* ACC (DCC) *on semiprime ideals if and only if R satisfies* ACC (DCC) *on G-invariant semiprime ideals. Moreover, if R has semiprime Krull dimension $\leqq \alpha$ on G-invariant semiprime ideals, then R has semiprime Krull dimension $\leqq \alpha$.*

Proof. The lattice of semiprime ideals of R is modular because it can be embedded in the lattice of sets of prime ideals by sending a semiprime ideal I to the complement of the set of prime ideals containing it. Hence the lattice of semiprime ideals is a sublattice of a modular lattice and hence is modular. The result now follows from the theorem.

Note 2. Since the set of semiprime ideals forms a distributive lattice, the proof can be obtained more easily. In fact, the proof of the theorem can be obtained more easily for distributive lattices. For, instead of requiring the $2^{|G|-1}$ G-invariant G-lattice polynomials, one only needs the $|G| = n$ G-invariant G-lattice symmetric polynomials, viz.,

$$s_i(x) = s_i(x^{g_1}, x^{g_2}, \ldots, x^{g_n}), \quad i = 1, 2, \ldots, n \quad \text{where}$$
$$s_i(x_1, x_2, \ldots, x_n) = \vee_{j_1 \leqslant \ldots \leqslant j_i \leqq n}(x_{j_1} \wedge \ldots \wedge x_{j_i}).$$

COROLLARY 2.6. *Let R be a ring and let G be a finite group of automorphisms acting on R such that $|G|$ is invertible in R. If $R*G$ satisfies the* ACC (DCC) *on semiprime ideals, then R satisfies the* ACC (DCC) *on semiprime ideals.*

Proof. By Corollary 2.5 it suffices to show that R satisfies the ACC (DCC) on G-invariant semiprime ideals. To this end, let, say $I_1 \subset I_2 \subset I_3 \subset \ldots$ be an ascending chain of G-invariant semiprime ideals of R. Because of the G-invariance, $I_1 * G \subset I_2 * G \subset I_3 * G \subset \ldots$ is an ascending chain of two-sided ideals in $R*G$. We claim that each I_j*G is semiprime. Note that I_j*G is the kernel of the canonical map $R*G \to (R/I_j)*G$. Since $|G|$ is invertible in R, R/I_j is a semiprime ring with no $|G|$-torsion. Therefore, by Fisher–Montgomery [**6**, Theorem 7], $(R/I_j)*G$ is semiprime. Wherefore, I_j*G is a semiprime ideal of $R*G$. The rest of the proof is now evident.

Added in proof. (a) The question asked in the "Remark" following Corollary 2.3 has been answered affirmatively by B. Lemonnier in Theorem 5.3 in Comm. in Algebra *6* (16) (1978), 1647–1665.

(b) Lorenz and Passman point out that Corollary 2.6 holds without the assumption that "$|G|$ is invertible" in R. It follows from results in their paper *Prime ideals in Crossed Products of Finite Groups*, Israel J. Math. (to appear).

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