Hausdorff dimension of escaping sets
of meromorphic functions II

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Abstract. A function which is transcendental and meromorphic in the plane has at least
two singular values. On the one hand, if a meromorphic function has exactly two singular
values, it is known that the Hausdorff dimension of the escaping set can only be either 2
or 1/2. On the other hand, the Hausdorff dimension of escaping sets of Speiser functions
can attain every number in [0, 2] (cf. [M. Aspenberg and W. Cui. Hausdorff dimension
of escaping sets of meromorphic functions. *Trans. Amer. Math. Soc.* **374**(9) (2021),
6145–6178]). In this paper, we show that number of singular values which is needed to
attain every Hausdorff dimension of escaping sets is not more than 4.

1. *Introduction and main results*
In this paper we study the dynamics of transcendental meromorphic functions. It is well
known that many dynamical behaviours of the function depend, to certain extent, on the
iterative behaviours of its singular values. By definition, $a \in \widehat{\mathbb{C}}$ is a singular value of a
meromorphic function f if it is either a critical or asymptotic value of f.

Much recent attention in transcendental dynamics is directed to the so-called *Speiser
class* $\mathcal{S}$, consisting of meromorphic functions with a finite number of singular values; see,
for instance, [**Bis15**, **Bis17**, **EL92**, **GK86**]. These functions are called *Speiser functions*.
Many familiar functions belong to this class, including, for example, the exponential
family, the cosine family and the tangent family. Speiser functions are studied in great
detail and provide dynamical behaviours similar to those of polynomial and rational maps.

We intend to study functions in the Speiser class with only few singular values.
Even with this restriction, the classes of functions are quite diverse; see discussion in

the next section. One can thus reasonably expect that very rich and diverse dynamical behaviours could occur in this case. Our main result will confirm this in a strong sense. Recall that for a meromorphic function f, the *escaping set* $\mathcal{I}(f)$ of f is the set of points which tend to ∞ under iteration. This set plays a fundamental role in recent studies of transcendental dynamics. Starting with McMullen [McM87], a wide range of research focuses on estimating the Hausdorff dimensions of escaping sets; see, for instance, [Bar08, BKS09, RS10, Sch07] for some entire functions and [BK12, Cui21b, GK18] for certain meromorphic functions. Some of these papers also treat special Speiser functions. A natural question arises: what are the possible values of the Hausdorff dimensions of escaping sets for Speiser functions? This was resolved recently by the present authors in [AC21]: any number in $[0, 2]$ can be achieved. In a larger setting (that is, those meromorphic functions with a bounded set of finite singular values), Bergweiler and Kotus proved a similar result [BK12].

This paper is a natural continuation of [AC21] and focuses on exploring a relation between the number of singular values and possible attainable Hausdorff dimensions for escaping sets. For convenience, we will use $\mathcal{S}_q$ to denote Speiser functions with exactly q singular values on $\widehat{\mathbb{C}}$.

Our starting point is the following theorem, which brings together results of several authors; see [GK18, McM87] (and also [Cui21b, Theorem 1]). By dim E we mean the Hausdorff dimension of the set E.

THEOREM A.

$$\{\dim \mathcal{I}(f) : \ f \in \mathcal{S}_2\} = \{1/2, \ 2\}.$$

Meromorphic functions in $\mathcal{S}_2$ have explicit formulas; see Theorem 2.1 in the next section for a simple proof. As the number of singular values increases, the varieties of functions are also increasing. Thus one can reasonably expect that a more flexible result would hold. This is indeed the case, as shown by the following result.

THEOREM 1.1.

$$\{\dim \mathcal{I}(f) : \ f \in \mathcal{S}_4\} = [0, 2].$$

The existing gap between Theorem A and Theorem 1.1 is the class $\mathcal{S}_3$. It is plausible that the above theorem holds in $\mathcal{S}_3$. However, our construction will not give this.

Another point that we would like to address concerns a question for the invariance of Hausdorff dimensions of escaping sets. To be more specific, it asks whether two quasi-conformally equivalent functions will have escaping sets of equal Hausdorff dimension. We say that two Speiser functions f and g are quasiconformally equivalent if there are quasiconformal mappings $\varphi, \psi : \mathbb{C} \to \mathbb{C}$ such that $\varphi \circ f = g \circ \psi$; see [EL92]. (We can also define topological equivalence by requiring φ and ψ to be homeomorphisms.) The above question was originally asked for entire functions which are not necessarily Speiser functions. Counterexamples were recently given in the meromorphic setting in [AC21]. Here we provide another class of counterexamples.

THEOREM 1.2. *There exist quasiconformally equivalent meromorphic functions* $f, g \in \mathcal{S}_4$ *for which* $\dim \mathcal{I}(f) \neq \dim \mathcal{I}(g)$.

We also remark that if two quasiconformally equivalent meromorphic functions belong to $\mathcal{S}_3$, then they are actually conformally equivalent (see [BC21, Lemma 2.3]). This implies immediately that two such functions have the same order of growth. It is plausible that two such functions will have escaping sets of the same Hausdorff dimension. In this sense, the above Theorem 1.2 may be optimal.

1.1. *Structure of the paper.* In §2 we briefly discuss transcendental meromorphic functions with few singular values. In §3 we construct meromorphic functions with four singular values of arbitrary order. Finally, in §4 we present an outline of the estimate for the Hausdorff dimension of their escaping sets.

2. *Speiser functions with few singular values*

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be transcendental and meromorphic. We say that c is a *critical value* of f if c has a preimage with zero spherical derivative. With this definition, ∞ will be a critical value if there are any multiple poles. $a \in \widehat{\mathbb{C}}$ is an *asymptotic value* of f if there exists a curve γ tending to ∞ such that $f(\gamma)$ tends to a. As a simple example, 0 and ∞ are asymptotic values of e^z. A value s is called a *singular value* if it is either a critical or asymptotic value. See [BE95] for a classification of singularities of the inverse of a meromorphic function. Singular values play a vital role in the dynamics of meromorphic functions; we refer to [Ber93] for more details and explanations.

Meromorphic functions with two singular values. The following simple fact concerning meromorphic functions with two singular values is folklore and it is not easy to locate a reference. Therefore, the outline of a proof is presented for completeness. Recall that $\mathcal{S}_q$ denotes the class of Speiser functions with exactly q singular values.

THEOREM 2.1. *Let* $f \in \mathcal{S}_2$. *Then* f *is of the form* $M \circ \exp \circ A$, *where* M *is Möbius and* A *is linear.*

Sketch of proof. Without loss of generality, we assume that the two singular values of f are 0 and ∞. Otherwise we consider $M_1 \circ f$, where M_1 is a Möbius transformation sending the two singular values of f to 0 and ∞. Then

$$f : \mathbb{C} \setminus \{f^{-1}(0), \ f^{-1}(\infty)\} \to \widehat{\mathbb{C}} \setminus \{0, \infty\}$$

is a covering map. Note that the fundamental group of $\widehat{\mathbb{C}} \setminus \{0, \infty\}$ is isomorphic to $\mathbb{Z}$. The transcendence of f then implies that the fundamental group of $\mathbb{C} \setminus \{f^{-1}(0), \ f^{-1}(\infty)\}$ is trivial and thus $\mathbb{C} \setminus \{f^{-1}(0), \ f^{-1}(\infty)\}$ is simply connected such that the above f is a universal covering to $\widehat{\mathbb{C}} \setminus \{0, \infty\}$. Since meromorphic functions are discrete maps, we have that $\mathbb{C} \setminus \{f^{-1}(0), \ f^{-1}(\infty)\} = \mathbb{C}$. This implies that both singular values are actually omitted. Note that the exponential map exp is a holomorphic universal covering from $\mathbb{C}$ to $\widehat{\mathbb{C}} \setminus \{0, \infty\}$. Now it follows from the essential uniqueness of the universal covering spaces that there exists a holomorphic homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ such that $f = \exp \circ \phi$. Since
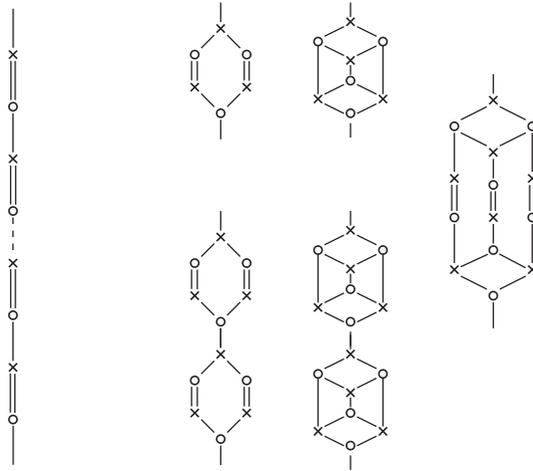
FIGURE 1. Producing meromorphic functions in $\mathcal{S}_3$ by using line complexes. Replacing the dashed line in the left picture by any one of the finite graphs on the right or any finite combination of these graphs will give functions with three singular values which are not topologically equivalent.

a holomorphic homeomorphism of the complex plane must be a linear map, we thus have

$$f(z) = e^{Az+B},$$

where $A(\neq 0)$ and B are complex constants. This completes the proof. $\qquad\square$

It follows immediately that if $f \in \mathcal{S}_2$, then the singular values of f are both asymptotic values. Moreover, if one of the asymptotic values is at ∞, then f is of the form λe^z; if both of them are finite, then f can be written as $M(e^z)$ where M is Möbius, sending 0 and ∞ to two finite points. This, together with the results obtained in [**GK18**, **McM87**], shows that Theorem A holds; see also [**Cui21a**].

The next natural step would be to ask if a similar result stated in Theorem 2.1 will hold if one has more singular values. This need not be true. In fact, even for meromorphic functions with three singular values, one cannot expect a similar function-theoretic rigidity to that of Theorem 2.1. Recent work of Bishop [**Bis15**] shows that there are uncountably many *essentially different* entire functions with two critical values. More precisely, the classes of topologically equivalent entire functions with two critical values are uncountable. See also [**Cui21a**]. One can also use the so-called Maclane–Vinberg method to construct entire functions with two singular values; see [**BFRG15**, Observation 5.2]. For general meromorphic functions with three singular values, one can resort to the theory of line complexes to construct such functions; see [**GO08**, Ch. 7]. Without giving a detailed account of line complexes, we mention that one can show that there exist infinitely many non-equivalent meromorphic functions in $\mathcal{S}_3$ by modifying the left graph in Figure 1: Replacing the dashed line by one of finite graphs shown on the right of Figure 1 will produce meromorphic functions with three singular values. It is clear that there are infinitely many such functions, since one can consider any finite combination of these finite graphs which are then used to replace the dashed line. The meromorphic functions
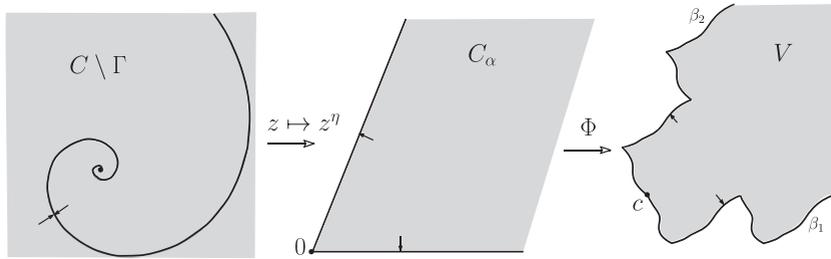
FIGURE 2. Illustration of the construction. We have put $\eta = \alpha\mu/(2\pi)$. The map Φ is quasiconformal, which helps to remove the discontinuity arising on the logarithmic spiral Γ.

produced are those with rational Schwarzian derivatives. For a complete treatment of these functions, we refer to [**GO08**, Ch. 7] and [**Elf34**].

3. *The construction*

For $\delta \in (0, 2\pi]$, put

$$\mathbb{C}_\delta = \{z = re^{i\theta} : r > 0, \ 0 < \theta < \delta\}.$$

In particular, if $\delta = 2\pi$, then $\mathbb{C}_\delta$ is the slit plane $\mathbb{C} \setminus \mathbb{R}^+$. Choosing the natural branch of the logarithm, for any $\alpha \in (0, 2\pi]$, we set

$$h : \mathbb{C}_{2\pi} \to \mathbb{C}_\alpha,$$
$$z \mapsto z^{\alpha/2\pi}$$

which then defines a conformal map.

To construct meromorphic functions in $\mathcal{S}_4$, we will consider the restriction of some carefully chosen Weierstraß elliptic function $\wp$ in the sector domain $\mathbb{C}_\alpha$. Then the function defined as $\wp \circ h$ will be meromorphic in $\mathbb{C}_{2\pi}$. We will then need to extend this function across the positive real axis in order to have a meromorphic function in the plane. However, the extension need not be continuous; in other words, for $x \in \mathbb{R}^+$, $\wp \circ h(x)$ need not coincide when one approaches x respectively from the upper and lower half planes. This will be resolved by introducing a spiral map and then using a quasiconformal surgery. See Figure 2 for an illustration.

Remark 3.1. The spiral map we will introduce will help us to achieve every possible finite order. In a recent paper [**BE17**], Bergweiler and Eremenko also used this idea to solve an open problem in the theory of complex differential equations.

From now on we fix one α and use the corresponding function h as defined above. Then in the sector $\mathbb{C}_\alpha$ we consider a Weierstraß elliptic function $\wp$ with two periods 1 and τ, where τ will satisfy the following condition:

$$\arg \tau = \begin{cases} \alpha & \text{if } \alpha < \pi, \\ \alpha - \pi & \text{if } \alpha > \pi. \end{cases} \tag{3.1}$$

When $\alpha = \pi$, the function defined as $\wp \circ h$ extends continuously across the positive real axis and thus gives a function meromorphic in the plane (which is actually $\wp(\sqrt{z})$). This is an easy case to deal with, so we will assume in what follows that $\alpha \neq \pi$. The main point of condition (3.1) is to make sure that the two prime periods of the chosen Weierstraß$\wp$-function lie on two boundaries of $\mathbb{C}_\alpha$. Now we put

$$e_1 = \wp\left(\frac{1}{2}\right), \quad e_2 = \wp\left(\frac{1+\tau}{2}\right), \quad e_3 = \wp\left(\frac{\tau}{2}\right),$$

which are *finite* critical values of $\wp$ (with another critical value at ∞).

One can then see immediately that

$$g_1 := \wp \circ h : \mathbb{C}_{2\pi} \to \mathbb{C}$$

is well defined and meromorphic in the slit plane. However, g_1 may not be able to extend continuously across the positive real axis, as we mentioned above. One of the main objectives in the construction we will make is to circumvent this problem.

By choosing the natural branch of the power map, let

$$p : \mathbb{C}_{2\pi} \to \mathbb{C}, \quad z \mapsto z^{1/\mu}.$$

It can readily be seen that if μ has real part equal to 1 then the image of $\mathbb{C} \setminus \mathbb{R}^+$ is an open set whose complement is a logarithmic spiral. Let us denote this spiral by Γ. Now note that if $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ are two sequences of complex numbers approaching a point $z \in \mathbb{R}^+$ from different sides of the real axis, the limits of $p(x_n)$ and $p(y_n)$ as $n \to \infty$ may well be different. So the map p may not have a continuous extension to $\mathbb{R}^+$. For our purposes, we put

$$\mu = 1 - i\,\frac{\log|\tau|}{\alpha}.$$

Denote by q the inverse of p; that is, $q(z) = z^\mu$. Then the function

$$g_2 : \mathbb{C} \setminus \Gamma \to \mathbb{C}_{2\pi},$$
$$z \mapsto g_1(q(z))$$

is well defined and meromorphic in the complement of a logarithmic spiral Γ. It may not extend continuously across Γ, but by the choice of μ and τ, with a simple computation, we can extend g_2 continuously to a discrete set of Γ whose points will be mapped by $h \circ q$ to poles of $\wp$. We show below how to remove the discontinuities between this discrete set of points on Γ by using a quasiconformal surgery. In short, we will construct a quasiconformal self-map Φ of $\mathbb{C}_\alpha$ such that the new defined function $\wp \circ \Phi \circ h \circ q$ extends continuously across Γ.

Let P denote the parallelogram formed by four vertices 0, $1/2$, $(1 + \tau)/2$ and $\tau/2$. It follows from the basic properties of Weierstraß elliptic functions that $\delta := \wp(\partial P)$ is a simple closed curve on $\widehat{\mathbb{C}}$, which passes through ∞ on both sides such that $\widehat{\mathbb{C}} \setminus \delta$ consists of two domains A and B. Suppose without loss of generality that $A = \wp(P)$. Note that all critical values of $\wp$ lie on δ. Now we can choose an analytic closed curve γ in $\widehat{\mathbb{C}}$ such that $\gamma \cap \delta = \{e_2, \infty\}$. To achieve this, we first choose an analytic curve γ_1 lying entirely in A with two endpoints being e_2 and ∞; similarly, an analytic curve γ_2 is chosen to lie
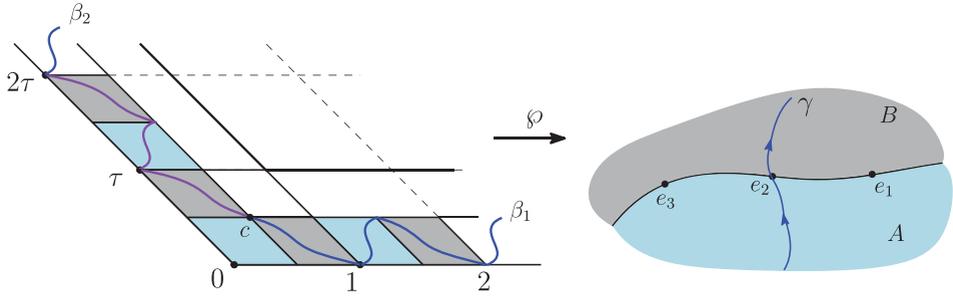
FIGURE 3. The curve γ is chosen such that it passes through the critical value e_2 and ∞. Then we choose two suitable preimages β_1 and β_2 of γ such that they are periodic and both start from c, which is a preimage of the critical value e_2.

completely in B with two endpoints e_2 and ∞. Then γ is defined as the union of γ_1 and γ_2 together with their common endpoints. See Figure 3 for an illustration. Now we consider suitable preimage of γ under the function $\wp$. More precisely, we have the following result; compare this with Proposition 3.1 of [AC21]. Put

$$c = \frac{1+\tau}{2}.$$

PROPOSITION 3.1. *Let γ be as above. Then there exist two piecewise analytic curves β_i with $\wp(\beta_i) = \gamma$ for all i, such that the following assertions hold.*

- $\beta_1 \cap \beta_2 = \{c\}$.
- β_1 *starts from the point c and is periodic with period 1, that is, $z \in \beta_1$ implies that $z + 1 \in \beta_1$. Moreover, β_1 passes through poles at n of $\wp$ for all $n \geq 1$.*
- β_2 *starts from c and is periodic with period τ, that is, $z \in \beta_2$ implies that $z + \tau \in \beta_2$. Moreover, β_2 passes through poles $n\tau$ for all $n \geq 1$.*

Proof. For convenience, we put $\gamma_1 = \gamma \cap A$ and $\gamma_2 = \gamma \cap B$. We will use $W + c$ as the translation by a complex number c for any set $W \subset \mathbb{C}$; in other words, $W + c = \{z + c : z \in W\}$. Recall that P is the parallelogram formed by four points $0, \ 1/2, \ (1+\tau)/2$ and $\tau/2$. Now by elementary properties of Weierstraß elliptic functions we know that $\wp : P + \frac{1}{2} \to B$ is conformal. Therefore, γ_2 has a preimage denoted by β_1^1 in $P + 1/2$ which is an analytic curve connecting $(1+\tau)/2$ and 1. The latter property follows easily since $\gamma_2 \subset B$ connects e_2 and ∞. Similarly, since $\wp : P + 1 \to A$ is conformal we see immediately that γ_1 has a preimage β_1^2 in $P + 1$ which is an analytic curve connecting 1 and $(3+\tau)/2$. Now by periodicity of $\wp$, the curve $\beta_1^1 \cup \beta_1^2$ and its translations by $n \in \mathbb{N}$ will be mapped conformally onto β. Now we define

$$\beta_1 = \bigcup_{n \geq 0} (\beta_1^1 \cup \beta_1^2 + n).$$

It is clear that β_1 satisfies all the required properties.

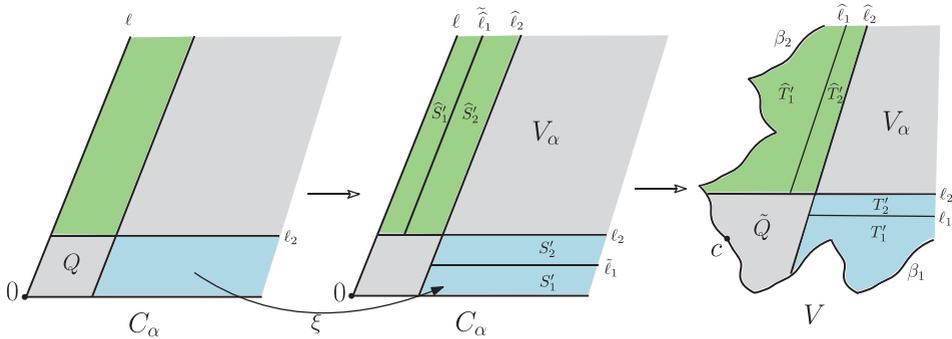In the same way we can define β_2 as claimed. We omit the details here. □

FIGURE 4. Sketch of the construction of the quasiconformal map Φ. S_1' and S_2' are the restrictions of S_1 and S_2 to the sector $\mathbb{C}_\alpha$, while T_1' and T_2' are the corresponding images of S_1' and S_2' under the maps $\phi_{1,1}$ and $\phi_{1,2}$. The same applies to $\widehat{S}_1'$, $\widehat{S}_2'$ and $\widehat{T}_1'$, $\widehat{T}_2'$.

We denote by V the domain contained in $\mathbb{C}_\alpha$ and bounded by β_1 and β_2. In what follows we concentrate on the construction of a quasiconformal mapping

$$\Phi : \mathbb{C}_\alpha \to V$$

such that the function $\wp \circ \Phi \circ h \circ q$ extends continuously across the logarithmic spiral Γ and thus gives a function continuous throughout the whole plane. See Figure 4 for the construction of Φ.

By periodicity of Weierstraß elliptic functions and the construction of β_1, we can define a periodic curve $\tilde{\beta}_1$ which is the extension of β_1 along the direction of the negative real axis. Now choose a real number $a > 0$ such that $a > \max_{z \in \tilde{\beta}_1} \operatorname{Im}(z)$. Define

$$T_1 = \{z : \operatorname{Im}_{w \in \tilde{\beta}_1, \operatorname{Re} w = \operatorname{Re} z}(w) < \operatorname{Im}(z) < a\}.$$

Then by [AC21, Lemma 3.1], there is a real number $a' > 0$ such that with

$$S_1 = \{z : 0 < \operatorname{Im}(z) < a'\}$$

there is a conformal map

$$\phi_{1,1} : S_1 \to T_1$$

which fixes three boundary points 0 and $\pm\infty$ and is periodic with period 1. In other words,

$$\phi_{1,1}(z + 1) = \phi_{1,1}(z) + 1$$

for any $z \in S_1$. Moreover, $\phi_{1,1}$ extends to the boundary as a piecewise diffeomorphism.

Choose another real number b satisfying $b > \max\{a, \ a'\}$ and put

$$S_2 = \{z : a' < \operatorname{Im}(z) < b\}$$

and

$$T_2 = \{z : a < \operatorname{Im}(z) < b\}.$$

We now want to define a quasiconformal map from S_2 to T_2 which interpolates between the identity map on the upper boundary of S_2 and the extension of $\phi_{1,1}$ on the lower boundary

of S_2. For convenience, we denote by ℓ_1 the horizontal curve $\{z : \text{Im}(z) = a\}$ and by $\tilde{\ell}_1$ the curve $\{z : \text{Im}(z) = a'\}$. We also put $\ell_2 = \{z : \text{Im}(z) = b\}$. Suppose that χ_1 is the boundary extension of $\phi_{1,1}$ to $\tilde{\ell}_1$. Then it follows from the construction of $\phi_{1,1}$, and the Schwarz reflection principle (and the fact that $\tilde{\ell}_1$ and ℓ_1 are straight lines), that $\chi_1 : \tilde{\ell}_1 \to \ell_1$ is analytic. Moreover, $\chi_1(z + 1) = \chi_1(z) + 1$ for $z \in \mathbb{R}$. By considering

$$\widetilde{\chi}_1(z) = \chi_1(z + ia') - ia$$

we see that $\widetilde{\chi}_1 : \mathbb{R} \to \mathbb{R}$ is increasing and analytic. We also define

$$\widetilde{\chi}_2(z) = z + i(a' - a).$$

Note that $\widetilde{\chi}_1$ is obtained from χ_1 and the identity map by moving the lower boundaries of S_2 and T_2 to the real axis. Suppose the new strips are S_2'' and T_2'', respectively. Then

$$L(x + iy) = \left(1 - \frac{y}{b - a'}\right)\widetilde{\chi}_1(x) + \frac{y}{b - a'}\widetilde{\chi}_2(x) + i\,\frac{b - a}{b - a'}y$$

is the linear interpolation between $\widetilde{\chi}_1$ and $\widetilde{\chi}_2$. This is actually a quasiconformal map, as can be seen by checking the Jacobian of L, which is

$$\frac{b - a}{b - a'}\left(\left(1 - \frac{y}{b - a'}\right)\widetilde{\chi}_1'(x) + \frac{y}{b - a'}\widetilde{\chi}_2'(x)\right).$$

It is strictly bigger than zero since both $\widetilde{\chi}_1$ and $\widetilde{\chi}_2$ are increasing. Together with the periodicity, this implies that L is quasiconformal. Now we define

$$\phi_{1,2} : S_2 \to T_2,$$
$$z \mapsto L(z - ia') + ia.$$

This is a quasiconformal map.

Along the τ-direction, we use a similar idea to that above to obtain a quasiconformal map. More precisely, we first consider a periodic curve $\tilde{\beta}_2$ which is the natural extension of β_2. With

$$\ell = \{z : \arg(z) = \arg(\tau) \text{ or } \arg(\tau) - \pi\},$$

we define, for some real numbers c', c and d, where $c = (1 + \tau)/2$ and c', d will be determined later,

$$\widetilde{\widetilde{\ell}}_1 = \ell + c',$$
$$\widehat{\ell}_1 = \ell + c,$$
$$\widehat{\ell}_2 = \ell + d.$$

We also denote by $\widehat{S}_1$ the strip between ℓ and $\widetilde{\widetilde{\ell}}_1$, and by $\widehat{T}_1$ the domain bounded by $\tilde{\beta}_2$ and $\widehat{\ell}_1$. Moreover, $\widehat{S}_2$ will be the strip bounded by $\widetilde{\widetilde{\ell}}_1$ and $\widehat{\ell}_2$, while $\widehat{T}_2$ is bounded by $\widehat{\ell}_1$ and $\widehat{\ell}_2$.

Now c' and c are chosen (similarly to a' and a above) such that there exists a conformal map

$$\phi_{2,1} : \widehat{S}_1 \to \widehat{T}_1$$

which is periodic with period τ and fixes three boundary points 0 and $\pm\infty$ (here $\pm\infty$ are understood as infinity along two directions of ℓ). In the same way as the map $\phi_{1,2}$ was defined, we can define a quasiconformal map

$$\phi_{2,2} : \widehat{S}_2 \to \widehat{T}_2.$$

We omit the details here. For later purposes, we put

$$S = \overline{S_1 \cup S_2} \cap \mathbb{C}_\alpha, \quad T = \overline{T_1 \cup T_2} \cap V,$$

and

$$\widehat{S} = \overline{\widehat{S}_1 \cup \widehat{S}_2} \cap \mathbb{C}_\alpha, \quad \widehat{T} = \overline{\widehat{T}_1 \cup \widehat{T}_2} \cap V.$$

Moreover,

$$V_\alpha = \mathbb{C}_\alpha \setminus (S \cup \widehat{S}) = \mathbb{C}_\alpha \setminus (T \cup \widehat{T}).$$

We now consider restrictions of the maps constructed above on the domain $\mathbb{C}_\alpha$. More precisely, we define

$$\phi_1 : S \longrightarrow T; \quad z \longmapsto \begin{cases} \phi_{1,1}(z) & \text{if } z \in S_1, \\ \phi_{1,2}(z) & \text{if } z \in S_2. \end{cases}$$

In a similar way, we have

$$\phi_2 : \widehat{S} \longrightarrow \widehat{T}; \quad z \longmapsto \begin{cases} \phi_{2,1}(z) & \text{if } z \in \widehat{S}_1, \\ \phi_{2,2}(z) & \text{if } z \in \widehat{S}_2. \end{cases}$$

Finally, we define

$$\phi_3 : V_\alpha \longrightarrow V_\alpha,$$
$$z \longmapsto z.$$

Note that the half-strips S and $\widehat{S}$ overlap on a parallelogram

$$Q = S \cap \widehat{S}$$

which contains the origin on the boundary. Both ϕ_1 and ϕ_2 are defined on Q, but they do not necessarily coincide there. Away from Q, in $\mathbb{C}_\alpha \setminus Q$ we have a well-defined quasi-conformal map, which is ϕ_1 in $S \setminus Q$, ϕ_2 in $\widehat{S} \setminus Q$ and the identity on V_α. However, this quasiconformal map still does not satisfy our requirement to remove the aforementioned discontinuity. To proceed, we need to change ϕ_1 or ϕ_2 further. We will change ϕ_1 in what follows.

To this end, we first define suitable inverse branches of $\wp$. Note that $\wp(\tilde{\beta}_i) = \gamma$. Let ψ_i denote an inverse branch of $\wp$ from γ into $\tilde{\beta}_i$, for $i = 1, 2$. Recall that $p : \mathbb{C}_{2\pi} \mapsto \mathbb{C} \setminus \Gamma$, where $p(z) = z^{1/\mu}$. Put $\tilde{p}(z) = p(z^{2\pi/\alpha})$, which maps $\mathbb{C}_\alpha$ onto $\mathbb{C} \setminus \Gamma$. We can extend $\tilde{p}$ to its boundary continuously. Moreover, from the definition of μ, we have that $x \in \mathbb{R}$ and $\tau x \in \ell = \{z : \arg(z) = \arg(\tau)\}$ are both mapped by $\tilde{p}$ onto the same point in Γ. So, in a sense, the 'transition function' $t(x) = \tau x$, for $x \in \mathbb{R}$, identifies points on the lines

bounding $\mathbb{C}_\alpha$ which correspond to the same origin in Γ. Now the desired 'correction function' is defined as

$$\kappa : \mathbb{R} \longrightarrow \mathbb{R},$$
$$x \longrightarrow \phi_1^{-1} \circ \psi_1 \circ \wp \circ \phi_2(\tau x).$$

Since $\tau x \in \ell$, the above function κ is well defined. Roughly speaking, the function κ fixes the difference between ϕ_1 and ϕ_2. With this function, we can now remove the discontinuity by considering a linear interpolation between κ on the real axis and the identity on the horizontal line ℓ_2. More precisely, we define

$$\xi_1(x + iy) = \left(1 - \frac{y}{b}\right)\kappa(x) + \frac{y}{b} x + iy \quad \text{for } 0 \le y \le b.$$

This map is quasiconformal, as one can check that the Jacobian of ξ is non-zero almost everywhere. Moreover, we define

$$\xi_2(x + iy) = x + iy \quad \text{for } y \ge b.$$

In this way, we have just constructed a quasiconformal map of the upper half-plane by setting

$$\xi : \mathbb{H}^+ \longrightarrow \mathbb{H}^+, \quad z \longmapsto \begin{cases} \xi_1(z) & \text{if } 0 \le \operatorname{Im}(z) \le b, \\ \xi_2(z) & \text{if } \operatorname{Im}(z) \ge b. \end{cases}$$

We still need to define a map on the parallelogram Q. Denote by I_1 the side of Q on the real axis (that is, $I_1 = [0, d]$), and by I_2 the side on the line ℓ (that is, $I_2 = \{re^{i\alpha} : 0 \le r \le b/\sin(\alpha)\}$). The other two sides of Q are denoted by I_3, which is parallel to I_1, and I_4, which is parallel to I_2. We put $\hat{I}_1 = \xi(I_1)$, $\hat{I}_2 = I_2$, $\hat{I}_3 = I_3$ and $\hat{I}_4 = \xi(I_4)$. Then the $\hat{I}_i$, $i = 1, \ldots, 4$, form a quadrilateral, denoted by $\hat{Q}$. We continue to define a new quadrilateral $\tilde{Q}$ formed by the $\tilde{I}_i$, $i = 1, \ldots, 4$, where

$$\tilde{I}_i = \phi_1(\hat{I}_i) \quad \text{for } i = 1, 4,$$

and

$$\tilde{I}_i = \phi_2(\hat{I}_i) \quad \text{for } i = 2, 3.$$

Now we can define a boundary map between Q and $\tilde{Q}$ by using $\phi_1 \circ \xi$ on $I_1 \cup I_4$ and $\phi_2 \circ \xi$ on $I_2 \cup I_3$. The boundary map extends to the interior of Q quasiconformally; see [**BF14**, Lemma 2.24]. So we have a quasiconformal map

$$h : Q \to \tilde{Q}.$$

We can now define our promised map

$$\Phi : \mathbb{C}_\alpha \longrightarrow V, \quad z \longmapsto \begin{cases} \phi_1(\xi(z)) & \text{if } z \in S \setminus Q, \\ \phi_2(z) & \text{if } z \in \widehat{S} \setminus Q, \\ h(z) & \text{if } z \in Q, \\ z & \text{if } z \in V_\alpha. \end{cases}$$

One can then check that the map

$$G(z) := \wp \circ \Phi \circ h \circ q(z) \tag{3.2}$$

extends continuously across the logarithmic spiral Γ and thus gives us, by construction, a quasimeromorphic function of the plane.

So by the measurable Riemann mapping theorem (cf. [**Ahl06**]) there exist a quasiconformal homeomorphism Ψ and a meromorphic function f such that $G = f \circ \Psi$. By our construction, G is quasiconformal only in $(h \circ q)^{-1}(W)$, where $W = \mathbb{C}_\alpha \setminus V_\alpha$.

It follows from our construction that f has exactly four critical values and no asymptotic values. In other words, $f \in \mathcal{S}_4$. Moreover, f has only double poles.

To derive some asymptotic properties of f, we will need the well-known Teichmüller–Wittich–Belinskii theorem concerning conformality of a quasiconformal mapping at a point. We refer to [**Ahl06**, **LV73**] for background on quasiconformal mappings and also for this result. For our purpose, a stronger result is required. We first recall some relevant notions. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping. Then the dilatation of φ at a point z is

$$K_\varphi(z) := \frac{|\varphi_z| + |\varphi_{\bar{z}}|}{|\varphi_z| - |\varphi_{\bar{z}}|} = \frac{1 + |\mu_\varphi|}{1 - |\mu_\varphi|},$$

where $\mu_\varphi := \varphi_{\bar{z}}/\varphi_z$ is the complex dilatation of φ.

The above-mentioned result is stated as follows; see [**AC21**, Lemma 2.2].

PROPOSITION 3.2. *Let $\varphi : \mathbb{C} \to \mathbb{C}$ be quasiconformal. Put $A = \{z \in \mathbb{C} : \mu_\varphi \neq 0\}$. If $\iint_{A \setminus \mathbb{D}} dx\, dy/(x^2 + y^2) < \infty$, then $\varphi(z) \sim z$ as $z \to \infty$. Upon normalization, one has $\varphi(z) = z + o(z)$ as $z \to \infty$.*

Note that W is the union of two half-strips, which means that the set $W \setminus \mathbb{D}$ has finite logarithmic area, that is, the integration above holds. So Ψ satisfies the conditions in Proposition 3.2. So we have, up to normalization,

$$\Psi(z) = z + o(z) \quad \text{as } z \to \infty. \tag{3.3}$$

3.1. *Distribution of poles of G.* Now fix $R > 0$ large. We will be interested in counting the number of poles in the closed disk $\overline{D}(0, R)$. This will follow from the following area formula for a domain bounded by two logarithmic spirals. More precisely, let σ be a (non-zero) complex number and $R > 0$. Put $\psi(z) = z^\sigma$ for $z \in \mathbb{C}_{2\pi}$ with chosen principle branch. Let Γ_β and Γ_γ be the ψ-images of two radial lines of arguments β and γ respectively, where $0 \leq \beta < \gamma \leq 2\pi$. See Figure 5. Denote by $A := A_{\gamma-\beta}$ the bounded region bounded by Γ_β, Γ_γ and the circle $\{z : |z| = R\}$, and by $B := B_{\gamma-\beta}$ the preimage of A under ψ. Then we have the following formula for the (Euclidean) area of A and B.

PROPOSITION 3.3.

$$\text{area } A = \frac{(\gamma - \beta)|\sigma|^2}{2\,\mathrm{Re}\,\sigma} R^2;$$

$$\text{area } B = \frac{\mathrm{Re}\,\sigma}{4\,\mathrm{Im}\,\sigma} (e^{2\gamma(\mathrm{Im}\,\sigma/\mathrm{Re}\,\sigma)} - e^{2\beta(\mathrm{Im}\,\sigma/\mathrm{Re}\,\sigma)}) \cdot R^{2/\mathrm{Re}(\sigma)}.$$
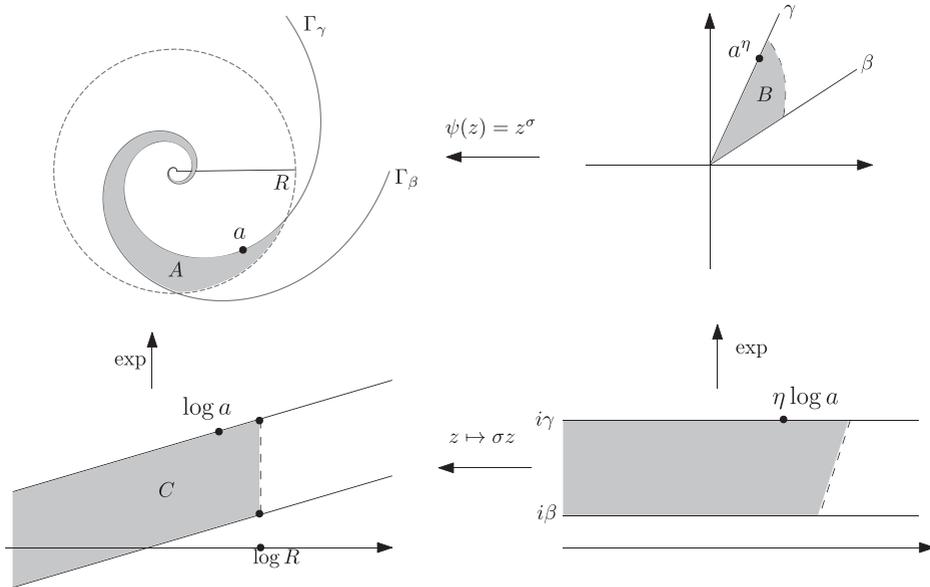
FIGURE 5. The area of the domain A bounded by two logarithmic spirals can be computed by using a logarithmic change of variable.

Proof. Let C be the shaded domain in Figure 5, which is mapped by the exponential map to A. Then

$$\text{area } A = \iint_C |(e^z)'|^2 |\,dz|^2$$

$$= \int_{-\infty}^{\log R} e^{2x} \int_{(\beta-x\,\text{Im}(1/\sigma))/\text{Re}(1/\sigma)}^{(\gamma-x\,\text{Im}(1/\sigma))/\text{Re}(1/\sigma)} dy\,dx = \frac{(\gamma-\beta)|\sigma|^2}{2\,\text{Re}\,\sigma} R^2.$$

Similarly, by using the map $e^{z/\sigma}$ we can obtain the area formula for B. We omit the details here. $\square$

We will also use in what follows some standard notions and notation from Nevanlinna theory [**GO08**, **Hay64**, **Nev70**]. In particular, $n(r, f)$ denotes the number of poles of f in the disk $\overline{D}(0, r)$, $m(r, f)$ the proximity function and $T(r, f)$ is the Nevanlinna characteristic of f. The order of a meromorphic function f is defined by

$$\rho(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

By using Proposition 3.3, we have the following estimate.

PROPOSITION 3.4.

$$n(r, f) = O(r^{(\alpha^2 + (\log |\tau|)^2)/\pi\alpha}), \tag{3.4a}$$

$$\rho(f) = \frac{\alpha^2 + (\log |\tau|)^2}{\pi\alpha}. \tag{3.4b}$$

Proof. To estimate the number of poles of f in a certain disk, by (3.3) it suffices to estimate the poles for the map G for a sufficiently large disk. This can be obtained by comparing the area of the disk and the area of a preimage of a parallelogram for the Weierstrass elliptic function under the spiral function $h \circ q(z) = z^\eta$, where $\eta = \mu\alpha/2\pi$. Then it follows from Proposition 3.3 that

$$n(r, G) \sim C\, r^{\alpha^2 + (\log |\tau|)^2/\pi\alpha},$$

for large r and for some positive constant C. This gives (3.4a).

To obtain (3.4b), we need a result of Teichmüller [**Tei37**] which states that for a meromorphic function $f \in S$, if ∞ is not an asymptotic value and the multiplicities of poles are bounded, then $m(r, f)$ is bounded. This means that the order of f can be estimated by using the integrated counting function $N(r, f)$ of f, where $N(r, f) = T(r, f) - m(r, f)$. Then classical results on the comparison on the growth scale of $N(r, f)$ and $n(r, f)$ show that

$$\rho(f) = \limsup_{r \to \infty} \frac{\log n(r, f)}{\log r}.$$

Now (3.4b) follows from our estimate of $n(r, f)$. $\square$

In the above constructions, we have from the beginning fixed the parameters α and τ. If we now vary these parameters, we can achieve every finite order. To be more specific, we have the following proposition.

PROPOSITION 3.5. *For any given $\rho \in (0, \infty)$, there exist $\alpha \in (0, 2\pi]$ and τ satisfying (3.1) such that*

$$\rho = \frac{\alpha^2 + (\log |\tau|)^2}{\pi\alpha}. \tag{3.5}$$

Moreover, there is a meromorphic function $f \in S_4$ such that $\rho(f) = \rho$.

We now put

$$\eta = \frac{\alpha\mu}{2\pi} = \frac{1}{2\pi}(\alpha - i \log |\tau|). \tag{3.6}$$

Later on we will also need to estimate $|a^\eta|$ in terms of the modulus of a. Note that η defined above is a complex number. We first prove the following result. For simplicity, we also use ρ as given in (3.5).

PROPOSITION 3.6. *For $a \in \mathbb{C}$, we have*

$$|a^\eta| = O(|a|^{\rho/2}).$$

Proof. Let S be the standard strip $S = \{z = x + iy : 0 < y < 2\pi\}$ and put $P = (1/\mu)S = \{z : \mu z \in S\}$. First, recall that exp maps the oblique strip P onto $\mathbb{C} \setminus \Gamma$. It has a continuous extension to the boundary, by the definition of $\mu = 1 - i\kappa$ (whose real part is equal to 1). Here $\kappa = (\log |\tau|)/\alpha$. To make it injective, let us extend exp only to the lower boundary of the strip P, so that, for instance, the preimage of $z = 1$ under exp has its preimage at

the origin, that is, the argument of this preimage is 0. Then the argument of an arbitrary complex number $a \in \mathbb{C}$ is the imaginary part of the 'spiral branch' $\log_P(a)$ of a; that is, $\log_p$ is this particular inverse of exp mapping $\mathbb{C} \setminus \Gamma$ onto P.

By definition $a^\mu = e^{\mu \log a} = e^{(1-i\kappa)(\ln r + i\theta(r))}$, where $\theta = \theta(r)$ now depends on $r :=$ $|a|$ in such a way that, if $\tilde{\theta} \in [0, 2\pi)$ is the natural argument for a, then $\theta(r) = \tilde{\theta} + 2\pi k$, where k is the unique integer such that

$$\tilde{\theta} + 2\pi k \in [\kappa \log r, \kappa \log r + 2\pi).$$

Hence there is some $\delta \in [0, 2\pi)$ such that

$$k = \frac{\kappa \log r - \tilde{\theta} + \delta}{2\pi}.$$

Replacing the expression for k in $\theta(r)$, and with $\kappa = (\log |\tau|)/\alpha$, we have

$$|a^\eta| = |a^{\mu\alpha/2\pi}| = r^{\alpha/2\pi} e^{\theta(r) \log |\tau|/2\pi} = r^{(\alpha^2 + (\log |\tau|)^2)/2\pi\alpha} e^{\delta \log |\tau|/2\pi}. \qquad \square$$

The strategy below follows in a similar way to [AC21, §3]. We first estimate asymptotic behaviours of the constructed function f near its poles, which will be useful in estimating the Hausdorff dimension of escaping sets later on. Then we show that by varying parameters α and τ we obtain functions in $\mathcal{S}_4$ which are actually equivalent. This is crucial for completing the proof of Theorem 1.2.

3.2. *Local behaviours near poles.* We first consider the local behaviours of the quasimeromorphic map G near its poles. In the following, $A \sim B$ means that A and B have comparable modulus. Let z_0 be a pole of G. By (3.2), $\Phi(h(q(z_0)))$ is a pole of the function $\wp$. Put $\zeta = \Phi(h(q(z)))$ and $\zeta_0 = \Phi(h(q(z_0)))$. Since ζ_0 is a pole of $\wp$, we see that there exists a constant C such that

$$\wp(\zeta) \sim \left(\frac{C}{\zeta - \zeta_0}\right)^2 \quad \text{as } \zeta \to \zeta_0. \tag{3.7}$$

By the construction of Φ, we have that

$$\zeta - \zeta_0 \sim C'(h(q(z)) - h(q(z_0))) = C'(z^{\alpha\mu/(2\pi)} - z_0^{\alpha\mu/(2\pi)}) \quad \text{as } z \to z_0,$$

where C' is some constant. This, together with (3.2) and (3.7), shows that

$$G(z) \sim \left(\frac{C''}{z^{\alpha\mu/(2\pi)} - z_0^{\alpha\mu/(2\pi)}}\right)^2 \quad \text{as } z \to z_0, \tag{3.8}$$

where C'' depends only on C and C'. Note that z_0 is a double pole of G. We may thus assume that

$$G(z) \sim \left(\frac{a(z)}{z - z_0}\right)^2 \quad \text{as } z \to z_0 \tag{3.9}$$

for some function a which is holomorphic near in some neighbourhood of z_0 and, moreover, $a(z_0) \neq 0$. By comparing (3.8) and (3.9), we see that

$$a(z) \sim C'' \frac{z - z_0}{z^{\alpha\mu/(2\pi)} - z_0^{\alpha\mu/(2\pi)}} \quad \text{as } z \to z_0.$$

Recall that η is defined in (3.6). By taking limit and using L'Hospital's rule, we obtain

$$a(z_0) \sim C''' z_0^{1-\alpha\mu/(2\pi)} = C''' z_0^{1-\eta}.$$

Denote $w = \Phi(z)$ and $w_0 = \Phi(z_0)$. Then w_0 is a double pole of f. Assume that

$$f(w) \sim \left(\frac{b(w)}{w - w_0} \right)^2 \quad \text{as } w \to w_0.$$

Here $b(w)$ is a function holomorphic in w_0 and $b(w_0) \neq 0$. So with (3.9) we see that

$$b(w) \sim a(z) \frac{w - w_0}{z - z_0}$$

as $z \to z_0$. Note that (3.3) implies that $w \to w_0$ as $z \to z_0$. Again, (3.3) will also imply that

$$b(w_0) \sim a(z_0) \lim_{z \to z_0} \frac{w - w_0}{z - z_0} = a(z_0).$$

It follows that near a pole w_0 of f, we have the following asymptotic relation:

$$f(w) = \left(\frac{C_1 w_0^{1-\eta}}{w - w_0} \right)^2 \quad \text{as } w \to w_0, \tag{3.10}$$

where C_1 is some constant.

3.3. *Equivalence.* By choosing distinct parameters (α_i, τ_i) we may obtain that the ρ_i as defined in (3.4b) are different for $i = 1, 2$. So by the construction we can have two functions $f_i \in \mathcal{S}_4$ whose orders are ρ_i. Following the idea of proof given in [**AC21**, Theorem 3.2], we see that f_1 is quasiconformally equivalent to f_2. We omit this proof here, merely stating the result below. We leave the details to the interested reader.

PROPOSITION 3.7. *f_1 is quasiconformally equivalent to f_2.*

One may notice that the above constructions only give meromorphic functions in $\mathcal{S}_4$ of finite but non-zero order. To achieve zero or full Hausdorff dimension of escaping sets, we may need functions of zero or infinite order. We mention several examples below which will suffice for our purposes.

3.4. *Zero order.* To have a zero-order meromorphic function in $\mathcal{S}_4$, one can consider the one used in [**AC21**, §4.1]. Roughly speaking, the function f is obtained by precomposing a suitably chosen Weierstraß elliptic function with an inverse branch of the function cosh. The obtained function is meromorphic in a certain slit plane. One can then use some basic properties of these two functions to show that it can be extended continuously across the slit and thus gives a meromorphic function in the plane. That this function belongs to $\mathcal{S}_4$

is clear since all singular values are just critical values of the Weierstraß elliptic function. It is also clear that f has zero order by checking the counting function of poles, which can be computed explicitly. We omit the details here.

3.5. *Infinite order.*   As for infinite-order functions in $\mathcal{S}_4$, consider the function

$$f(z) = \wp(2\pi \cosh(z)),$$

where $\wp$ is a Weierstrass elliptic function with two periods 2π and τ such that τ is not a real multiple of 2π. It follows by construction that f has four critical values which are exactly the critical values of $\wp$ and has no asymptotic values. That $\rho(f) = \infty$ follows directly from [**EF64**, Corollary 1.2]. This can also be obtained by checking the counting functions of poles.

4. *Estimate of the dimension*

We will need to estimate the Hausdorff dimension of the escaping sets for the Speiser functions in $\mathcal{S}_4$ constructed in the previous section, which then finishes the proof of Theorem 1.1. This will follow from the following estimate.

PROPOSITION 4.1. *Let $\rho \in [0, \infty)$. Then there exists a meromorphic function $f \in \mathcal{S}_4$ such that* dim $\mathcal{I}(f) = (2\rho/(1 + \rho))$.

Before we prove this, we state the following direct consequence.

COROLLARY 4.1. *For any $d \in [0, 2)$, there exists $f \in \mathcal{S}_4$ such that* dim $\mathcal{I}(f) = d$.

Therefore, Theorem 1.1 is proved except for the case where $d = 2$, which will be discussed in the final part of this section. By combining this with Proposition 3.7, Theorem 1.2 follows.

The rest of the proof will be devoted to the proof of the above Proposition 4.1. We only give a sketch of the proof, as the estimate of the Hausdorff dimension for escaping sets for our constructed functions will use the same idea as in [**AC21**]: the upper bound follows from a result of Bergweiler and Kotus [**BK12**], while the lower bound uses the spherical version of a well-known result of McMullen [**McM87**].

Proof of Proposition 4.1. For any given $\rho \in [0, \infty)$, there exist α and μ satisfying

$$\rho = \frac{\alpha^2 + (\log|\tau|)^2}{\pi\alpha}.$$

The constructions in §3 then gives us a Speiser function f in $\mathcal{S}_4$ whose order is ρ. This function f has only critical values and no asymptotic values. Moreover, all poles have the multiplicity 2.

Upper bound. It is clear that the above function f satisfies the conditions of [**BK12**, Theorem 1.1]. So we have

$$\dim \mathcal{I}(f) \le \frac{2\rho}{1 + \rho}.$$

Lower bound. Suppose that a_j are the poles of f, where $\cdots \leq |a_j| \leq |a_{j+1}| \leq \cdots$. Then it follows from (3.10) and Proposition 3.6 that

$$f(z) \sim \left(\frac{b_j}{z - a_j} \right)^2 \quad \text{as } z \to z_j,$$

where

$$|b_j| \sim |a_j^{1-\eta}| \sim |a_j|^{1-\rho/2}.$$

Since f is Speiser, we may take a large $R_0 > 0$ such that $D(0, R_0)$ contains all singular values of f. Now with $B(R) = \widehat{\mathbb{C}} \setminus \overline{D}(0, R)$ and $R > R_0$, each component of $f^{-1}(B(R))$ is bounded, simply connected and contains one pole of f. Let U_j be the component containing the pole a_j. By using Koebe's distortion and one-quarter theorem, we may obtain

$$D\left(a_j, \frac{|b_j|}{4\sqrt{R}} \right) \subset U_j \subset D\left(a_j, \frac{2|b_j|}{\sqrt{R}} \right). \tag{4.1}$$

See [**AC21**, §4] for more details. Moreover, if g_j is an inverse branch of f from some domain Ω to U_j, where $\Omega \subset B(R)$, then

$$|g'_j(z)| \leq B_1 \frac{|b_j|}{|z|^{3/2}} \quad \text{for } z \in \Omega. \tag{4.2}$$

Here $B_1 > 0$ is some constant.

We denote by $\mathrm{diam}(E)$ the diameter of the set E in the plane and by $\mathrm{diam}_\chi(E)$ the spherical diameter of E. The above estimates (4.1) and (4.2) will give us good control over the sizes of the pullbacks of U_k under f for large k. More precisely, for sufficiently large k, we have

$$\mathrm{diam}\, g_j(U_k) \leq \sup_{z \in U_k} |g'_j(z)|\, \mathrm{diam}\, U_k \leq B_1 \frac{|b_j|}{|a_k|^{3/2}} \frac{|b_k|}{\sqrt{R}},$$

and if the indices $j_1, \ldots, j_\ell$ are chosen such that U_{j_k} is contained in $B(R)$, where $k = 1, \ldots, \ell$, we obtain, in terms of the spherical metric,

$$\mathrm{diam}_\chi (g_{j_1} \circ g_{j_2} \circ \cdots \circ g_{j_{\ell-1}})(U_{j_\ell}) \leq B_1^{\ell-1} \frac{32}{\sqrt{R}} \prod_{k=1}^{\ell} \frac{|b_{j_k}|}{|a_{j_k}|^{3/2}}. \tag{4.3}$$

We now consider the set

$$\mathcal{I}_R(f) = \{z \in \mathcal{I}(f) : f^n(z) \in B(R) \text{ for all } n \in \mathbb{N}\}.$$

In other words, we are considering those escaping points whose iterates always stay in $B(R)$. Apparently, this is a subset of $\mathcal{I}(f)$. Let E_l be the collection of all components V of $f^{-l}(B(R))$ for which $f^k(V) \subset B(R)$ holds for $0 \leq k \leq l-1$. We will estimate the sizes of components of E_l by using (4.3). For such a component V, by definition there exist $j_1, \ldots, j_{l-1}$ such that

$$f^k(V) \subset U_{j_{k+1}} \quad \text{for } k = 0, 1, \ldots, l-1.$$

So, using (4.3), one can have, for some constants B_2 and B_3,

$$\mathrm{diam}_\chi(V) \leq B_1^{\ell-1} \frac{32}{\sqrt{R}} \prod_{k=1}^{\ell} \frac{|b_{j_k}|}{|a_{j_k}|^{3/2}} = B_1^{\ell-1} \frac{32}{\sqrt{R}} \prod_{k=1}^{\ell} \frac{|B_2|^\ell}{|a_{j_k}|^{1/2+\rho/2}} \leq \left(\frac{B_3}{|R|^{1/2+\rho/2}}\right)^\ell.$$

Put

$$d_\ell = \left(\frac{|B_3|}{|R|^{1/2+\rho/2}}\right)^\ell. \tag{4.4}$$

In addition to the term d_ℓ defined above, McMullen's lower-bound estimate for the Hausdorff dimension also involves a lower-bound estimate for the density of $\overline{E}_{l+1}$ in V. Here $\overline{E}_\ell$ represents the union of all elements of E_ℓ. We will also define $E = \bigcap_\ell \overline{E}_\ell$. For this purpose, we consider an annulus $A(s) := \{z : s < |z| < 2s\}$ which is contained in $B(R)$, that is, $s > R$. Then the number of U_j contained in $A(s)$ is $B_4(n(2s, f) - n(s, f)) = B_5 s^\rho$, where B_4 and B_5 are positive numbers. So we have

$$\mathrm{diam}\, U_j \geq \frac{|b_j|}{2\sqrt{R}} = B_6 \frac{|a_j|^{1-\rho/2}}{2\sqrt{R}} \geq \frac{B_7}{s^{\rho/2-1/2}}.$$

where B_6, B_7 are constants. Therefore,

$$\mathrm{dens}(\overline{E}_1, A(s)) = \frac{\mathrm{area}(\overline{E}_1 \cap A(s))}{\mathrm{area}\, A(s)} \geq \frac{B_5 s^\rho \pi (B_7/s^{\rho/2-1/2})^2}{3\pi s^2} = B_8\, s,$$

where $B_8 > 0$ is a constant. By repeating the argument used in [**AC21**, §4], which we do not repeat here, we have the following estimate, for some constant $B_9 > 0$:

$$\mathrm{dens}_\chi(\overline{E}_{\ell+1}, V) \geq \frac{B_9}{R} =: \Delta_\ell. \tag{4.5}$$

Now we can apply McMullen's result by using (4.4) and (4.5) to obtain

$$\dim E \geq 2 - \limsup_{\ell\to\infty} \frac{\sum_{j=1}^{\ell+1} |\log \Delta_j|}{|\log d_\ell|} \geq 2 - \frac{\log B_9 - \log R}{\log B_3 - ((1/2) + (\rho/2)) \log R}.$$

With $R \to \infty$, we have $\dim E \geq 2\rho/(1 + \rho)$. The next step is to use this estimate to give the estimate for the Hausdorff dimension of the escaping set by taking a sequence (R_k) which tends to infinity increasingly and consider those points whose k-iterate lies in $B(R_k)$. This goes in the same way as in the aforementioned reference and so we omit the details. We conclude directly that

$$\dim \mathcal{I}(f) \geq \frac{2\rho}{1 + \rho}.$$

Combined with the upper bound discussed above, we have thus finished the proof. $\quad\square$

To complete the proof of Theorem 1.1, we still need to find a function in $\mathcal{S}_4$ with a full-dimensional escaping set. For this purpose, we put

$$f_1(z) = \wp_1(2\pi \cosh(z))$$

This function belongs to $\mathcal{S}_4$, as mentioned before.

PROPOSITION 4.2.

$$\dim \mathcal{I}(f_1) = 2.$$

The proof of this result is obtained in the same manner as in [AC21, §4.3]. We leave the details to the interested reader.

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