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Examples of unstable Hamiltonian-minimal Lagrangian tori in $\mathbf{C}^2$

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Abstract. A new family of Hamiltonian-minimal Lagrangian tori in the complex Euclidean plane is constructed. They are the first known unstable ones and are characterized in terms of being the only Hamiltonian-minimal Lagrangian tori (with non-parallel mean curvature vector) in $\mathbf{C}^2$ admitting a one-parameter group of isometries.

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1. Introduction

It is known that the only *stable* compact minimal submanifolds of the complex projective space $\mathbf{CP}^n$ are the complex ones (see [4]). In particular, compact minimal Lagrangian submanifolds are unstable. In [7], Oh introduced the notion of Hamiltonian stability for Lagrangian minimal submanifolds in Kaehler manifolds, as those ones such that the second variation of volume is nonnegative for Hamiltonian deformations. He proved that the totally geodesic $\mathbf{RP}^n$ and the Clifford torus in $\mathbf{CP}^n$ are Hamiltonian stable Lagrangian minimal submanifolds. He also conjectured that the Clifford torus in $\mathbf{CP}^n$ is even volume minimizing under Hamiltonian deformations, such as it happens with $\mathbf{RP}^n$. In this context, the second author proved in [10] that the Clifford torus is the only Hamiltonian stable minimal Lagrangian torus in $\mathbf{CP}^2$.

While minimal submanifolds are critical points of the functional volume for any (compactly supported) variation, motivated by the previous study in [7], Oh introduces in [8] the notion of Hamiltonian-minimal (H-minimal briefly) Lagrangian submanifolds as critical points of the functional volume for Hamiltonian deformations, states the Euler–Lagrange Equation of this variational problem and derives a second variation formula for H-minimal Lagrangian submanifolds, which generalizes the well-known one for the minimal case. Applying it, he proves that the

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standard tori $S^1(r_1) \times \cdots \times S^1(r_n)$ in the complex Euclidean space $\mathbf{C}^n$ (which are H-minimal but non minimal, in fact they have parallel mean curvature vector) are all of them not only Hamiltonian stable but also local minima of volume under Hamiltonian deformations, so that he extends his above conjecture to all these tori.

Also, Oh comments in [8] Section 2 that H-minimal Lagrangian submanifolds seem to exist more often than minimal Lagrangian submanifolds do, but no more examples, after those with parallel mean curvature vector, were known so far.

If we pay our attention in the special case of dimension two, we firstly find out that Lagrangian minimal surfaces in the complex Euclidean plane $\mathbf{C}^2$ are essentially (see [1]) complex curves. On the other hand, the result of Minicozzi [5] proving that the Oh's Conjecture about the minimizing area property of the Clifford torus in $\mathbf{C}^2$ would follow from the well-known Willmore's Conjecture restricted for Lagrangian tori, shows the hardness of Oh's Conjecture.

Our contribution in this paper consists of an Existence and Uniqueness Theorem about H-minimal Lagrangian tori in $\mathbf{C}^2$. In fact, we make a construction of a three-parameter family (Theorem 1)

$$\mathcal{F}^{\alpha}_{\theta,\beta} \colon \mathbf{R}^2 \to \mathbf{C}^2,$$

with

$$(\theta, \beta, \alpha) \in [0, \tfrac{1}{2}\pi) \times (0, \tfrac{1}{2}\pi) \times (\Leftrightarrow\tfrac{1}{2}\pi, \tfrac{1}{2}\pi); \quad \theta < \beta, \quad |\alpha| < \beta,$$

of new examples of Lagrangian H-minimal (conformal) immersions, all of them with non-parallel mean curvature vector, such that $\mathcal{F}^{\alpha}_{\theta,\beta}$ produces a torus $T^{\alpha}_{\theta,\beta}$ if and only if

$$\left(\frac{\sin\alpha}{\sin\beta}, \frac{\cos\alpha}{\cos\beta}\right) \in \mathbf{Q}^2.$$

The Lagrangian tori with parallel mean curvature vector can clearly be seen as the limits $T^{\theta}_{\theta,\theta}$.

About the construction, we remember that any round circle on $\mathbf{S}^2$ is H-minimal. In this sense, H-minimal Lagrangian submanifolds can be considered as a high dimensional symplectic generalization of such curves. Using exactly any parallel in $\mathbf{S}^2$, taking its horizontal lift by the Hopf fibration to $\mathbf{S}^3 \subset \mathbf{C} \oplus \mathbf{C}$ (well-defined up to rotations), $\gamma(y) = (\gamma_1(y), \gamma_2(y))$, and using a couple of regular curves $\varsigma_j = \varsigma_j(x)$, $j = 1, 2$, in $\mathbf{C}^*$ satisfying certain conditions, we can describe this family in the following easy way

$$\mathcal{F}^{\alpha}_{\theta,\beta}(x, y) = (\varsigma_1(x)\gamma_1(y), \varsigma_2(x)\gamma_2(y)).$$

Precisely, the first above rationality condition just means that the horizontal lift $(\gamma_1(y), \gamma_2(y))$ is a closed curve in $\mathbf{S}^3$.

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We point out that the immersions $\mathcal{F}^{\alpha}_{\theta,\beta}$ with $\theta=\alpha$ correspond to the particular case $\varsigma_1(x)=\varsigma_2(x)$, and then this construction coincides with the made in ([9], Prop. 3) by Ros and the second author.

Next, in Theorem 2 we prove that our tori $T^{\alpha}_{\theta,\beta}$ are the only Lagrangian H-minimal tori (with non-parallel mean curvature vector) in $\mathbf{C}^2$ admitting a one-parameter group of isometries.

Finally, in Proposition 3 we make a small contribution to Oh's Conjecture proving that the tori $T^{\alpha}_{\theta,\beta}$ are Hamiltonian unstable and their Hamiltonian nullity is positive.

2. Hamiltonian-minimal Lagrangian surfaces

Let $\mathbf{C}^2$ be the two-dimensional complex Euclidean plane endowed with a canonical structure of Kähler manifold $(\langle\,,\,\rangle, J)$, where we denote by $\langle\,,\,\rangle$ the Euclidean metric in $\mathbf{C}^2\equiv\mathbf{R}^4$ and by J a canonical complex structure. The Kähler form Ω is defined by $\Omega(X,Y)=\langle X, JY\rangle$, for any tangent vector fields X and Y.

Let Σ be an orientable surface and $\phi\colon\Sigma\to\mathbf{C}^2$ a Lagrangian immersion, i.e. an immersion with $\phi^*\Omega=0$. We also denote by $\langle\,,\,\rangle$ the induced metric in Σ by the Euclidean one. Then $\phi^*T\mathbf{C}^2=T\Sigma\oplus T^{\perp}\Sigma$, where $T\Sigma$ and $T^{\perp}\Sigma$ are the tangent and normal bundles of ϕ respectively. Let $\widetilde{\nabla}$ denote the connection on $\phi^*T\mathbf{C}^2$ induced by the Levi–Civita connection of $\mathbf{C}^2$, and $\widetilde{\nabla}=\nabla\oplus\nabla^{\perp}$ the corresponding decomposition.

The most elementary properties of ϕ are

(a) J defines a bundle-isomorphism from the tangent bundle to the normal bundle of ϕ such that

$$J\circ\nabla=\nabla^{\perp}\circ J.$$

(b) If σ is the second fundamental form of ϕ and A_η is the Weingarten endomorphism associated to a normal vector field η, then for any tangent vector fields X and Y

$$\sigma(X,Y)=JA_{JX}Y.$$

So, the trilinear form $C(X,Y,Z)=\langle\sigma(X,Y),JZ\rangle$ is totally symmetric on $T\Sigma$.

(c) If H denotes the mean curvature vector of ϕ and α_H is the one-form on Σ dual to the tangent vector field JH, i.e. $\alpha_H=\Omega(\Leftrightarrow, H)$ (up to constants, α_H is the well-known *Maslov form*, see [6]), then from (a), (b) and the Codazzi Equation of ϕ, we have that α_H is a closed one-form.

Studying the problem of minimizing volume of Lagrangian submanifolds under Hamiltonian deformations in Kaehler manifolds, Oh [8] introduced the notion of *Hamiltonian-minimal* (abbreviated as H-*minimal*) Lagrangian submanifold

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(E-minimal Lagrangian submanifolds, in the nomenclature of Chen and Morvan in [2]).

DEFINITION 1. A Lagrangian immersion $\phi\colon \Sigma \to \mathbf{C}^2$ is said to be Hamiltonian-minimal (or H-*minimal*) if it is a critical point of the area functional restricted to (compactly supported) Hamiltonian variations of ϕ, i.e. variations with normal variational vector field ξ such that the one-form $\alpha_\xi = \Omega(\Leftrightarrow, \xi)$ on Σ is exact.

The Euler–Lagrange equation for the variational problem of the H-minimal Lagrangian submanifolds was deduced by Oh in [8] (see (ii) in the following proposition).

PROPOSITION 1. *For a Lagrangian immersion* $\phi\colon \Sigma \to \mathbf{C}^2$ *the following properties are equivalent*:

(i) ϕ *is* H-*minimal*.
(ii) α_H *is coclosed, i.e.* $\delta\alpha_H = 0$, *where* δ *is the codifferential operator in* Σ.

In particular, if Σ *is a* compact *orientable surface then* (ii) *is equivalent to that* α_H *is a harmonic* 1-*form, i.e.* $\triangle\alpha_H = 0$, *and so the genus of* Σ *will be greater than or equal to one.*

The second variation operator for Hamiltonian deformations of a H-minimal Lagrangian immersion $\phi\colon \Sigma \to \mathbf{C}^2$ can be seen as the quadratic form associated to an operator acting on the compactly supported exact one-forms on the surface (or the gradients of functions on the surface) and it is given by (see Theorem 3.4 in [8])

$$Q(\nabla f) = \int_\Sigma 4\langle JH, \nabla f\rangle^2 \Leftrightarrow \langle \nabla(\triangle f), \nabla f\rangle \Leftrightarrow 4\langle \sigma(\nabla f, \nabla f), H\rangle,$$

where $f \in C_0^\infty(\Sigma)$ and ∇f and $\triangle f$ denote the gradient and Laplacian of the function f in the surface Σ respectively.

H-minimal Lagrangian surfaces in $\mathbf{C}^2$ can be also characterized in terms of their Gauss maps. In what follows we will use the notation and ideas of Harvey and Lawson ([3]). Let $\mathrm{Lag} = \mathcal{L}(\mathbf{C}^2)$ be the space of oriented Lagrangian planes in $(\mathbf{C}^2, \Omega)$, whose Riemannian structure is defined by identifying Lag with the symmetric space $U(2)/\mathrm{SO}(2)$, where $U(2)$ (respectively $\mathrm{SO}(2)$) is the unitary group (respectively special orthogonal group) of order 2.

Let

$$\nu\colon \Sigma \to \mathrm{Lag}$$

be the Gauss map of a Lagrangian immersion $\phi\colon \Sigma \to \mathbf{C}^2$ of an oriented surface Σ.

The determinant map

$$\det\colon \mathrm{Lag} \to \mathbf{S}^1$$

defines a $\mathrm{SU}(2)/\mathrm{SO}(2)$-fiber bundle over the circle $\mathbf{S}^1$, and so we obtain a map

$$g = \det \circ \nu \colon \Sigma \to \mathbf{S}^1 \subset \mathbf{C}.$$

If $\{z_1, z_2\}$ are standard coordinates on $\mathbf{C}^2$, then g is written as

$$g(p) = (\mathrm{d}z_1 \wedge \mathrm{d}z_2)(\mathrm{d}\phi_p(T_p\Sigma)).$$

A straightforward computation shows that for any vector field X on Σ

$$X(g) = \Leftrightarrow 2\alpha_H(X)ig$$

and derivating again we obtain

$$\Delta g + 4|\alpha_H|^2 g = \Leftrightarrow 2\delta\alpha_H ig.$$

So, from the above formulas, we get the following characterization.

PROPOSITION 2. *Let $\phi\colon \Sigma \to \mathbf{C}^2$ be a Lagrangian immersion of an oriented surface Σ. Then*

(a) *ϕ is minimal if and only if g is a constant map* [3, Prop. 2.17].
(b) *ϕ is* H-*minimal if and only if g is harmonic*.

Examples of H-minimal Lagrangian surfaces in $\mathbf{C}^2$ are, of course, the minimal ones and those with parallel mean curvature vector. The standard tori $S^1(r_1) \times S^1(r_2)$ are H-minimal but they are not minimal. But, so far, there were no other known examples. In this paper, we deal with a particular family of new H-minimal Lagrangian tori (the easiest possible topology), all of them with non-parallel mean curvature vector.

3. Existence

We start this section explaining a certain construction that will let us introduce later a family of H-minimal Lagrangian immersions in $\mathbf{C}^2$.

Let $\Pi\colon \mathbf{S}^3 \to \mathbf{S}^2$ be the Hopf fibration of $\mathbf{S}^2$ given by $\Pi(z,w) = (2z\overline{w}; |z|^2 \Leftrightarrow |w|^2)$, $(z,w) \in \mathbf{S}^3 \subset \mathbf{C}^2$ and let $\varphi \in (\Leftrightarrow\frac{1}{2}\pi, \frac{1}{2}\pi)$. If we consider the curve $\gamma\colon \mathbf{R} \to \mathbf{S}^3$ given by

$$\gamma(s) = (\cos\psi\, \mathrm{e}^{i\tan\psi s}, \sin\psi\, \mathrm{e}^{-i\cot\psi s}),$$

with $\psi = (\frac{1}{2}\pi \Leftrightarrow \varphi)/2$, then it is clear that $\Pi(\gamma(s)) = (\cos\varphi\, \mathrm{e}^{2is/\cos\varphi}, \sin\varphi)$ and also we have that $\langle\gamma', J\gamma\rangle = 0$. This just means that $\gamma = \gamma(s)$ is a *horizontal lift* of the parallel of latitude φ in $\mathbf{S}^2$. It is unique up to rotations in $\mathbf{S}^3$ and reparametrizations (here s is the arc parameter).

We are going to use it joint with a couple of regular curves in $\mathbf{C}^*$ in order to construct Lagrangian immersions in the complex Euclidean plane. We will sometimes follow the standard notation $s_\vartheta = \sin\vartheta$ and $c_\vartheta = \cos\vartheta$ for the sine and cosine of an angle ϑ respectively.

LEMMA 1. *Let* $\varsigma_1 = \varsigma_1(t)$, $\varsigma_2 = \varsigma_2(t)\colon \mathbf{R} \to \mathbf{C}^*$ *be regular curves in* $\mathbf{C}^*$ *such that* $\varsigma_1'\overline{\varsigma_1} = \varsigma_2'\overline{\varsigma_2}$ *and let* $\varphi \in (\Leftrightarrow\frac{1}{2}\pi, \frac{1}{2}\pi)$. *If we define* $F\colon \mathbf{R}^2 \to \mathbf{C}^2$ *by*

$$F(t,s) = (\cos\psi\, \varsigma_1(t)\, \mathrm{e}^{i\tan\psi\, s}, \sin\psi\, \varsigma_2(t)\, \mathrm{e}^{-i\cot\psi\, s}), \qquad \psi = (\tfrac{1}{2}\pi \Leftrightarrow \varphi)/2,$$

then

(1) $F\colon (\mathbf{R}^2, E\,\mathrm{d}t^2 + G\,\mathrm{d}s^2) \to \mathbf{C}^2$ *is a Lagrangian isometric immersion which is well-defined up to rotations in* $\mathbf{C}^2$, *where*

$$E = c_\psi^2|\varsigma_1'|^2 + s_\psi^2|\varsigma_2'|^2 \quad and \quad G = s_\psi^2|\varsigma_1|^2 + c_\psi^2|\varsigma_2|^2.$$

(2) $X = \sqrt{G/E}\;\partial_t$ (resp. $Y = \partial_s$) *is a closed and conformal* (resp. *Killing*) *vector field on* $(\mathbf{R}^2, E\,\mathrm{d}t^2 + G\,\mathrm{d}s^2)$.

(3) *The second fundamental form of* F *is determined* (*see* (b) *in Sect.* 2) *by*

$$C(\partial_t,\partial_t,\partial_t) = c_\psi^2\langle\varsigma_1'', J\varsigma_1'\rangle + s_\psi^2\langle\varsigma_2'', J\varsigma_2'\rangle,$$
$$C(\partial_t,\partial_t,\partial_s) = c_\psi s_\psi(|\varsigma_1'|^2 \Leftrightarrow |\varsigma_2'|^2),$$
$$C(\partial_t,\partial_s,\partial_s) = \langle\varsigma_1', J\varsigma_1\rangle = \langle\varsigma_2', J\varsigma_2\rangle,$$
$$C(\partial_s,\partial_s,\partial_s) = (s_\psi^4|\varsigma_1|^2 \Leftrightarrow c_\psi^4|\varsigma_2|^2)/c_\psi s_\psi.$$

(4) F *is* H-*minimal if and only if the function*

$$\sqrt{G/E}\left(\frac{C(\partial_t,\partial_t,\partial_t)}{E} + \frac{C(\partial_t,\partial_s,\partial_s)}{G}\right) = 2\langle H, JX\rangle$$

is constant.

Proof. A straightforward computation leads to

$$|F_t|^2 = c_\psi^2|\varsigma_1'|^2 + s_\psi^2|\varsigma_2'|^2, \qquad |F_s|^2 = s_\psi^2|\varsigma_1|^2 + c_\psi^2|\varsigma_2|^2,$$
$$\langle F_t, F_s\rangle = c_\psi s_\psi \Im(\varsigma_1'\overline{\varsigma_1} \Leftrightarrow \varsigma_2'\overline{\varsigma_2}), \qquad \langle F_t, JF_s\rangle = \Leftrightarrow c_\psi s_\psi \Re(\varsigma_1'\overline{\varsigma_1} \Leftrightarrow \varsigma_2'\overline{\varsigma_2}),$$

which proves (1).

The proof of (2) follows from

$$\nabla_{\partial_t}\partial_t = \frac{E'}{2E}\,\partial_t, \qquad \nabla_{\partial_t}\partial_s = \frac{G'}{2G}\,\partial_s, \qquad \nabla_{\partial_s}\partial_s = \Leftrightarrow\frac{G'}{2E}\,\partial_t,$$

where ∇ is the Levi–Civita connection of the induced metric $E\,\mathrm{d}t^2 + G\,\mathrm{d}s^2$.

Computing the second derivatives of F, it is an easy exercise to obtain the formulas given in (3).

Finally, using Proposition 1, F is H-minimal if and only if the divergence of the vector field JH vanishes. From (2) and (3) there exist certain functions $a(t)$ and $b(t)$ such that we can write $2H = a(t)JX + b(t)JY$. Then $2\operatorname{div} JH = \Leftrightarrow(X(a) +$

$a\,\mathrm{div}\,X$). Using the Christtofel symbols of ∇ we have that $\mathrm{div}\,X = \sqrt{G/E}\,G'/G$ and so we conclude that F is H-minimal if and only if $a' + aG'/G = 0$. Using (3), this last equation is equivalent to the assertion in (4). □

Using the above lemma, we introduce in the following result a three-parameter family of H-minimal immersions and state their main geometric properties. The parameter-count in the result below ignores reparametrizations and also congruences of $\mathbf{C}^2$.

THEOREM 1. *Let*

$$\Gamma = \left\{(\theta, \beta, \alpha) \in [0, \tfrac{1}{2}\pi) \times (0, \tfrac{1}{2}\pi) \times (\Leftrightarrow\tfrac{1}{2}\pi, \tfrac{1}{2}\pi); \theta < \beta,\ \ |\alpha| < \beta\right\}.$$

For each $(\theta, \beta, \alpha) \in \Gamma$*, there exists a Lagrangian H-minimal immersion* (*with non-parallel mean curvature vector*)

$$\mathcal{F}^{\alpha}_{\theta,\beta}\colon \mathbf{R}^2 \to \mathbf{C}^2,$$

that in suitable coordinates can be written as

$$\mathcal{F}^{\alpha}_{\theta,\beta}(x, y) = (\varsigma_1(x)\gamma_1(y), \varsigma_2(x)\gamma_2(y)),$$

where $\varsigma_j = \varsigma_j(x)\colon \mathbf{R} \to \mathbf{C}^*$, $j = 1, 2$, *are certain regular curves and* $(\gamma_1(y), \gamma_2(y))$ *is a horizontal lift to* $\mathbf{S}^3 \subset \mathbf{C}^2$ *by the Hopf fibration* $\Pi\colon \mathbf{S}^3 \to \mathbf{S}^2$ *of the parallel of* $\mathbf{S}^2$ *of latitude* $\varphi = \arcsin(\sin\alpha/\sin\beta)$ (y *is not necessarily the arc parameter*).

$\mathcal{F}^{\alpha}_{\theta,\beta}$ *satisfies the following properties*

(1) *The induced metric by* $\mathcal{F}^{\alpha}_{\theta,\beta}$ *is given by*

$$\langle\ ,\ \rangle = \varrho(x)|\mathrm{d}z|^2, \quad \textit{where} \quad \varrho(x) = \frac{c_\theta + \sqrt{c_\theta^2 \Leftrightarrow c_\beta^2}\cos(2c_\beta x)}{c_\beta^2}.$$

(2) *The second fundamental form* $\sigma = \sigma^{\alpha}_{\theta,\beta}$ *of* $\mathcal{F} = \mathcal{F}^{\alpha}_{\theta,\beta}$ *is determined by*

$$\sigma(\partial_z, \partial_z) = \frac{\mathrm{e}^{-i\alpha}}{2} J\mathcal{F}_z + \left(\frac{\mathrm{e}^{i(\theta-\alpha)}}{\varrho(x)} \Leftrightarrow \frac{\mathrm{e}^{-i\alpha}}{2}\right) J\mathcal{F}_{\overline{z}},$$

$$\sigma(\partial_z, \partial_{\overline{z}}) = \Re(\mathrm{e}^{i\alpha} J\mathcal{F}_z),$$

where $\Re$ *denotes real part and* $\partial_z = \frac{1}{2}(\partial_x \Leftrightarrow i\partial_y)$, $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$.

(3) $\{\varphi_t\colon (x, y) \mapsto (x, y + t),\ \ t \in \mathbf{R}\}$ *is a one-parameter group of isometries of the induced metric by* $\mathcal{F}^{\alpha}_{\theta,\beta}$ *that is the restriction, induced by* $\mathcal{F}^{\alpha}_{\theta,\beta}$, *of the one-parameter group of holomorphic isometries of* $\mathbf{C}^2$ *given by* $\{\mathrm{diag}(\mathrm{e}^{-i(s_\beta - s_\alpha)t}, \mathrm{e}^{i(s_\beta + s_\alpha)t}),\ \ t \in \mathbf{R}\}$.

(4) *The limit immersions* $\mathcal{F}_\theta = \mathcal{F}^\theta_{\theta,\theta}$, $0 < \theta \leqslant \frac{1}{4}\pi$, *have parallel mean curvature vector.* $\mathcal{F}_{\pi/4}$ *is the universal covering of the Clifford torus.*

(5) $\mathcal{F}^\alpha_{\theta,\beta}$ *descends to a torus if and only if*

$$\left(\frac{\sin\alpha}{\sin\beta}, \frac{\cos\alpha}{\cos\beta}\right) \in \mathbf{Q}^2.$$

Precisely, $\sin\alpha/\sin\beta \in \mathbf{Q}$ *just means that the lift* $(\gamma_1(y), \gamma_2(y))$ *is a closed curve in* $\mathbf{S}^3$.

Remark 1. In the particular case of the construction described in Theorem 1 with only one curve, i.e. $\varsigma_1(x) = \varsigma_2(x)$, we will obtain the immersions $\mathcal{F}^\alpha_{\alpha,\beta}$ (see the proof of the theorem below). Then this construction agrees with the made in [9, Prop. 3] by Ros and the second author.

Proof. For each $(\theta, \beta, \alpha) \in \Gamma$, we define

$$\begin{aligned} \gamma_1(y) &= \sqrt{\frac{s_\beta + s_\alpha}{2s_\beta}}\, \mathrm{e}^{-i(s_\beta - s_\alpha)y}, \\ \gamma_2(y) &= \sqrt{\frac{s_\beta \Leftrightarrow s_\alpha}{2s_\beta}}\, \mathrm{e}^{i(s_\beta + s_\alpha)y} \end{aligned} \tag{1}$$

and let $\varsigma_j(x) = \rho_j(x)\,\mathrm{e}^{i\nu_j(x)}$, $j = 1, 2$, with

$$\begin{aligned} \rho_1(x) &= \frac{\sqrt{(s_\beta \Leftrightarrow s_\alpha)\varrho(x) \Leftrightarrow s_{\theta-\alpha}}}{(s_\beta \Leftrightarrow s_\alpha)\sqrt{s_\beta + s_\alpha}}, \\ \rho_2(x) &= \frac{\sqrt{(s_\beta + s_\alpha)\varrho(x) + s_{\theta-\alpha}}}{(s_\beta + s_\alpha)\sqrt{s_\beta \Leftrightarrow s_\alpha}} \end{aligned} \tag{2}$$

and

$$\begin{aligned} \nu_1(x) &= c_\alpha x \Leftrightarrow (s_\beta c_{\theta-\alpha} \Leftrightarrow s_\theta) \int_0^x \frac{\mathrm{d}t}{(s_\beta \Leftrightarrow s_\alpha)\varrho(t) \Leftrightarrow s_{\theta-\alpha}}, \\ \nu_2(x) &= c_\alpha x \Leftrightarrow (s_\beta c_{\theta-\alpha} + s_\theta) \int_0^x \frac{\mathrm{d}t}{(s_\beta + s_\alpha)\varrho(t) + s_{\theta-\alpha}}. \end{aligned} \tag{3}$$

In this way we define $\mathcal{F}^\alpha_{\theta,\beta}(x, y) = (\varsigma_1(x)\gamma_1(y), \varsigma_2(x)\gamma_2(y))$.

First we observe that $\Pi(\gamma_1(y), \gamma_2(y)) = (\cos\varphi\,\mathrm{e}^{-2is_\beta y}, \sin\varphi)$, with $\varphi = \arcsin(\sin\alpha/\sin\beta)$. This shows that $(\gamma_1(y), \gamma_2(y))$ is a lift to $\mathbf{S}^3$ of the parallel in $\mathbf{S}^2$ of latitude φ, and it is easy to check that it is horizontal. We notice that the arc parameter s of $(\gamma_1(y), \gamma_2(y))$ is given by $\mathrm{d}s^2 = (s_\beta^2 \Leftrightarrow s_\alpha^2)\,\mathrm{d}y^2$.

On the other hand, a simple computation leads to $\varsigma_j'\overline{\varsigma_j} = \rho_j'\rho_j + i\nu_j'\rho_j^2$, $j = 1, 2$. From (2) and (3), using that $\rho_1^2 \Leftrightarrow \rho_2^2$ is clearly constant, it is easy to obtain that $\varsigma_1'\overline{\varsigma_1} = \varsigma_2'\overline{\varsigma_2}$. So we can use Lemma 1. Following its notation, we obtain here that $E = E(x) = \varrho(x)$ and $G = G(x) = \varrho(x)/(s_\beta^2 \Leftrightarrow s_\alpha^2)$ and then Lemma 1 says that $\mathcal{F}_{\theta,\beta}^{\alpha}$ is a Lagrangian immersion whose induced metric (Lemma 1,1) is given by

$$\langle\ ,\ \rangle = \varrho(x)\,\mathrm{d}x^2 + \frac{\varrho(x)}{s_\beta^2 \Leftrightarrow s_\alpha^2}\,\mathrm{d}s^2 = \varrho(x)(\mathrm{d}x^2 + \mathrm{d}y^2),$$

which proves (1).

Using now Lemma 1,3 it is then straightforward (after some computations) to arrive at

$$C(\partial_x, \partial_x, \partial_x) = c_\alpha\varrho(x) + c_{\theta-\alpha}, \qquad C(\partial_x, \partial_x, \partial_y) = \Leftrightarrow s_{\theta-\alpha},$$
$$C(\partial_x, \partial_y, \partial_y) = c_\alpha\varrho(x) \Leftrightarrow c_{\theta-\alpha}, \qquad C(\partial_y, \partial_y, \partial_y) = 2s_\alpha\varrho(x) + s_{\theta-\alpha}$$

and it is easy to check (2). Also we deduce that the mean curvature vector of $\mathcal{F}_{\theta,\beta}^{\alpha}$ is given by $H = (c_\alpha J\mathcal{F}_x + s_\alpha J\mathcal{F}_y)/\varrho(x)$ and so $\langle H, J\mathcal{F}_x\rangle = c_\alpha$ which proves that $\mathcal{F}_{\theta,\beta}^{\alpha}$ is H-minimal using Lemma 1,4.

The paragraph (3) follows from (1) and Lemma 1,2.

If we consider the limit immersions when $\theta = \beta = \alpha$, we arrive at

$$\mathcal{F}_\theta(x, y) \equiv \mathcal{F}_{\theta,\theta}^{\theta}(x, y) = \frac{1}{2\sqrt{c_\theta}}\left(\frac{\mathrm{e}^{2ic_\theta x}}{c_\theta}, \frac{\mathrm{e}^{2is_\theta y}}{s_\theta}\right), \tag{4}$$

which proves (4). We notice that in the coordinates $\tilde{x} = c_\theta x + s_\theta y$, $\tilde{y} = c_\theta x \Leftrightarrow s_\theta y$, we rewrite

$$\mathcal{F}_\theta(\tilde{x}, \tilde{y}) = \frac{1}{2\sqrt{c_\theta}}\left(\frac{\mathrm{e}^{i(\tilde{x}+\tilde{y})}}{c_\theta}, \frac{\mathrm{e}^{i(\tilde{x}-\tilde{y})}}{s_\theta}\right).$$

To prove (5), we look for conditions on the parameters of $\mathcal{F}_{\theta,\beta}^{\alpha}$ that insure that $\mathcal{F}_{\theta,\beta}^{\alpha}$ is doubly-periodic.

We first observe that if $(\lambda, \mu) \neq (0, 0)$ is a period of $\mathcal{F}_{\theta,\beta}^{\alpha}$, then λ is a period of $\varrho(x)$ and so, using (1), there exists $m \in \mathbf{Z}$ such that $\lambda = m\pi/c_\beta$.

From (1), (2) and (3), we deduce then that $(m\pi/c_\beta, \mu) \neq (0, 0)$ is a period of $\mathcal{F}_{\theta,\beta}^{\alpha}$ if and only if it satisfies the system of congruences

$$\begin{aligned} &\nu_1\left(\frac{\pi}{c_\beta}\right)m \Leftrightarrow (s_\beta \Leftrightarrow s_\alpha)\mu \equiv 0 \mod 2\pi, \\ &\nu_2\left(\frac{\pi}{c_\beta}\right)m + (s_\beta + s_\alpha)\mu \equiv 0 \mod 2\pi. \end{aligned} \tag{5}$$

When $x \in (\Leftrightarrow \pi/2c_\beta, \pi/2c_\beta)$, the integral functions which appear in (3) are of the type a_j arctan$(b_j \tan(c_\beta x))$, for certain constants $a_j, b_j, j = 1, 2$, depending on θ, β, α, and the corresponding computation leads to $\nu_j(\pi/c_\beta) = \pi(c_\alpha \Leftrightarrow c_\beta)/c_\beta$, $j = 1, 2$. Thus it is easy to check from (5) that $(m\pi/c_\beta, \mu) \neq (0, 0))$ is a period of $\mathcal{F}^\alpha_{\theta,\beta}$ if and only if

$$\exists k_1, k_2 \in \mathbf{Z}/\mu = (k_2 \Leftrightarrow k_1)\frac{\pi}{s_\beta}, \qquad m = \frac{(k_1 + k_2) + \frac{s_\alpha}{s_\beta}(k_1 \Leftrightarrow k_2)}{\frac{c_\alpha}{c_\beta} \Leftrightarrow 1}. \tag{6}$$

When writing $k = k_1 + k_2$ and $n = k_2 \Leftrightarrow k_1$, it follows that $\mu = n\pi/s_\beta$ and (6) is reduced to

$$\exists k \in \mathbf{Z} \Big/ \left(\frac{c_\alpha}{c_\beta} \Leftrightarrow 1\right) m = k \Leftrightarrow \frac{s_\alpha}{s_\beta} n. \tag{7}$$

Let $\Lambda^\alpha_{\theta,\beta} = \{(m, n) \in \mathbf{Z} \times \mathbf{Z}$ satisfying $(7)\}$ a free subgroup of $(\mathbf{Z} \times \mathbf{Z}, +)$. We have just shown that $\mathcal{F}^\alpha_{\theta,\beta}(x_1, y_1) = \mathcal{F}^\alpha_{\theta,\beta}(x_2, y_2) \Leftrightarrow (c_\beta(x_2 \Leftrightarrow x_1)/\pi, s_\beta(y_2 \Leftrightarrow y_1)/\pi) \in \Lambda^\alpha_{\theta,\beta}$. Hence, $\mathcal{F}^\alpha_{\theta,\beta}$ descends to a torus if and only if rank $\Lambda^\alpha_{\theta,\beta} = 2$. Let us see that this is equivalent to that s_α/s_β and c_α/c_β are rational numbers.

Firstly, if rank $\Lambda^\alpha_{\theta,\beta} = 2$ and (m_1, n_1), (m_2, n_2) are the generators of $\Lambda^\alpha_{\theta,\beta}$, since $m_1 n_2 \Leftrightarrow m_2 n_1 \neq 0$ it is easy to see from (7) that s_α/s_β, $c_\alpha/c_\beta \in \mathbf{Q}$.

Conversely, if $s_\alpha/s_\beta = r/s$, $(r, s) = 1$ and $c_\alpha/c_\beta = p/q$, $(p, q) = 1$, then it is not difficult to check from (7) that $\Lambda^\alpha_{\theta,\beta} = \text{span}\{(0, s), (q, s)\}$, taking $k = r$ and $k = r + p \Leftrightarrow q$ in (7) respectively. □

4. Uniqueness

If we denote

$$\Gamma^* = \{(\theta, \beta, \alpha) \in \Gamma; (\sin\alpha/\sin\beta,\ \cos\alpha/\cos\beta) \in \mathbf{Q}^2\},$$

the immersions $\mathcal{F}^\alpha_{\theta,\beta}\colon \mathbf{R}^2 \to \mathbf{C}^2$, with $(\theta, \beta, \alpha) \in \Gamma^*$, are doubly-periodic and then descend to tori (Theorem 1,5). These new examples of H-minimal Lagrangian tori can be characterized as follows.

THEOREM 2. *Let $\phi\colon \Sigma \to \mathbf{C}^2$ a Lagrangian* H*-minimal* (*with non-parallel mean curvature vector*) *immersion of a torus admitting a one-parameter group of isometries. Then the universal covering of ϕ will be congruent to $\mathcal{F}^\alpha_{\theta,\beta}$, for some* $(\theta, \beta, \alpha) \in \Gamma^*$.

Proof. We also denote by ϕ the lift of $\phi\colon \Sigma \to \mathbf{C}^2$ to the universal covering of the torus Σ. Then $\phi\colon \mathbf{C} \to \mathbf{C}^2$ is also a Lagrangian H-minimal immersion admitting a one-parameter group of isometries.

Let Y be the non-trivial Killing vector field in $(\mathbf{C}, \langle\ ,\ \rangle)$ determined by the one-parameter group of isometries. Then it is easy to check that $X = \Leftrightarrow iY$ is a holomorphic vector field on **C**. So, we can normalize it in order to get $X = \partial_x$ and hence $Y = \partial_y$. Thus, we can write the induced metric in the coordinate $z = x + iy$ of **C** as a conformal metric to the Euclidean one in the following form

$$\langle\ ,\ \rangle = \mathrm{e}^{2u(x)}|\mathrm{d}z|^2. \tag{8}$$

Then the one-parameter group of isometries associated to ϕ is now given by the translations in the y-direction.

We now denote by $\partial_z = \frac{1}{2}(\partial_x \Leftrightarrow i\partial_y)$ and $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ the Cauchy–Riemann operators in **C** and extend **C**-linearly σ, J and $\langle\ ,\ \rangle$ to the complexified bundles. Using properties (a), (b) and (c) of Section 2 and the Coddazi Equation of ϕ, it follows that $(\nabla\sigma)(\partial_z, \partial_z, \partial_{\overline{z}}) = (\mathrm{e}^{2u}/2)\nabla^{\perp}_{\partial_z} H$ and, on the other hand, $\partial_z(\alpha_H(\partial_z)) = 2u_z\alpha_H(\partial_z) \Leftrightarrow \langle\nabla^{\perp}_{\partial_z} H, J\partial_z\rangle$. Thus we obtain that

$$\partial_{\overline{z}}\left(\langle\sigma(\partial_z, \partial_z), J\partial_z\rangle\right) = \mathrm{e}^{2u}(u_z\, \alpha_H(\partial_z) \Leftrightarrow \tfrac{1}{2}\partial_z(\alpha_H(\partial_z))). \tag{9}$$

Since ϕ is H-minimal the (real) 1-form α_H is harmonic (see Proposition 1) and so the function $\alpha_H(\partial_z)$ is holomorphic. As Σ is a torus, it is also bounded and so it is necessarily constant. We can write then $\alpha_H(\partial_z) = \Leftrightarrow\lambda\, \mathrm{e}^{-i\alpha}$, $\lambda > 0$, $\alpha \in [\Leftrightarrow\pi, \pi]$, since that $\lambda = 0$ implies that ϕ is minimal, which is impossible becouse of the compactness of Σ.

In addition, from (8) it follows that $(\mathrm{e}^{2u})_z = (\mathrm{e}^{2u})_{\overline{z}}$. Thus, using (9), we have that

$$\partial_{\overline{z}}(\langle\sigma(\partial_z, \partial_z), J\partial_z\rangle + \lambda\, \mathrm{e}^{-i\alpha}\, \mathrm{e}^{2u}/2) = 0$$

and also in the same way, we can conclude that $\langle\sigma(\partial_z, \partial_z), J\partial_z\rangle + \lambda\, \mathrm{e}^{-i\alpha}\, \mathrm{e}^{2u}/2 = \mu\, \mathrm{e}^{i(\theta-\alpha)}$, $\mu \geqslant 0$, $\theta \Leftrightarrow \alpha \in [\Leftrightarrow\pi, \pi]$.

Therefore, since the equality $\alpha_H(\partial_z) = \Leftrightarrow\lambda\, \mathrm{e}^{-i\alpha}$ implies that $\langle\sigma(\partial_z, \partial_{\overline{z}}), J\partial_z\rangle = \lambda\, \mathrm{e}^{-i\alpha}\, \mathrm{e}^{2u}/2$, we deduce that the second fundamental form of ϕ is written in the following way

$$\begin{aligned} \sigma(\partial_z, \partial_z) &= \lambda\, \mathrm{e}^{-i\alpha} J\phi_z + \left(\frac{2\mu\, \mathrm{e}^{i(\theta-\alpha)}}{\mathrm{e}^{2u}} \Leftrightarrow \lambda\, \mathrm{e}^{-i\alpha}\right) J\phi_{\overline{z}}, \\ \sigma(\partial_z, \partial_{\overline{z}}) &= 2\lambda\, \Re(\mathrm{e}^{i\alpha} J\phi_z), \end{aligned} \tag{10}$$

where $\Re$ denotes real part. From the Gauss Equation of ϕ we deduce that the Gauss curvature K of ϕ is given by

$$K = 8\mu(\lambda\cos\theta\, \mathrm{e}^{-4u} \Leftrightarrow \mu\, \mathrm{e}^{-6u}), \tag{11}$$

from (8) we get that $\mathrm{e}^{-2u}u'' = \Leftrightarrow K$ and so we have proved that $u = u(x)$ satisfies the o.d.e.

$$u'' \Leftrightarrow 8\mu^2\, \mathrm{e}^{-4u} + 8\lambda\mu\cos\theta\, \mathrm{e}^{-2u} = 0. \tag{12}$$

Now we are going to prove that we can normalize the constants λ and μ. Firstly, it is not difficult to verify that if $\mu = 0$ then $K \equiv 0$ from (11) and ϕ has parallel mean curvature vector. So, we can consider $\mu > 0$ and we denote by ψ the corresponding immersion to the values $\lambda = \mu = \frac{1}{2}$. Using (12), its induced metric, that we will continue writting as $\mathrm{e}^{2u(x)}|\mathrm{d}z|^2$, satisfies

$$u'' \Leftrightarrow 2\,\mathrm{e}^{-4u} + 2\cos\theta\,\mathrm{e}^{-2u} = 0. \tag{13}$$

Using (10) and (12), it happens then that our starting immersion $\phi = \phi(z)$ induces the metric $(\mu/\lambda)\,\mathrm{e}^{2u}|\mathrm{d}z|^2$ (being already $u = u(x)$ solution to (13)) and it is congruent to the immersion $\sqrt{\mu/4\lambda^3}\psi(\lambda z)$. So, up to dilations, it is enough to consider the values $\lambda = \mu = \frac{1}{2}$ and we can work only with the immersion ψ, that we rename again by ϕ, and with the o.d.e. (13).

On the other hand, since Σ is a torus, the induced metric must be periodic. If $\cos\theta \leqslant 0$, from (13) it would follow that $u'' > 0$ and obviusly u would not can be periodic. Hence $\cos\theta > 0$ and so $\theta \in (\Leftrightarrow\frac{1}{2}\pi, \frac{1}{2}\pi)$. Note that if we change θ by $\Leftrightarrow\theta$ the o.d.e. (13) remains invariant.

In order to analyze the possible values of θ and α, it is convenient to denote our immersion ϕ by $\phi \equiv \phi_\theta^\alpha(z)$, whose Frenet Equations (see (10)) are given by

$$\begin{aligned} \phi_{z\,z} &= u'\phi_z + \frac{\mathrm{e}^{-i\alpha}}{2}J\phi_z + \left(\frac{\mathrm{e}^{i(\theta-\alpha)}}{\mathrm{e}^{2u}} \Leftrightarrow \frac{\mathrm{e}^{-i\alpha}}{2}\right)J\phi_{\overline{z}}, \\ \phi_{z\,\overline{z}} &= \Re(\mathrm{e}^{i\alpha}J\phi_z). \end{aligned} \tag{14}$$

From (14) it is now easy to check that $\phi \equiv \phi_\theta^\alpha(z)$ satisfies the same Frenet Equations that $\phi_{-\theta}^{-\alpha}(\overline{z})$ and $\phi_\theta^{\alpha+\pi}(\Leftrightarrow z)$; the same happens with $\phi_\theta^{-\alpha}(z)$ and $\phi_\theta^{\pi-\alpha}(\Leftrightarrow z)$. Hence we can restrict to consider $\theta \in [0, \frac{1}{2}\pi)$ and $\alpha \in [\Leftrightarrow\frac{1}{2}\pi, \frac{1}{2}\pi]$.

Now we continue the study of the periodic solutions of (13). The energy integral for (13) is given by

$$u'^{\,2} + \mathrm{e}^{-4u} \Leftrightarrow 2\cos\theta\,\mathrm{e}^{-2u} = \text{constant} = A. \tag{15}$$

There is no restriction by imposing the initial condition $u'(0) = 0$ (any periodic solution has critical points) and then it is easy to check from (15) that $A \geqslant \Leftrightarrow\cos^2\theta$, holding the equality only for the constant solution $\mathrm{e}^{2u(x)} \equiv 1/\cos\theta$.

Analyzing the nature of the possible solutions of (13), we find out that $\mathrm{e}^{2u(x)}$ is expressed in terms of hyperbolic functions (respectively polynomial of second degree) if $A > 0$ (respectively if $A = 0$) and we must conclude that it is necessary that $A < 0$ in order to obtain periodic solutions to it. If we put $A = \Leftrightarrow\cos^2\beta$, with $\beta \in [0, \frac{1}{2}\pi)$ and $\beta \geqslant \theta$ so that $\Leftrightarrow\cos^2\theta \leqslant A < 0$, it is an exercise to verify that the explicit solutions to (13) are given by $\mathrm{e}^{2u(x)} = (c_\theta + \sqrt{c_\theta^2 \Leftrightarrow c_\beta^2}\cos(2c_\beta x))/c_\beta^2$, that is just the expression for $\varrho(x)$ in Theorem 1,1. Note that $\beta = \theta$ implies that $u(x)$ is constant and then ϕ has parallel mean curvature vector.

We point out that, from (14), the second fundamental form of ϕ is also the same that in Theorem 1,2.

To finish this proof, it still remains to verify that (θ, β, α) belongs to Γ^*. Let us see now that $|\alpha| < \beta$. It is easy to deduce from the Frenet Equations (14) of ϕ that $\phi(\Leftrightarrow, y)$ satisfies, using (15), the o.d.e.

$$\phi_{yy} \Leftrightarrow 2 \sin \alpha J \phi_y + (\cos^2 \alpha \Leftrightarrow \cos^2 \beta)\phi = 0. \tag{16}$$

If $\cos^2 \alpha < \cos^2 \beta$, by integrating (16) we would obtain that ϕ depends on the variable y in terms of hyperbolic functions, which is impossible if Σ is a torus. Then $\cos^2 \alpha \geqslant \cos^2 \beta$, or equivalently $|\alpha| \leqslant \beta$. If the equality holds, from (16) it follows that $\phi_{yy} \Leftrightarrow 2 \sin \alpha J \phi_y = 0$. Using this in (14) we arrive at the limit case $\alpha = \beta = \theta$ with $u(x)$ constant and hence the immersion has parallel mean curvature vector. Note that $\beta = 0$ implies $\theta = \alpha = 0$, obtaining then a right-circular cylinder.

As a summary, we have proved that our immersion ϕ has the same first and second fundamental forms that $\mathcal{F}^{\alpha}_{\theta,\beta}$ and the parameters (θ, β, α) associated to ϕ are in Γ. Moreover, since Σ is a torus, using Theorem 1,5 it is clear that (θ, β, α) are also in Γ^*, what finishes the proof. □

We will denote by $T^{\alpha}_{\theta,\beta}$ the tori described by the doubly-periodic immersions $\mathcal{F}^{\alpha}_{\theta,\beta}$, $(\theta, \beta, \alpha) \in \Gamma^*$ and by $\Lambda^{\alpha}_{\theta,\beta}$ the lattice granted by $\mathcal{F}^{\alpha}_{\theta,\beta}$. Let $\Pi^{\alpha}_{\theta,\beta}$ be a fundamental region (a parallelogram) of the lattice $\Lambda^{\alpha}_{\theta,\beta}$. In the following result we collect some interesting geometric properties of the tori $T^{\alpha}_{\theta,\beta}$.

PROPOSITION 3. (a) *The area of the torus $T^{\alpha}_{\theta,\beta}$ is equal to*

$$A = \frac{\cos\theta}{\cos^2\beta} A(\Pi^{\alpha}_{\theta,\beta})$$

and the Willmore functional of the torus $T^{\alpha}_{\theta,\beta}$ is equal to

$$W = A(\Pi^{\alpha}_{\theta,\beta}),$$

where $A(\Pi^{\alpha}_{\theta,\beta})$ is the area of the fundamental parallelogram $\Pi^{\alpha}_{\theta,\beta}$.

(b) *The Hamiltonian nullity of the tori $T^{\alpha}_{\theta,\beta}$ is positive.*

(c) *The tori $T^{\alpha}_{\theta,\beta}$ are Hamiltonian unstable.*

Proof. Since the immersion $\mathcal{F}^{\alpha}_{\theta,\beta}$ is conformal, the area of a constructed torus $T^{\alpha}_{\theta,\beta}$ is given by

$$A = \iint_{\Pi^{\alpha}_{\theta,\beta}} \varrho(x)\, \mathrm{d}x\, \mathrm{d}y.$$

It is now possible to perform the above integration using the explicit expression for $\varrho(x)$ in Theorem 1,1 and the generators of $\Lambda^{\alpha}_{\theta,\beta}$ which are given in the proof of Theorem 1,5.

From Theorem 1,2 it is deduced that $|H|^2 = 1/\varrho(x)$ and so the Willmore functional, $W = \int |H|^2$, is just the area of $\Pi^{\alpha}_{\theta,\beta}$.

It is an easy exercise to see that the quadratic form Q of the second variation of the area (given in Sect. 2) satisfies that $Q(\nabla\varrho) = 0$, what proves (b).

Finally, it is not difficult to check that

$$\begin{aligned} Q(\nabla\varrho'(x)) &= \Leftrightarrow\frac{64}{\pi}(c_\theta^2 \Leftrightarrow c_\beta^2)A(\Pi^{\alpha}_{\theta,\beta})\int_0^{\pi}(c_\theta\varrho(x)^{-2} \Leftrightarrow c_\beta^2\varrho(x)^{-1})\,\mathrm{d}x \\ &= \Leftrightarrow 64c_\beta(c_\theta^2 \Leftrightarrow c_\beta^2)^2 A(\Pi^{\alpha}_{\theta,\beta}) < 0, \end{aligned}$$

which proves (c). □

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