THE CHORDS OF THE NON-RULED QUADRIC IN $PG(3, 3)$

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1. Introduction. In the preceding paper, Tutte described the forty-five chords of an "ellipsoid" in the finite space $PG(3, 3)$, showing that they may be regarded as the edges of a remarkable graph whose group is the group of automorphisms of the symmetric group $\mathfrak{S}_6$. The object of this sequel is to relate Tutte's idea to Sylvester's combinatorial investigation of the fifteen "duads" and fifteen "synthemes" formed by six symbols, and to Richmond's discussion of the figure of six points in a projective 4-space. (The *syntheme* 12,34,56 is made up of the three *duads* 12, 34, 56.)

2. The relevant work of Edge. Edge (**4**, pp. 271, 275) observed that the non-ruled quadric in $PG(3, 3)$ consists of ten points, and that the remaining thirty points in the space fall into two sets of fifteen, called "positive" and "negative," which are in (3, 3) correspondence: each positive point is joined by chords to three negative points, and likewise for the other sign. Tutte (**7**) observed that consequently the thirty "exterior" points and forty-five chords form a graph of degree three, which he identified with his 8-*cage* (**3**, p. 442).[1]

The present note achieves this identification more quickly by a notational device. Since the thirty vertices of the 8-cage (Figure 1)[2] correspond to the duads and synthemes of six digits (**6**, p. 92), we merely have to associate the same symbols with the thirty points of $PG(3, 3)$ not lying on the quadric. This was actually done by Edge (**4**, p. 275) when he found that each of the fifteen negative points is the sole common vertex of two of the six "negative pentagons" (complete pentagons whose ten edges are tangents), and that the fifteen positive points "answer one to each of the fifteen synthemes of negative pentagons." However, a more direct procedure is to replace Edge's six pentagons in $PG(3, 3)$ by six points in $PG(4, 3)$.

3. Richmond's contribution. The background for this idea was supplied by Richmond (**5**, pp. 127–129), who observed that the duads and synthemes provide an ideal notation for the "edges" and "transversals" of the *hexastigm*

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[1]I take this opportunity to correct an error on page 441: In the last two sentences of §8, the word "equilateral" occurs three times. The second and third occurrences should be deleted.

[2]For simplicity, only the "duad" vertices have been marked. The synthemes can be immediately inferred; for example, the vertex that appears between 14 and 23 (near the top) is (14,23,56).

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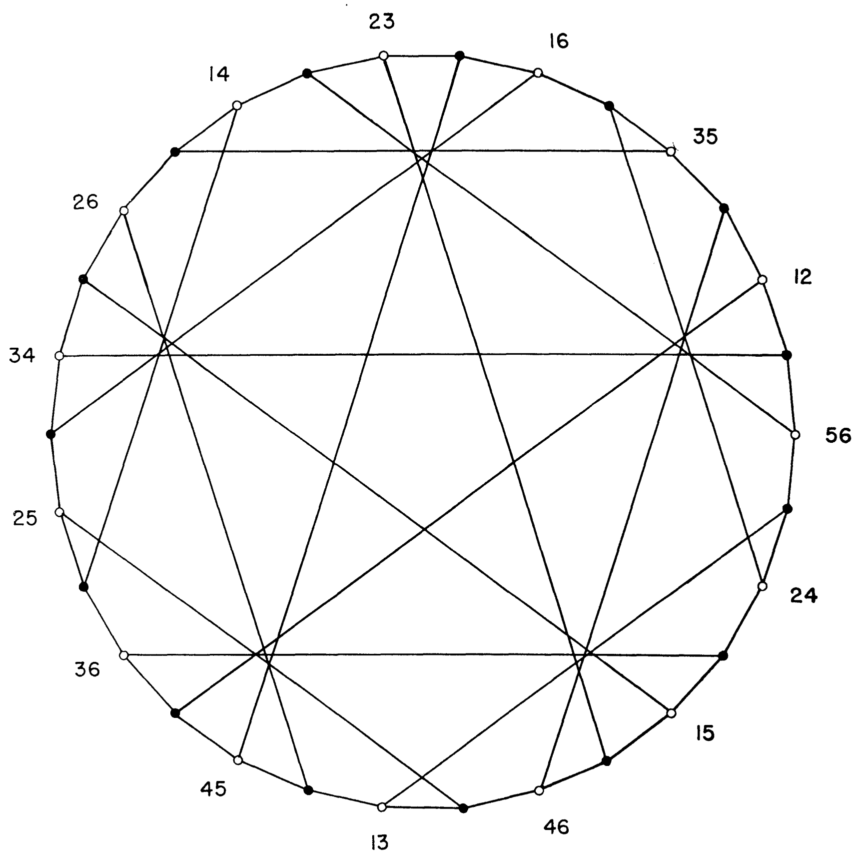

FIGURE 1

formed by six points of general position in any projective 4-space. The six *vertices 1, . . . , 6* of the hexastigm are joined in sets of two, three, or four by:

15 *edges* such as the line *12*,
20 *faces* such as the plane *123*,
15 *spaces* such as the hyperplane *1234*.

Each edge meets the opposite space in a *diagonal point* such as

$$P_{12} = 12 \cdot 3456.$$

Three edges which together involve all the six vertices are met by a unique line called a *transversal*; for example, the line

$$P_{12}P_{34}P_{56} = 3456 \cdot 5612 \cdot 1234$$

is the transversal of the edges *12*, *34*, *56*. The (3, 3) correspondence between edges and transversals (or between duads and synthemes) is seen in the fact that each transversal meets three edges while each edge belongs to three transversals. Since the three transversals that involve the edge *12* all meet it in the same point P_{12}, the fifteen diagonal points and the fifteen transversals form a configuration 15_3 of the kind used by Baker in two of his frontispieces **(1**; **2)**.

There are also fifteen *harmonic points* such as Q_{12}, the harmonic conjugate of P_{12} with respect to *1* and *2*, and ten points of "minor importance" **(5**, p. 128**)** such as

$$P_{123} = P_{456} = 123 \cdot 456,$$

the intersection of two opposite faces. The forty-five joins of these last ten points pass by threes through the fifteen harmonic points; for example, the lines

$$P_{145}P_{245}, \quad P_{135}P_{235}, \quad P_{134}P_{234}$$

all pass through Q_{12}. Other triads of these forty-five lines meet the transversals; for example, the lines

$$P_{134}P_{234}, \quad P_{123}P_{124}, \quad P_{125}P_{345}$$

all meet the transversal $P_{12}P_{34}P_{56}$.

To check these results we may use Möbius's barycentric calculus **(1**, p. 97; **2**, p. 115**)**, which allows us to "weight" the six vertices in such a way that

$$1 + 2 + 3 + 4 + 5 + 6 = 0.$$

Then

$$P_{12} = 1 + 2, \quad P_{34} = 3 + 4, \quad P_{56} = 5 + 6;$$
$$Q_{12} = 1 - 2; \quad P_{123} = 1 + 2 + 3, \quad P_{1jk} - P_{2jk} = 1 - 2 = Q_{12};$$
$$P_{134} + P_{234} = P_{12} + 2P_{34}, \; P_{123} + P_{124} = 2P_{12} + P_{34}, \; P_{125} - P_{345} = P_{12} - P_{34}.$$

4. Six points in $PG(4, 3)$. We shall find that, when the co-ordinate field is restricted to $GF(3)$, three new features present themselves:

I. *The ten "minor" points P_{ijk} all lie in a hyperplane.*
II. *They form a non-ruled quadric in this 3-space.*
III. *The three chords that meet a transversal all meet it in the same point.*

Using co-ordinates $(x_1, x_2, x_3, x_4, x_5)$ (mod 3), we take the vertices *1*, . . . , *5*, *6* to be

$$(1, 0, 0, 0, 0), \ldots, (0, 0, 0, 0, 1), (1, 1, 1, 1, 1).$$

Then the spaces *3456* and *1234* are

$$x_1 = x_2 \quad \text{and} \quad x_5 = 0,$$

the diagonal points P_{12} and P_{56} are

$$(1, 1, 0, 0, 0) \text{ and } (1, 1, 1, 1, 0),$$

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the transversal $P_{12} P_{34} P_{56}$ is

$$x_1 = x_2, \quad x_3 = x_4, \quad x_5 = 0,$$

and the faces *456* and *123* are

$$x_1 = x_2 = x_3 \qquad \text{and} \qquad x_4 = x_5 = 0.$$

Proof of I. The point *123·456* (denoted by P_{123} or P_{456}) is

$$(1, 1, 1, 0, 0).$$

Permuting the five co-ordinates, we obtain the ten "minor" points, all lying in the hyperplane

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0 \qquad \pmod 3.$$

Proof of II. These ten are the only points (in the 3-space) that lie on the quadric

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0,$$

which is non-ruled since its section by the tangent plane $x_1 + x_2 + x_3 = 0$ consists of the point of contact $(1, 1, 1, 0, 0)$ alone. In fact, our equation agrees with Edge's "ellipsoid"

$$x^2 + y^2 + z^2 + t^2 + (x + y + z + t)^2 = 0$$

(**4**, p. 274).

Proof of III. Since the chords lie in the 3-space while the diagonal points are outside, those chords which meet any one transversal must pass through the point in which that transversal meets the 3-space. An alternative proof is provided by the observation that, since $2 = -1$, the three symbols

$$P_{12} + 2P_{34}, \quad 2P_{12} + P_{34}, \quad P_{12} - P_{34}$$

(as well as $P_{34} - P_{56}$, $P_{56} - P_{12}$, etc.) all refer to the same point. Since this is the point of intersection of the 3-space $\Sigma x = 0$ and the transversal $P_{12} P_{34} P_{56}$, it is naturally denoted by the syntheme (12,34,56).

5. Conclusion. In this geometry every line contains just four points. In particular, the remaining two points on the chord $P_{134} P_{234}$ are

$$P_{134} - P_{234} = Q_{12} \qquad \text{and} \qquad P_{134} + P_{234} = (12,34,56).$$

The thirty points of $PG(3, 3)$ not on the non-ruled quadric are thus denoted by the fifteen duads such as 12 (meaning Q_{12}) and the fifteen synthemes such as (12,34,56). Each of the forty-five chords joins one "duad" point to one "syntheme" point, and the duad belongs to the syntheme. In other words, as Tutte observed, if we disregard the ten points that lie on the quadric itself, the forty-five chords are the edges of the 8-cage.

Edge's designation of the two sets of fifteen points as *negative* and *positive* can be justified by examining the co-ordinates. The "duad" points, such as

$$Q_{12} = (1, 2, 0, 0, 0), \quad Q_{56} = (1, 1, 1, 1, 2),$$

are "negative" since $\Sigma x^2 = -1$; the "syntheme" points, such as

$$(12, 34, 56) = (1, 1, 2, 2, 0),$$

are "positive" since $\Sigma x^2 = 1$.

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