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The Concept of Symmetry

> Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection.
>
> — *Hermann Weyl*
> (*Symmetry* [1952])

Since birth, all of us have grown up with the concept of symmetry rooted in our minds; broadly, symmetry can be defined as the repeating property of phenomena in time and space with regularity. Regardless of the field considered, this definition branches out into several different interpretations: for example in the arts, it assumes the meaning of "sense of harmonious and beautiful proportions." Specifically, symmetries can be used to obtain ideal representations of figures, respecting proportions. On the other hand, in Mathematics and Physics, they represent invariances under some transformations.

Certainly, many aspects of our world and life are ruled by symmetry relations from the smallest to the largest scale. It comes natural, indeed, wondering whether the symmetry of structures is a fundamental and essential property, or whether it is just a mere tool, only aimed at schematizing the laws of Nature and the corresponding dynamics. Is it actually surprising that the Universe enjoys certain symmetries, once we created the concept of symmetry by looking at the Universe itself? In other words: Are symmetries constructions of our mind or fundamental structures that reality is endowed with?

Reality, in which we exist, continuously manifests many expressions of symmetry, even in everyday life, such as the indistinguishability between right and left, or up and down. One may think that the latter could be distinguished by the natural propensity of objects to fall down from the top to the bottom. However, this is only a consequence of gravitational interaction, which when far away from massive bodies (as well as planets, stars, or galaxies) is not manifesting. Although at small scales the Universe seems to be inhomogeneous and anisotropic, such asymmetry disappears at properly large scales (cosmological scales), where no preferred direction occurs. The assumption of homogeneity and isotropy was

https://doi.org/10.1017/9781009208727.002 Published online by Cambridge University Press

proposed in the first half of the twentieth century and it is known as the "Cosmological Principle"; we will largely use such a principle throughout the second part of the book, where we will assume the space-time metric to be isotropic and homogeneous. Just as the large-scale Universe is dominated by symmetry properties, so also molecules and atoms assume certain preferred symmetry configurations, such as the invariance under translations or rotations. The latter concept is particularly useful in Chemistry since it reduces the complexity of the whole system made of many-bodies interactions.

Can the idea of symmetry be considered as the lens through which we watch the world, or is it a concept deep-rooted in our world, regardless of human perception and conception? In other words: Might it be just something created by the human mind with no real confirmation in Nature? In order to better explain this point, let us consider the symmetry of geometric figures. A "regular polygon" can be thought of as a figure having equal angles and equal sides. Equivalently, a regular polygon with n sides can be thought of as a geometric structure that is invariant under rotations of $\frac{2k\pi}{n}$ radians around its center, with k a natural number. In this way, the square is *at least* symmetric under rotation of $\frac{\pi}{2}$ radians, the pentagon under rotation of $\frac{2}{5}\pi$ radians, and so (see Fig. 1.1). In the limit $n \to \infty$ one gets a geometric figure that is invariant under any rotation of any arbitrary angle: the circle.

However, it has to be clear that this is just a mental abstraction, since in nature neither perfect circles nor regular polygons exist. Any geometric figure we find in everyday life is just similar to the hypothetical definition that we keep in our mind, since no figure having exactly the same sides or the same angles can be found in nature. Equivalently, we may argue that no natural symmetry can be completely perfect, so that any time a symmetry property is identified in nature, it is automatically addressed to something that seems to respect the idea of symmetry we created in our mind. For example, from this point of view, the Cosmological Principle would represent only an approximation of the perfect symmetry in the Universe; molecules would not be invariant under rotations of given angles, but their rotated configuration would just look similar to the initial one, instead of exactly identical. According to this vision, symmetry might be only a human creation not describing the real world, which we recognize whenever our observations look similar to what we cataloged as "symmetric" in our thought. The same argument can also be applied to Mathematics and other

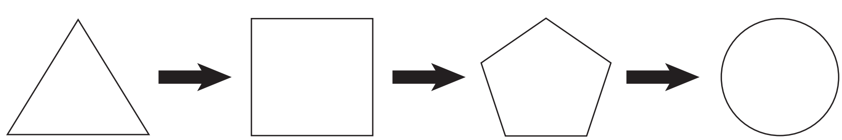

Figure 1.1 Polygons invariant under rotations of $\frac{2\pi}{3}$ rad, $\frac{\pi}{2}$ rad, $\frac{2\pi}{5}$, and any angle, respectively.

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fields, where the utility of symmetries is undoubtedly independent of this issue. Just as it is true that a human being is too much influenced by the Universe in which she/he lives to answer these questions, so is it true that, although imperfect or ideal, symmetries play a crucial role in the comprehension of Physics; without them, it would be impossible to get several results that constitute the basis of the whole of Science. Besides simplifying dynamics of systems, continuous symmetries are always linked to quantities that remain constants in time or space. These quantities allow one to introduce cyclic variables, which reduces the complexity of the equations of motion. In Biology, they can be used to schematize animal and plant species, which can be classified with respect to their radial, bilateral, or spherical symmetry. In Chemistry, the stereochemistry studies the arrangements in space of atoms or atomic groups making up the molecules (molecular configuration and conformation) and the relationships between these structures and chemical properties. From this definition, it is easy to understand the importance of symmetries. Those just mentioned are only some examples of the importance of symmetry in Science.

In the next section, we discuss what is meant by "symmetry" in Physics, analyzing then some symmetry groups. For an exhaustive treatment of philosophical and physical aspects of symmetries, we suggest Refs. [1, 2], while mathematical aspects are considered in Ref. [3], in the context of differential equation solutions. Specifically, the latter reference uses mathematical techniques – based on symmetries – to find out solutions of nonlinear differential equations.

1.1 Symmetries in Physics

The concept of symmetry in Physics plays such an important role that it is hard to think that symmetric properties of the systems are the result of mere coincidence. Over the years, the concept has been so exaggerated as to begin to think that the asymmetries that we observe in the current Universe may be attributed to spontaneous symmetry breakings in early epochs. The best known theory that pursues such a prescription is called *supersymmetry* (SUSY), according to which for each particle there corresponds a symmetric partner with the same mass but different spin. This supersymmetric situation held in the early Universe. Subsequently, after a symmetry breaking, the mass of supersymmetric particles changed, unlike other quantum properties, which remained unchanged. Such supersymmetric particles are the best candidates to solve many problems of String Theory, Grand Unification of physical interactions, and Standard Model of Particles, but so far they have never been observed. The formal elegance of the theory and its promising results are the real reason for its reputation, but as long as SUSY is not supported by experimental observations it cannot be considered fully valid. Currently we have no evidence of such spontaneously broken symmetry in the early Universe. For this reason, the detection of new particles that

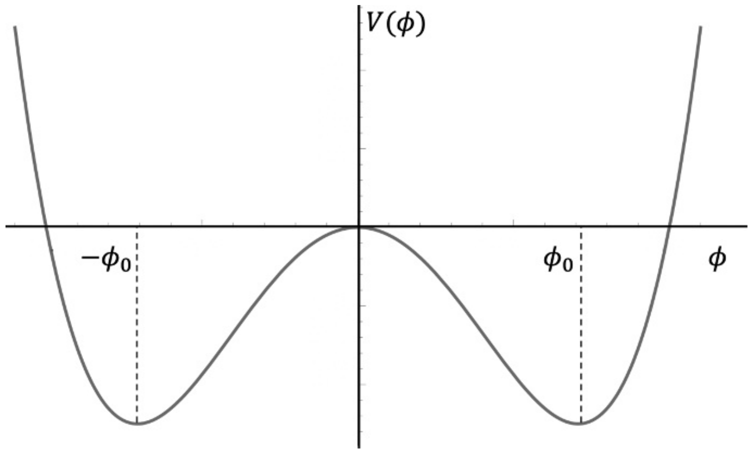

Figure 1.2 $\mu^2\phi^2 - \lambda\phi^4$ potential with $\mu^2 < 0$

can potentially confirm the validity of the theory is a very demanding research area.

Another example of the importance of symmetry is represented by the Brout–Englert–Higgs Mechanism (better known as *Higgs mechanism*). It is mostly used to explain the currently observed mass of elementary particles, which is attributed to a spontaneous symmetry breaking in the early epoch of the Universe. The fundamental mechanism states that when the symmetry of a certain system is spontaneously broken, a massive (Higgs boson) and a massless (Goldstone boson) particle arise. A standard example is given by the potential $V(\phi) = \mu^2\phi^2 - \lambda\phi^4$, whose graphical representation is shown in Fig. 1.2.

The symmetry can be broken after shifting the minimum by means of a reparameterization of the scalar field, as pointed out in Section 3.4. The Higgs mechanism is a clear example of how the presence of symmetry (and its breaking) allows one to fix dynamics and to give an overall picture of very complicated problems like the generation of particle masses.

Another important example of how symmetry is related to physical properties comes from the study of functions in Mathematics. One of the most basic distinctions in the framework of real functions depending on real variables is provided by the symmetry of such functions with respect to the ordinate axis or the origin point. In the former case, a function $f(x)$ respecting the symmetry condition $f(x) = f(-x)$ is said to be *even*; in the latter case, it is said to be *odd*. In the analysis of the function trend, such properties may largely simplify the treatment, allowing one to focus the procedure on particular intervals of the real axis rather than studying the trend for any real value of x. In this way, the integral of an even function over the real axis can be restricted to the positive axis and vice versa. By means of this manipulation, it is possible to compute, for example, the integral of the Gaussian function within the interval $[0, \infty[$. More precisely,

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the integral can be extended to the entire real domain and calculations can be computed in the complex plane. In the same context, many difficult integrals or differential equations can be easily calculated by considering the symmetry of the integrating function. This finds wide application in Quantum Mechanics, where the integral over the three-dimensional space of the spherical harmonics vanishes everywhere due to parity properties.

In Classical Mechanics, symmetries with respect to space or time translations allow one to define constants of motion that can be used to find the final configuration of a system, once the initial conditions are known. Without making use of invariance properties, it is still possible to solve differential equations of dynamics. Though this procedure has the great advantage of providing dynamics at any time, it turns out to be complicated because, sometimes, it leads to analytically unsolvable equations of motion. The invariance under time translation yields a time-independent Hamiltonian, while the invariance under space translations leads to the conservation of conjugate momenta. As shown in Section 3.2.1, these quantities can be thought as the on-diagonal components of the energy-momentum tensor.

Moreover, in Section 3.2.2, we will also show that, from the invariance of a system under rotations, another constant of motion arises: the *Angular Momentum*. As in Classical Mechanics, the Angular Momentum conservation can easily explain several phenomena, thereby avoiding a large amount of computations. In Quantum Mechanics, it plays a fundamental role in the development of theoretical atomic structure, including the theory of spin.

In the nineteenth century it was not clear whether conservation laws were valid even at microscopic scales, or if they were only the macroscopic result of microscopic averages. According to the currently accepted view, they hold even at the microscopic level and represent an essential tool in all fields of Science.

Another famous result based on symmetry considerations is the Gauss theorem, by means of which it is possible to get the electric field of given configurations without calculating the contribution of each charge. When a one-dimensional distribution is considered, such as a charged line, the surrounding electric field only depends on the radial distance from the line itself. It can be calculated either by integrating the contribution provided by any charge or by using the Gauss theorem. To compare the two procedures, let us consider a line with homogeneous linear density of charge λ and length h. The electric field is provided by an integration over the infinite set of charges at the distance r, namely:

$$\begin{cases} E_z = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \cos\theta\,\frac{dq}{R^2} = \frac{1}{4\pi\epsilon_0}\int \frac{z\,dq}{R^3} = \frac{\lambda}{4\pi\epsilon_0}\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{zdz}{\sqrt{(x^2+z^2)}^3} \\ E_x = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \sin\theta\,\frac{dq}{R^2} = \frac{1}{4\pi\epsilon_0}\int \frac{x\,dq}{R^3} = \frac{\lambda}{4\pi\epsilon_0}\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{xdz}{\sqrt{(x^2+z^2)}^3}. \end{cases} \tag{1.1}$$

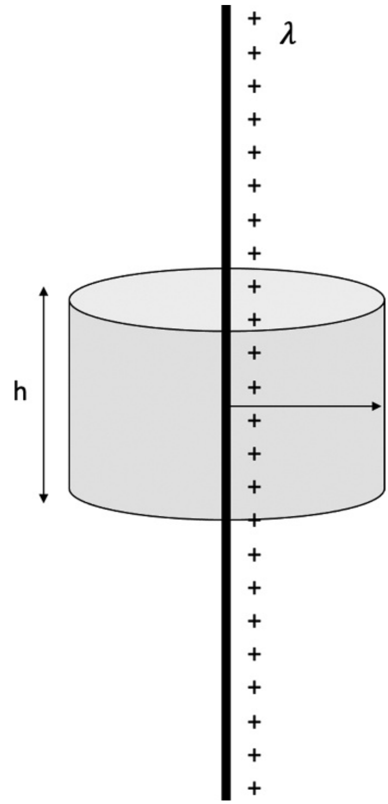

Figure 1.3 Closed cylindrical surface enclosing the charged line

After solving the integrals, it turns out that in the limit $h \to \infty$, the total electric field can be written as

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0|r|}\mathbf{e}_r, \tag{1.2}$$

with $\mathbf{e}_r$ being the unitary vector, which labels the field direction. The same result can be achieved by taking into account the Gauss theorem. Considering a cylindrical surface that encloses the line (as shown in Fig. 1.3), we notice that, in the case of infinite length, the total flux of the electric field over the base areas vanishes due to the same amount of incoming and outgoing contributions. Therefore only lateral areas contribute to the total flux, and the Gauss theorem can be written as

$$\int_{Cylinder} \mathbf{E} \cdot \mathbf{n}\, dS = \frac{\lambda h}{\epsilon_0}. \tag{1.3}$$

Thus the integral can be easily recast as a product, since the electric field is constant over the surface chosen. It finally reads

$$|E| = \frac{\lambda}{2\pi\epsilon_0|r|}. \tag{1.4}$$

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The role of symmetries in this treatment lies behind the choice of the cylindrical surface. Though the theorem holds for every closed surface, most of them make the procedure even more complicated. The closed surface that best fits the standard of this example must satisfy the symmetry of the system. When the line is supposed to be infinitely long, the system acquires a cylindrical symmetry, so that a cylindrical surface allows one to neglect the flux over the base areas.

This concept can be further stressed by considering the two-dimensional case. The electric field generated by a continuous surface with charge surface density σ can be straightforwardly obtained by means of the Gauss theorem as follows:

$$\Phi_E = \frac{q}{\epsilon_0} \rightarrow 2|E|S = \frac{\sigma S}{\epsilon_0} \rightarrow |E| = \frac{\sigma}{2\epsilon_0}. \tag{1.5}$$

Nevertheless, it is worth noticing that a careful choice of the closed surface is always important. It must respect the symmetry of the distribution considered; otherwise no simplification can be provided by the theorem.

So far we have dealt with continuous symmetries generated by continuous transformations. Another physically relevant example is given by a set of three *discrete* transformations, very useful in Particle Physics.

The first one is the so-called *Charge Conjugation* or *C-Parity*, whose action of the corresponding operator $\widehat{C}$ transforms a particle into the related antiparticle. A physical system that is symmetric with respect to such transformation is said to be *C-symmetric*. The only interaction violating the C-symmetry is the Weak Interaction. It goes without proving that once the C-Parity acts twice to a given state $|n>$, the final state must be the same as the initial one. Therefore the charge operator must have only two eigenvalues: ± 1.

Another discrete transformation is the so-called *Parity Transformation*, which reverses the spatial coordinates of the system. For the same reason as before, the eigenvalues of the corresponding operator can assume only the values ± 1. This transformation is very important since it allows one to distinguish scalar from pseudoscalar or vector from pseudovector. Scalar quantities that change sign under parity transformations are called *pseudoscalars*, while vector fields that do not change sign under parity transformations are called *pseudovectors*. Even in this case, the only fundamental force that violates parity symmetry is the Weak Interaction.

The last discrete transformation is the *Time Reversal*. It is a symmetry transformation of time $t \rightarrow -t$ whose violation can be attributed to the second principle of thermodynamics violation. Since the symmetry under time reversal (T-symmetry) implies the conservation of entropy, the macroscopic Universe is not symmetric under such transformation. However, at very small scales, some systems exhibit symmetric properties under time reversal.

In the twentieth century, it was supposed that the laws of physics were invariant under CP transformations, which can be obtained by combining charge conjugation and parity transformation. However, in 1964 the *CP* symmetry violation was discovered in the decays of neutral kaons. Nowadays CP symmetry is still deeply studied due to its quantum and cosmological implications and because it might potentially explain the dominance of matter with respect to antimatter. Even though the attempt to find a discrete symmetry for all physical systems failed in 1964, soon after it turned out that this feature could be attributed to CPT transformation. It represents the only discrete symmetry in nature, whose violation would imply the Lorentz invariance violation, as shown by Greenberg in [4]. Therefore it automatically follows that a system violating the CP symmetry must also violate T-symmetry. Basically, according to CPT symmetry, a hypothetical universe where matter is replaced with antimatter, where particles and fields have opposite positions, and where time flows in the opposite direction would evolve exactly like our observed Universe.

In light of these examples, given the large and different amounts of symmetries in Physics, it is worth distinguishing all possible cases that can characterize a given system. Continuous symmetries, first, are symmetries that can be described by infinitesimal transformations generated by continuous parameters. On the other hand, discrete symmetries describe noncontinuous changes in the system as well as parity or charge conjugation. Moreover, continuous symmetries can be further split into *local* symmetries, which depend on the space-time point of the given manifold, and *global* symmetries, which are independent of the local position. Invariance under translations or Lorentz invariance belongs to the latter category, and gauge invariance to the former. This last distinction is fundamental since, as pointed out in Chapter 4, it is responsible for the impossibility of treating gravity under the formalism of the Yang–Mills theory. Furthermore, an internal transformation acting on the field $\phi(x^a)$ does not involve the coordinates x^a, so that its variation reads as $\phi(x^a) \to \tilde{\phi}(x^a)$. On the contrary, an external transformation changes also the coordinates, so that the given scalar field turns out to transform as $\phi(x^a) \to \tilde{\phi}(\tilde{x}^a)$.

The need of treating gravity as a gauge theory comes from the fact that gauge invariance seems to be a crucial property toward the construction of a self-consistent theory of Quantum Gravity. Indeed, it turns out that theories that are not invariant under gauge transformations do not have any predictive power at UV scales. Nevertheless they may be perfectly acceptable theories at lower scales. The main issue lies behind the diffeomorphism invariance enjoyed by the gravitational action, which is an external symmetry. On the other hand, the nongravitational fundamental interactions are symmetric under local gauge transformations and this provides a non-Abelian structure that gravity in four dimensions cannot show. Reversing the argument, the gravitational action of GR cannot be written as a local gauge-invariant action (Yang–Mills theory) for the diffeomorphism group. Nevertheless, it is still possible to locally link the curved space-time to a flat tangent space-time point by point. This procedure makes standard GR invariant under the local Lorentz group (LG), where tetrad fields and spin connections are used to label the geometry. For a detailed discussion see

Section 5.2, of Chap. 5, where we show how to recast the gravitational interaction as a local gauge theory.

To conclude this short overview of symmetries in Physics, if the action is invariant under some transformation that does not act on the Euler–Lagrange (EL) equations, the system is said to enjoy an *on-shell* symmetry; otherwise, if the transformation changes the EL equations, it is said to be *off-shell*. To better clarify this point, let us consider a field transformation $\phi \to \tilde{\phi}$, whose Lagrangian variation yields a set of EL equations plus a total derivative TD: $\delta\mathcal{L} \to EL + TD$. On-shell symmetry occurs if $EL = TD = 0$; otherwise when $EL = -TD$ we have an off-shell symmetry. In the former case, the total derivative can be intended as a conserved quantity $TD = \partial_a j^a$, called Noether current (see the following chapters for details).

In what follows, we shall overview the main features of the symmetry groups that play a crucial role in the applications of the Noether Theorem. We mainly analyze the unitary $U(n)$ group, the LG – isomorphic to $O(1,3)$ – the translation and rotation group, and the Poincaré group.

1.1.1 The Unitary Group

The n-parameters unitary group $U(n)$ and the special unitary group $SU(n)$ represent an essential step in the framework of symmetries in fundamental Physics. Formally, a $n \times n$ matrix M is said to be *unitary* if $M^\dagger M = \mathbb{1}$, where $M^\dagger$ is the conjugate transpose of M. These matrices, whose determinant may only assume the two values ± 1, form the so-called $U(n)$ group. The set of unitary matrices with positive determinant form in turn the *special* subgroup $SU(n)$. It is the $n^2 - 1$ dimensional subgroup of the n^2 dimensional $U(n)$, which is, in turn, a subgroup of the general $GL(n, \mathbb{C})$. Of particular interest are the $U(1)$ transformations, which form the only Abelian subgroup of $U(n)$ with vanishing structure constants of the corresponding Lie algebra. $U(1)$ transformations are fundamental in all physical interactions since all Lagrangians depending on scalar combinations of the variables are $U(1)$-invariant. From the definition of $SU(n)$, it follows that the $U(n)$ group can be written as the semi-direct product between $SU(n)$ and $U(1)$. Other important subgroups of $U(n)$ having several physical implications are the $SU(3)$ group (whose algebra is defined by the Gell-Man matrix) for Quantum Chromodynamics and the $SU(2)$ group for the Electroweak Interaction. Regarding the latter, it can be straightforwardly shown that the Pauli Matrices satisfy all the constraints imposed by the $SU(2)$ Lie algebra and therefore can be chosen as generators of $SU(2)$ transformations. Being represented by all the 2×2 matrices with unitary determinant, the generator G of $SU(2)$ must satisfy the relations

$$G^\dagger G^{-1} = \mathbb{1}, \qquad Tr[G] = 0. \tag{1.6}$$

The second condition can be easily demonstrated by considering an element g of $SU(2)$ that weakly differs from the unitary matrix:

$$g = \mathbb{1} + iG \rightarrow det(g) = 1 + iTr[G] \rightarrow Tr[G] = 0. \tag{1.7}$$

Since the Pauli Matrices, σ^a, are traceless and Hermitian, they can be arbitrarily chosen as a basis for $SU(2)$. In this way, the generator G is represented by any linear combination of σ^a. Moreover, by means of the definition $G^\alpha = \sigma^\alpha/2$ the Lie algebra of $SU(2)$ turns out to be

$$[G^a, G^b] = i\epsilon^{abc} G_c. \tag{1.8}$$

Notice that, as better pointed out in Section 1.1.3, the above algebra is formally equivalent to that of the $SO(3)$ group.

1.1.2 The Translation Group

Let us now consider another set of transformations that act on the positions of all points. It is the Abelian subgroup of plane isometries. In order to find the transformation generator, we take into account the infinitesimal translation

$$x^a \rightarrow x^a + \delta x^a, \tag{1.9}$$

with respect to which a scalar field $\phi(x^a)$ changes as

$$\tilde{\phi}(x^a) = \phi(x^a + \delta x^b) \sim \phi(x^a) + \delta x^b \partial_b \phi(x^a). \tag{1.10}$$

The transformation, therefore, can be written as

$$T\phi = \phi + \partial_b \phi \delta x^b \rightarrow T = \mathbb{1} + \delta x^b \partial_b, \tag{1.11}$$

and coming back to the corresponding finite quantity, the preceding translation takes the form

$$T = e^{-i\epsilon^a \mathcal{T}_a}, \tag{1.12}$$

where $\epsilon^a \equiv \delta x^a$ and $\mathcal{T}_a$ the group generator is defined as $\mathcal{T}_a \equiv i\partial_a$.

1.1.3 The Rotation Group

Other transformations change the angles of vectors but preserve the distances. The Lie group that corresponds to such transformations is the so-called orthogonal group $O(n)$, where n is the number of dimensions. It admits as a subgroup the *special orthogonal* group $SO(n)$, called *Rotation Group*, whose determinant of the corresponding matrices is equal to $+1$. In addition, it can be connected to the preceding unitary group according to the relation $O(n) \supset U(n) \supset SU(n)$. The 3-D rotation group $SO(3)$ consists of 3×3 symmetric matrices whose elements represent the given rotation. In general, the elements of $O(n)$ are combinations of rotations.

To introduce the main features of the $SO(n)$ group, let $\mathcal{R}$ be an infinitesimal rotation transforming the vector r into another vector, $\tilde{r}$:

$$\tilde{r} = \mathcal{R}(\theta)\, r. \tag{1.13}$$

A finite rotation around the direction j can be written as $\mathcal{R}_j(\theta) = e^{-i\hat{L}_j\theta}$, where $\hat{L}_j$ is the generator of the transformation. By means of the preceding relation, it is possible to write the corresponding infinitesimal transformation as

$$\mathcal{R}_j(\delta\theta) = 1 - i\,\delta\theta\,\hat{L}_j. \tag{1.14}$$

The rotation transforms the vector $r(x_i, x_k, x_j)$ to $\tilde{r}(x_i, x_k, x_j) = r(x_i - x_k d\theta, x_k + x_i d\theta, x_j)$, which, up to the first order, can be written as follows[1]:

$$\begin{aligned} &\tilde{r}(x_i, x_k, x_j) = r(x_i, x_k, x_j) + d\theta\,(x_i\partial_k - x_k\partial_i)\, r(x_i, x_k, x_j), \\ &\delta r = d\theta\,(x_i\partial_k - x_k\partial_i)\, r(x_i, x_k, x_j). \end{aligned} \tag{1.15}$$

Equation (1.15) shows that the infinitesimal generator of the rotation is $\hat{L}_j = (x_i\partial_k - x_k\partial_i)$, with corresponding Lie algebra given by

$$[\hat{L}_i, \hat{L}_j] = i\,\epsilon_{ijk}\hat{L}_k. \tag{1.16}$$

As is well known, $\hat{L}_j$ represents the j-component of the orbital angular momentum operator. Note that the Lie algebra is formally equivalent to that of $SU(2)$ in Eq. (1.8). This means that the two groups are locally equivalent. Indeed, the $SU(2)$ group can be understood as the universal cover of $SO(3)$.

1.1.4 The Lorentz Group

Once the three-dimensional rotations have been defined and the main features of the $O(n)$ group have been set up, we can introduce the Lorentz transformations (LTs), which form the so-called *Lorentz group*, isomorphic to $O(1,3)$. Considering the properties of LTs (hereafter denoted by the symbol Λ), one can straightforwardly prove that they respect all the axioms needed to form a group. Specifically, they are *closed* since two consecutive LTs provide another LT, namely

$$\Lambda_1, \Lambda_2 \in LG \rightarrow \Lambda_1\Lambda_2 \in LG\,. \tag{1.17}$$

The *associativity* and the existence of the *identity* transformation can be similarly proven. Moreover, from the possibility of going back to the starting reference frame, it follows that LTs are also *invertible*. We mainly focus on the only connected subgroup of LG, namely the *Restricted Lorentz Group* (RLG), isomorphic

[1] In this paragraph and in 3.2.2, we only consider the three-dimensional case; thus the Einstein convention on indexes is not used.

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to $SO(1,3)$ and characterized by the conditions $|\Lambda| = +1$ and $\Lambda_{00} \geq 1$. With the aim of obtaining the generator of the RLG, let us consider a LT weakly differing from the identity:

$$\Lambda^c_d = \delta^c_d + \lambda^c_d. \tag{1.18}$$

From the definition of LT, it follows that $\eta_{cd}\Lambda^c_a\Lambda^d_b = \eta_{ab}$, where η_{cd} is the Minkowski tensor. Substituting the preceding relation into Eq. (1.18), we obtain

$$\eta_{cd}(\delta^c_a + \lambda^c_a)(\delta^d_b + \lambda^d_b) = \eta_{ab}, \tag{1.19}$$

so that, neglecting higher-than-first-order terms, Eq. (1.19) yields:

$$\eta_{cd}(\delta^c_a\delta^d_b + \delta^c_a\lambda^d_b + \lambda^c_a\delta^d_b) = \eta_{ab} \to \lambda_{ab} = -\lambda_{ba}. \tag{1.20}$$

Finally, by means of the definition

$$\left(\mathcal{J}^{ab}\right)^c_{\ d} = \frac{i}{2}\left(\eta^{ac}\delta^b_d - \eta^{bc}\delta^a_d\right), \tag{1.21}$$

the infinitesimal LT reads

$$\Lambda^c_d = \delta^c_d - i\lambda_{ab}\left(\mathcal{J}^{ab}\right)^c_{\ d}. \tag{1.22}$$

The corresponding finite transformation therefore can be written as $\Lambda = e^{-i\lambda_{ab}\mathcal{J}^{ab}}$, with $\mathcal{J}$ being the generator of the RLG, satisfying the following Lie algebra:

$$\left[\mathcal{J}^{ab}, \mathcal{J}^{cd}\right] = \frac{i}{2}\left(\eta^{ad}\mathcal{J}^{bc} + \eta^{bc}\mathcal{J}^{ad} - \eta^{ac}\mathcal{J}^{bd} - \eta^{bd}\mathcal{J}^{ac}\right). \tag{1.23}$$

It is possible to split the contributions of boosts from those of rotations, by means of the definitions

$$\begin{cases} \mathcal{R}^i = \epsilon^{ij\ell}\mathcal{J}_{j\ell} \\ \mathcal{B}^i = \mathcal{J}^{0i}. \end{cases} \tag{1.24}$$

In this way, the general LT takes the form

$$\Lambda = \exp\left\{-i\left(\lambda_{0i}\mathcal{B}^i - \epsilon_{ij\ell}\lambda^{j\ell}\mathcal{R}^i\right)\right\}. \tag{1.25}$$

It is worth noticing that LTs can be intended as the combination of three-dimensional rotation and boosts. The LG plays a role of special interest in fundamental physics because LTs can be used to define the rank of tensor fields. In particular, depending on how tensor fields transform under LTs, the following distinctions can be made:

- The spin-n *tensor field* under LTs transforms as an n-rank tensor, namely

$$\phi^{b_1b_2b_3\ldots.b_n}(x^a) \xrightarrow{LT} \tilde{\phi}^{b_1b_2b_3\ldots.b_n}(\tilde{x}^a) = \Lambda^{b_1}_{c_1}\Lambda^{b_2}_{c_2}\Lambda^{b_3}_{c_3}\ldots\ldots\Lambda^{b_n}_{c_n}\phi^{c_1c_2c_3\ldots.c_n}(x^a). \tag{1.26}$$

Two main subcases follow:

1. The spin-0 *scalar field* is defined as the invariant under LTs, that is,

$$\phi(x^a) \xrightarrow{LT} \tilde{\phi}(\tilde{x}^a) = \phi(x^a). \tag{1.27}$$

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2. The spin-1 *vector field* transforms as

$$\phi^b(x^a) \xrightarrow{LT} \tilde{\phi}^b(\tilde{x}^a) = \Lambda^b_c \phi^c(x^a). \tag{1.28}$$

- The spin-$\frac{1}{2}$ *spinor field*, whose transformation is defined through the Dirac matrices, is

$$\Phi(x^a) \xrightarrow{LT} \tilde{\Phi}(\tilde{x}^a) = e^{-\frac{i}{4}\lambda_{ab}\sigma^{ab}}\Phi(x^a), \tag{1.29}$$

where Φ is the spinor field, $\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b]$, and γ are the Dirac matrices.

1.1.5 The Poincaré Group

The Poincaré group is an extension of the preceding LG. It consists of the semi-direct product $T \otimes O(1,3)$ and forms a 10-parameter Lie group. Four parameters belong to the translations group, and the remaining six to the LG. When the Poincaré transformation acts on a vector field ϕ^c, the latter transforms as

$$\tilde{\phi}^c = \Lambda^c_d \phi^d + t^c. \tag{1.30}$$

The Poincaré group finds the most application in Quantum Mechanics, where the relativistic single-particle states are described by an irreducible unitary representation of the covering group of the Poincaré group. The Poincaré transformation can be obtained by merging the generic translation in Eq. (1.12), with an LT (1.22):

$$PT = \exp\left\{-i\left(\lambda_{ab}\mathcal{J}^{ab} + \epsilon_c \mathcal{T}^c\right)\right\}. \tag{1.31}$$

The corresponding Lie algebra is ruled by the relations

$$\begin{cases} [\mathcal{T}^a, \mathcal{J}^{cd}] = -\dfrac{i}{2}\eta^{a[c}\mathcal{T}^{d]} \\ [\mathcal{J}^{ab}, \mathcal{J}^{cd}] = i\left(\eta^{ca}\mathcal{J}^{db} - \eta^{cb}\mathcal{J}^{da} + \eta^{db}\mathcal{J}^{ca} - \eta^{da}\mathcal{J}^{cb}\right) \\ [\mathcal{T}^a, \mathcal{T}^b] = 0. \end{cases} \tag{1.32}$$

The summary proposed here is intended to point out the most important symmetry groups. Their main features are worth noticing in view of applications considered in Chapter 3, where the Noether Theorem will be applied to Lagrangian densities to find out the corresponding conserved quantities.

https://doi.org/10.1017/9781009208727.002 Published online by Cambridge University Press