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A REVERSE HÖLDER TYPE INEQUALITY FOR THE LOGARITHMIC MEAN AND GENERALIZATIONS

JOHN MALONEY[1], JACK HEIDEL[1] and JOSIP PEČARIĆ[2]

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Abstract

An inequality involving the logarithmic mean is established. Specifically, we show that

$$L(c,x)^{\frac{\ln(c/x)}{\ln(c/a)}} L(x,a)^{\frac{\ln(x/a)}{\ln(c/a)}} < L(c,a),$$

where $0 < a < x < c$ and $L(x,y) = \frac{y-x}{\ln x - \ln y}$, $0 < x < y$. Then several generalizations are given.

1. Introduction

The logarithmic mean,

$$L(x,y) = \frac{y-x}{\ln x - \ln y}, \qquad 0 < x < y$$

has many applications in statistics and economics [9]. It is well known, and easily established [1, 3, 7, 10] that

$$G(x,y) \le L(y,x) \le A(y,x),$$

where $G(y,x) = \sqrt{xy}$ is the geometric mean and $A(x,y) = (x+y)/2$ is the arithmetic mean. In fact, writing $A(x,y) = M_1(y,x)$, where

$$M_p(y,x) = \left(\frac{y^p + x^p}{2}\right)^{1/p},$$

[1]Department of Mathematics, University of Nebraska at Omaha, Omaha Nebraska 68182, USA
[2]Faculty of Textile Technology, Zagreb Croatia

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it is known [7] that $M_{p_1}(y,x) \le M_{p_2}(y,x)$ for $p_1 \le p_2$. It is also known [4–6, 9, 12] that

$$L(y,x) \le M_{1/3}(y,x).$$

On the other hand, Hölder's inequality states that

$$M_1(y_1y_2, x_1x_2) \le M_p(y_1,x_1)M_q(y_2,x_2),$$

if $1/p + 1/q = 1$ with $p, q > 0$. It is thus curious that the logarithmic mean $L(y,x)$ satisfies the inequality

$$L(c,x)^{\frac{\ln(c/x)}{\ln(c/a)}} L(x,a)^{\frac{\ln(x/a)}{\ln(c/a)}} < L(c,a), \tag{1}$$

where $0 < a < x < c$ and it is noted that

$$\frac{\ln(c/x)}{\ln(c/a)} + \frac{\ln(x/a)}{\ln(c/a)} = 1.$$

It is the reverse Hölder type inequality (1) which is the subject of this note and will be established below. Relation (1) arises in a parameter identification problem for a fractal Michaelis-Mention equation [8].

In the following, use will be made of Jensen's inequality [11] which we now state for the reader's convenience.

JENSEN'S INEQUALITY. *If,*

(1) $w_i > 0\ \forall\ i = 1, 2, \dots, n,$

(2) $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n \in R,$

(3) $\Phi : [0,\infty) \to R$ *is a strictly convex function,*

then

$$\left(\sum_{i=1}^n w_i\right) \Phi\left(\frac{\sum_{i=1}^n w_i\alpha_i}{\sum_{i=1}^n w_i}\right) \le \sum_{i=1}^n w_i\Phi(\alpha_i)$$

and the inequality is strict unless $\alpha_0 = \alpha_1 = \alpha_2 = \cdots = \alpha_n$.

2. Main result

LEMMA 2.1. *Let* $g(u) = \frac{\ln u}{u-1}$, *where* $g(1) = 1$. *Then for all* $u > 0$

(i) g *is a strictly decreasing function of* u,

(ii) $\lim_{u\to 0^+} g(u) = \infty$, $\lim_{u\to\infty} g(u) = 0$, $\lim_{u\to 1} g(u) = 1$,

(iii) $g(1/u) = ug(u)$.

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PROOF.

$$g'(u) = \frac{1 - (1/u) - \ln u}{(1-u)^2}.$$

Set $z(u) = 1 - (1/u) - \ln u$. Then $z'(u) = (1/u)(1/u - 1)$ which is positive for $0 < u < 1$ and negative for $u > 1$. Thus $z(u)$ increases from $-\infty$ at $u = 0$ to 0 at $u = 1$ and then decreases to $-\infty$ as u tends to ∞. Thus $g'(u)$ is negative except at $u = 1$. This establishes (i). The limits in (ii) can be computed in the usual fashion using L'Hôpital's rule. For (iii) we have

$$g(1/u) = \frac{\ln(1/u)}{1/u - 1} = ug(u).$$

LEMMA 2.2. *Let* $f(x) = x - \ln x$. *Then*

(i) f *is decreasing on* $(0, 1)$ *and increasing on* $(1, \infty)$,

(ii) $\lim_{x\to 0^+} f(x) = \infty$, $f(1) = 1$ *and* $\lim_{x\to\infty} f(x) = \infty$,

(iii) *if* $\alpha > 0$, $x > 0$ *then* $f(\alpha x) = f(x)$ *for* $x = g(\alpha)$ *so that* $f(\alpha g(\alpha)) = f(g(\alpha))$.

PROOF. Parts (i) and (ii) can be established in the usual way. For (iii) we have

$$f(\alpha x) = f(x) \Rightarrow \alpha x - \ln(\alpha x) = x - \ln x \Rightarrow (\alpha - 1)x = \ln \alpha \Rightarrow x = g(\alpha).$$

Let $y(x)$ denote the left-hand side of (1), and set $\alpha = \ln c - \ln a$. Note that $y(x) > 0$ $\forall\, a < x < c$. Then

$$\begin{aligned}\alpha \ln y = {} & [\ln c - \ln x][\ln(c - x) - \ln(\ln c - \ln x)] \\ & + [\ln x - \ln a][\ln(x - a) - \ln(\ln x - \ln a)]\end{aligned}$$

and so

$$\begin{aligned}\frac{\alpha y'}{y} &= -\frac{1}{x}[\ln(c - x) - \ln(\ln c - \ln x)] + [\ln c - \ln x]\left[\frac{-1}{c - x} - \frac{-1/x}{\ln c - \ln x}\right] \\ &\quad + \frac{1}{x}[\ln(x - a) - \ln(\ln x - \ln a)] + [\ln x - \ln a]\left[\frac{1}{x - a} - \frac{1/x}{\ln x - \ln a}\right] \\ &= \frac{1}{x}\left[\ln\left[\frac{x - a}{\ln x - \ln a}\right]\right] + \left[\frac{\ln x - \ln a}{x - a}\right] - \frac{1}{x} \\ &\quad + \frac{1}{x} - \frac{\ln c - \ln x}{c - x} - \frac{1}{x}\ln\left[\frac{c - x}{\ln c - \ln x}\right] \\ &= \frac{1}{x}\ln\left[x\frac{a/x - 1}{\ln(a/x)}\right] + \frac{1}{x}\frac{\ln(a/x)}{a/x - 1} - \frac{1}{x}\frac{\ln(c/x)}{c/x - 1} - \frac{1}{x}\ln\left[x\frac{c/x - 1}{\ln(c/x)}\right] \\ &= \frac{1}{x}\,[f(g(a/x)) - f(g(c/x))] = \frac{1}{x}h(x).\end{aligned}$$

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Now $f(g(a/x))$ is an increasing function of x while $f(g(c/x))$ is a decreasing function of x so that $h(x)$ is an increasing function of x. Clearly $\alpha y'/y$ is zero at exactly one point which implies that y' is zero at exactly one point.

LEMMA 2.3. *y' is zero at the point $x = \sqrt{ac}$.*

PROOF. Now $f(g(c/x)) = f(g(a/x)) = f\left(\frac{a}{x}g(a/x)\right)$, from Lemma 2.3 (iii), so that $g(c/x) = (a/x)g(a/x) = g(x/a)$ by Lemma 2.2 (iii). Thus $c/x = x/a$ which gives $x = \sqrt{ac}$.

THEOREM 2.4. *For all values of* $0 < a < x < c$

$$\left(\frac{c-x}{\ln c - \ln x}\right)^{\ln c - \ln x}\left(\frac{x-a}{\ln x - \ln a}\right)^{\ln x - \ln a} < \left(\frac{c-a}{\ln c - \ln a}\right)^{\ln c - \ln a}. \tag{2}$$

PROOF. The result holds if and only if

$$(\ln c - \ln x)\ln\left(\frac{c-x}{\ln c - \ln x}\right) + (\ln x - \ln a)\ln\left(\frac{x-a}{\ln x - \ln a}\right) < (\ln c - \ln a)\ln\left(\frac{c-a}{\ln c - \ln a}\right).$$

Setting $x_0 = a$, $x_1 = x$, $x_2 = c$ and letting $w_i = \ln x_i - \ln x_{i-1}$, $\alpha_i = \frac{x_i - x_{i-1}}{\ln x_i - \ln x_{i-1}}$ and $\Phi(x) = -\ln x$, the result follows from the Jensen's inequality with $\leq$ rather than $<$.

But

$$\alpha y' = \frac{y}{x}[f(g(a/x)) - f(g(c/x))]$$

so that y' is negative on $[a, \sqrt{ac}]$ and positive on $[\sqrt{ac}, c]$. Strict inequality in Theorem 2.4 now follows from the previous results since the derivative is strictly negative on $[a, \sqrt{ac}]$ and positive on the interval $[\sqrt{ac}, c]$. Thus equality holds only at a and c.

3. Convexity

THEOREM 3.1. *The function*

$$y(x) = \left(\frac{c-x}{\ln c - \ln x}\right)^{\frac{\ln c - \ln x}{\ln c - \ln a}}\left(\frac{x-a}{\ln x - \ln a}\right)^{\frac{\ln x - \ln a}{\ln c - \ln a}} \tag{3}$$

is log-convex, and hence convex, on the interval $[a, \sqrt{ac}]$.

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PROOF. Let $w = \alpha \ln y$; then $w' = \alpha y'/y$ and hence from (2) $xw' = f(g(a/x)) - f(g(c/x))$ is an increasing function so that $w' + xw'' \geq 0$. Thus $xw'' \geq -w'$. Now on $[a, \sqrt{ac}]$, $w' \leq 0$ and so $w'' \geq 0$ so that w is convex (and hence log-convex) on the interval $[a, \sqrt{ac}]$.

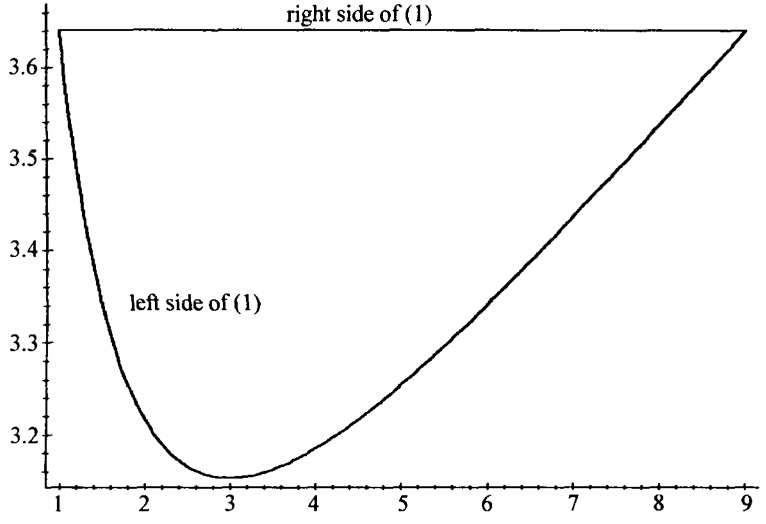

FIGURE 1. Graph of equation (1) with $a = 1$, $c = 9$.

Figure 1 indicates that the function is also probably convex on the interval $[\sqrt{ac}, c]$. However we have not been able to establish this even with the aid of the next result.

LEMMA 3.2. *The curve*

$$y(x) = \left(\frac{c-x}{\ln c - \ln x}\right)^{\frac{\ln c - \ln x}{\ln c - \ln a}} \left(\frac{x-a}{\ln x - \ln a}\right)^{\frac{\ln x - \ln a}{\ln c - \ln a}}$$

is invariant under the transformation $x \to ac/x$.

PROOF.

$$\begin{aligned} z(x) &= \left(\frac{c - ac/x}{\ln c - \ln(ac/x)}\right)^{\frac{\ln c - \ln(ac/x)}{\ln c - \ln a}} \left(\frac{ac/x - a}{\ln(ac/x) - \ln a}\right)^{\frac{\ln(ac/x) - \ln a}{\ln c - \ln a}} \\ &= \left(\frac{c(x-a)/x}{\ln c - \ln a - \ln c + \ln x}\right)^{\frac{\ln c - \ln(ac) + \ln x}{\ln c - \ln a}} \left(\frac{a(c-x)/x}{\ln(ac) - \ln x - \ln a}\right)^{\frac{\ln(ac) - \ln x - \ln a}{\ln c - \ln a}} \\ &= \left(\frac{c(x-a)/x}{\ln x - \ln a}\right)^{\frac{\ln x - \ln a}{\ln c - \ln a}} \left(\frac{a(c-x)/x}{\ln c - \ln x}\right)^{\frac{\ln c - \ln x}{\ln c - \ln a}} \\ &= (c/x)^{\frac{\ln x - \ln a}{\ln c - \ln a}} (a/x)^{\frac{\ln c - \ln x}{\ln c - \ln a}} y(x). \end{aligned} \tag{4}$$

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Now,

$$\frac{\ln x - \ln a}{\ln c - \ln a} + \frac{\ln c - \ln x}{\ln c - \ln a} = 1.$$

Thus, from (4)

$$\begin{aligned}(c/x)^{\frac{\ln x - \ln a}{\ln c - \ln a}} (a/x)^{\frac{\ln c - \ln x}{\ln c - \ln a}} &= (c/x)^{\frac{\ln x - \ln a}{\ln c - \ln a}} (a/x)^{1 - \frac{\ln x - \ln a}{\ln c - \ln a}} \\ &= \frac{(c/x)^{\frac{\ln x - \ln a}{\ln c - \ln a}}}{(a/x)^{\frac{\ln x - \ln a}{\ln c - \ln a}}} \frac{a}{x} \\ &= (a/x)\,(c/a)^{\ln(x/a)/\ln(c/a)} \\ &= \frac{a}{x}\frac{x}{a} = 1 \quad \text{since } b^x = e^{x \ln b}.\end{aligned}$$

Thus $z(x) = y(x)$ and the lemma is proved.

4. Generalizations and applications

The following theorems follow directly from Jensen's inequality and are generalizations of Theorem 2.1.

THEOREM 4.1. *If*

(1) $\Phi : [0, \infty) \to R$ *is a function,*

(2) $f, g : [0, \infty) \to R$ *are increasing functions,*

(3) $A_0 \le A_1 \le \cdots \le A_n$,

then

(1) *if* Φ *is convex then*

$$\begin{aligned}(g(A_n) - g(A_0))\Phi\left(\frac{f(A_n) - f(A_0)}{g(A_n) - g(A_0)}\right) \\ \le \sum_{i=1}^{n}(g(A_i) - g(A_{i-1}))\Phi\left(\frac{f(A_i) - f(A_{i-1})}{g(A_i) - g(A_{i-1})}\right),\end{aligned}$$

(2) *if* Φ *is concave then*

$$\begin{aligned}(g(A_n) - g(A_0))\Phi\left(\frac{f(A_n) - f(A_0)}{g(A_n) - g(A_0)}\right) \\ \ge \sum_{i=1}^{n}(g(A_i) - g(A_{i-1}))\Phi\left(\frac{f(A_i) - f(A_{i-1})}{g(A_i) - g(A_{i-1})}\right),\end{aligned}$$

(3) *if Φ is log-convex then*

$$\Phi\left(\frac{f(A_n)-f(A_0)}{g(A_n)-g(A_0)}\right)^{(g(A_n)-g(A_0))} \le \prod_{i=1}^{n} \Phi\left(\frac{f(A_i)-f(A_{i-1})}{g(A_i)-g(A_{i-1})}\right)^{(g(A_i)-g(A_{i-1}))},$$

(4) *if Φ is log-concave then*

$$\Phi\left(\frac{f(A_n)-f(A_0)}{g(A_n)-g(A_0)}\right)^{(g(A_n)-g(A_0))} \ge \prod_{i=1}^{n} \Phi\left(\frac{f(A_i)-f(A_{i-1})}{g(A_i)-g(A_{i-1})}\right)^{(g(A_i)-g(A_{i-1}))}.$$

PROOF. In Jensen's inequality set $w_i = g(A_i) - g(A_{i-1})$ and $\alpha_i = \frac{f(A_i)-f(A_{i-1})}{g(A_i)-g(A_{i-1})}$ and the result follows.

As a first application let $M, N : R \to R$ be differentiable functions with N strictly monotone. Given any two numbers a and b, there is a number c, according to the mean value theorem, such that

$$\frac{M(b)-M(a)}{N(b)-N(a)} = \frac{M'(c)}{N'(c)}$$

for some c, $a < c < b$. If c is uniquely determined then it is called the (M.N) mean-value mean of a and b [2]. In this case let H be the inverse of M'/N' and write

$$c = H\left(\frac{M(b)-M(a)}{N(b)-N(a)}\right).$$

If M and N are both increasing and H is either log-convex or log-concave, we can apply one of the inequalities in Theorem 4.1 to write

$$H\left(\frac{M(A_n)-M(A_0)}{N(A_n)-N(A_0)}\right) \le \prod_{i=1}^{n} H\left(\frac{M(A_i)-M(A_{i-1})}{N(A_i)-N(A_{i-1})}\right)^{\frac{N(A_i)-N(A_{i-1})}{N(A_n)-N(A_0)}},$$

or

$$H\left(\frac{M(A_n)-M(A_0)}{N(A_n)-N(A_0)}\right) \ge \prod_{i=1}^{n} H\left(\frac{M(A_i)-M(A_{i-1})}{N(A_i)-N(A_{i-1})}\right)^{\frac{N(A_i)-N(A_{i-1})}{N(A_n)-N(A_0)}},$$

where we have made the associations that $\Phi = h, f = M, g = N, A_n = b, A_0 = a$.

Now specializing to the case of $\Phi(x) = x$ (log-concave Φ) in Theorem 4.1 we obtain

$$\frac{f(A_n)-f(A_0)}{g(A_n)-g(A_0)} \ge \prod_{i=1}^{n} \left(\frac{f(A_i)-f(A_{i-1})}{g(A_i)-g(A_{i-1})}\right)^{\frac{g(A_i)-g(A_{i-1})}{g(A_n)-g(A_0)}}$$

and interchanging f and g we can write

$$\frac{f(A_n)-f(A_0)}{g(A_n)-g(A_0)} \leq \prod_{i=1}^{n}\left(\frac{f(A_i)-f(A_{i-1})}{g(A_i)-g(A_{i-1})}\right)^{\frac{f(A_i)-f(A_{i-1})}{f(A_n)-f(A_0)}}.$$

From these expressions we can obtain inequalities for Stolarsky's [2, 13] extended mean value

$$E_{r,s}(a,b)=\left(\frac{r(a^s-b^s)}{s(a^r-b^r)}\right)^{\frac{1}{s-r}}$$

by setting $f(x)=x^s/s$, $g(x)=x^r/r$, $A_n=b$, $A_0=a$ and then raising both sides to the power $1/(s-r)$. For $rs>0$

$$\left(\frac{b^s-u^s}{b^r-u^r}\right)^{\frac{b^r-u^r}{b^r-a^r}}\left(\frac{u^s-a^s}{u^r-a^r}\right)^{\frac{u^r-a^r}{b^r-a^r}} \leq \frac{b^s-a^s}{b^r-a^r} \leq \left(\frac{b^s-u^s}{b^r-u^r}\right)^{\frac{b^s-u^s}{b^s-a^s}}\left(\frac{u^s-a^s}{u^r-a^r}\right)^{\frac{u^s-a^s}{b^s-a^s}}$$

where $a<u<b$.

If $rs<0$, $f(x)=x^s/s$ and $g(x)=x^r/r$ are still both increasing functions and we have a similar inequality

$$\left(\frac{r(b^s-u^s)}{s(b^r-u^r)}\right)^{\frac{b^r-u^r}{b^r-a^r}}\left(\frac{r(u^s-a^s)}{s(u^r-a^r)}\right)^{\frac{u^r-a^r}{b^r-a^r}} \leq \frac{r(b^s-a^s)}{s(b^r-a^r)} \leq \left(\frac{r(b^s-u^s)}{s(b^r-u^r)}\right)^{\frac{b^s-u^s}{b^s-a^s}}\left(\frac{r(u^s-a^s)}{s(u^r-a^r)}\right)^{\frac{u^s-a^s}{b^s-a^s}}$$

where it is now necessary to include r/s or else reverse the inequality.

A further application is obtained by setting $f(x)=x$ and $g(x)=\ln x$ above to obtain

$$\left(\frac{A_n-A_0}{\ln(A_n)-\ln(A_0)}\right)^{\ln(A_n)-\ln(A_0)} \geq \prod_{i=1}^{n}\left(\frac{A_i-A_{i-1}}{\ln(A_i)-\ln(A_{i-1})}\right)^{\ln(A_i)-\ln(A_{i-1})}$$

and

$$\left(\frac{A_n-A_0}{\ln(A_n)-\ln(A_0)}\right)^{A_n-A_0} \leq \prod_{i=1}^{n}\left(\frac{A_i-A_{i-1}}{\ln(A_i)-\ln(A_{i-1})}\right)^{A_i-A_{i-1}}.$$

These two inequalities provide a direct generalization and converse to the main inequality (2) discussed in this paper.

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