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EMBEDDING OF ACCESSIBLE REGULAR CATEGORIES

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ABSTRACT. The purpose of this note is to prove that a regular accessible category has a full regular embedding into a set-valued functor category.

1. **Introduction** The main purpose of this paper is to extend the embedding theorem for small regular categories [Barr, 1971], to accessible categories.

THEOREM 1.1. *A regular accessible category has a full regular embedding into a set-valued functor category.*

1.2. An accessible category is like a locally presentable category in the sense of Gabriel and Ulmer, except that it is not necessarily complete (or, equivalently for accessible categories, cocomplete). More precisely, for some cardinal α, it has all colimits taken over diagrams which are α-filtered and all finite limits preserve these α-filtered colimits. Makkai and Paré have studied their properties at length in their forthcoming monograph, [Makkai & Paré, to appear].

It has long been clear that a great many categories have the property of being completely determined by some small subcategory. For example, the category of sets is the free colimit closure of the category of finite sets. The category of abelian groups is the free colimit closure of the category of finitely generated abelian groups. The accessible categories are the categories that are determined, in a similar but quite precise way, by some small subcategory.

In particular, any small category is accessible. This is not completely obvious. For example, not every filtered diagram of finite sets has a colimit which is finite. However every $\aleph_1$-filtered diagram of finite sets has a finite colimit so the category of sets is $\aleph_1$-filtered. This statement is equivalent to the assertion that if a colimit of finite sets is infinite, then already a countable subdiagram has an infinite colimit. (This is as good a time as any to mention that I am following the convention of Makkai & Paré by calling α-filtered a diagram in which every subdiagram of size $< \alpha$ has a cone over it.)

Makkai & Paré have shown that many of the properties that are valid for small categories are also true for accessible ones. Theorem 1.1 is one more example. This result

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implies of course that the abelian category embedding theorems of Mitchell [1965] and of Freyd-Mitchell-Heron [Freyd, 1964] are also valid for accessible categories.

Makkai has recently shown (using a very different argument) that a regular accessible category has a full exact embedding into a set-valued functor category, however with a large exponent. That result is phrased as a density theorem and the one proved here can also be so phrased. See [Makkai, to appear].

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2. **Embedding of small regular categories** The most important results we will need in order to carry out the proof are summarized in the following, found in [Barr, 1986].

PROPOSITION 2.1 *Let $\mathcal{C}$ be a small regular category and let $\mathcal{D}$ denote the opposite of the full subcategory* $\mathbf{FL}(\mathcal{C}, \mathbf{Set})$ *of finite-limit preserving set-valued functors on $\mathcal{C}$. Then*

1. *$\mathcal{D}$ is regular.*
2. *Every object D of $\mathcal{D}$ has a cover (a regular epimorphism) by an object P of $\mathcal{D}$ with the property that whenever $A \twoheadrightarrow B$ is a regular epimorphism in $\mathcal{C}$, then the induced* $\mathrm{Hom}_{\mathcal{D}}(P, A) \twoheadrightarrow \mathrm{Hom}_{\mathcal{D}}(P, B)$ *is surjective.*

Let us say that an object P with the property described in (2) above is **$\mathcal{C}$-projective**. It is important to note that the condition is on objects of $\mathcal{D}$ which are projective with respect to regular epimorphisms of $\mathcal{C}$.

2.2. To prove the full embedding theorem from just these facts, we begin by letting $\mathcal{P}$ be a full subcategory of $\mathcal{D}$ with the following three properties:

1. Every object of $\mathcal{P}$ is $\mathcal{C}$-projective.
2. Every object of $\mathcal{C}$ has a cover by some object of $\mathcal{P}$.
3. Given any pair of maps in $\mathcal{D}$, $P_0 \longrightarrow C \longleftarrow P_1$ whose sources lie in $\mathcal{P}$ and whose target is the same object of $\mathcal{C}$, there is a commutative square

$$\begin{array}{ccc} P_2 & \longrightarrow & P_1 \\ \downarrow & & \downarrow \\ P_0 & \longrightarrow & C \end{array}$$

with P_2 an object of $\mathcal{P}$. Moreover, if one of $P_0 \longrightarrow C$ or $P_1 \longrightarrow C$ is a regular epi, then the arrow opposite can be chosen to be one as well.

To show that such a subcategory exists, begin with a set of $\mathcal{C}$-projective covers of the objects of $\mathcal{C}$ and enlarge it by taking any $\mathcal{C}$-projective cover of any pullback of the form $P_1 \times_C P_2$ with P_1 and P_2 in $\mathcal{P}$ and C in $\mathcal{C}$. Doing this repeatedly will result in a subcategory with the desired properties. The last part of (3) follows from the regularity of $\mathcal{D}$.

Then I claim that the functor $\Phi : \mathcal{C} \longrightarrow \mathbf{Set}^{\mathcal{P}^{\mathrm{op}}}$ that takes C to $\mathrm{Hom}(-, C)$ is full, faithful and regular. The regularity is evident from (1), the faithfulness from (2) and the fullness is a consequence of (3) as I will now outline. Interestingly, the $\mathcal{C}$-projectivity of the objects of $\mathcal{P}$ plays no direct role in proving fullness.

Suppose that $\phi : \Phi(C) \longrightarrow \Phi(B)$ is a natural transformation. Naturality means that for each $P \in \mathrm{Ob}(\mathcal{P})$ there is a function $\phi(P) : \mathrm{Hom}_{\mathcal{D}}(P, C) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(P, B)$ with the property that for $g : Q \longrightarrow P$ in $\mathcal{P}$, $\phi(P)(c) \circ g = \phi(Q)(c \circ g)$ for all $c \in \Phi(C)(P) = \mathrm{Hom}_{\mathcal{D}}(P, C)$. We will write Hom for $\mathrm{Hom}_{\mathcal{D}}$ and ϕ for both $\phi(P)$ and $\phi(Q)$ to simplify notation.

Now given such a ϕ, choose a presentation

$$P_1 \underset{d^1}{\overset{d^0}{\rightrightarrows}} P_0 \xrightarrow{c_0} C$$

This means that P_0 is a cover of C and that P_1 is a cover of the kernel pair of $P_0 \longrightarrow C$. Since a cover is a regular epimorphism, this implies the sequence is a coequalizer in $\mathcal{D}$. Then $\phi(c_0) \in \mathrm{Hom}(P_0, B)$, that is $\phi(c_0) : P_0 \longrightarrow B$ and

$$\phi(c_0) \circ d^0 = \phi(c_0 \circ d^0) = \phi(c_0 \circ d^1) = \phi(c_0) \circ d^1$$

whence there is a unique arrow $f : C \longrightarrow B$ with $f \circ c_0 = \phi(c_0)$. Now let $c : P \longrightarrow C$ be an element of $\Phi(C)(P)$. There is a square

$$\begin{array}{ccc} Q & \xrightarrow{g} & P \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle c} \\ P_0 & \xrightarrow[c_0]{} & C \end{array}$$

and we have that

$$\phi(c) \circ g = \phi(c \circ g) = \phi(c_0 \circ h) = \phi(c_0) \circ h = f \circ c_0 \circ h = f \circ c \circ g$$

and g is epi, whence $\phi(c) = f \circ c$. This shows that ϕ is composition with f, which is fullness. □

3. **Properties of accessible categories.** Now I want to show that this argument can be modified to deal with the case of an accessible regular category. To do this, we let $\mathcal{A}$ be such a category.

We find it useful to use the terminological convention that when $\mathcal{C}$ is a subcategory of $\mathcal{A}$, we call the functor $Y : \mathcal{A} \longrightarrow \mathbf{Set}^{\mathcal{P}^{\mathrm{op}}}$ defined by $YA = \mathrm{Hom}(-, A)$ a Yoneda embedding, even though, strictly speaking, this term is correct only when $\mathcal{A} = \mathcal{C}$.

Makkai and Paré have shown that any α-accessible category $\mathcal{A}$ includes a small subcategory $\mathcal{C}$ such that the Yoneda embedding $Y : \mathcal{A} \longrightarrow \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ embeds $\mathcal{A}$ as the full subcategory of $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ consisting of the functors that are α-filtered colimits of functors representable by objects of $\mathcal{C}$. See [Makkai & Paré], Proposition 2.1.8.

They have further shown that the left Kan extension to $\mathcal{A}$ of every functor $\mathcal{C} \longrightarrow \mathbf{Set}$ preserves α-filtered colimits (such functors are called *α-flat*). See Proposition 2.4.3 of loc. cit., as well as the proof of that proposition.

PROPOSITION 3.1. *Let $\mathcal{A}$ be an accessible category. Then there is a cardinal α with the property that $\mathcal{A}$ is α-accessible and the full subcategory $\mathcal{A}_\alpha$ of α-accessible objects is regular and closed under subobjects in $\mathcal{A}$.*

PROOF. It follows from Proposition 5.2 of [Makkai, to appear], that for all suitable large α we can suppose that $\mathcal{A}_\alpha$ is closed in $\mathcal{A}$ under finite limits (in fact, limits of size $< \alpha$ insofar as they exist). Since α is also closed under colimits of size $< \alpha$ (an immediate consequence of Proposition 2.3.11 of [Makkai & Paré]), we conclude that $\mathcal{A}_\alpha$ is closed under the pullbacks used in the definition of regularity and hence is regular. Now choose a cardinal β for which $\mathcal{A}_\beta$ is regular and then an α sharply greater than β (see loc. cit., Definition 2.3.1) such that every subobject of an object of $\mathcal{A}_\beta$ is α-accessible. I claim that every subobject of an α-accessible object is also α-accessible which will complete the argument. So let A be α-accessible and $B \rightarrowtail A$. Using Proposition 2.3.11 of loc. cit., we see that A is an α-filtered of β-accessible objects, say $A = \mathrm{colim} A_i$. If we let $B_i = B \times_A A_i$, then from Proposition 5.2 of [Makkai], it follows that $B = \mathrm{colim}\, B_i$, an α-filtered colimit of α-accessible objects and hence by Proposition 2.3.11 of [Makkai & Paré], B is α-accessible. □

4. Proof of the main theorem.

PROPOSITION 4.1. *If $\Phi : \mathcal{C} \longrightarrow \mathbf{Set}$ is left exact, then its unique α-flat extension to $\mathcal{A}$ is also left exact.*

PROOF. A left exact functor is a filtered colimit of representable functors. But the left Kan extension of a representable functor is representable (by the same object) and left Kan extension, being a left adjoint, preserves the colimit, so the Kan extension is still a filtered colimit of representable functors and is thus left exact. □

Let $\Phi : \mathcal{C} \longrightarrow \mathbf{Set}^{\mathcal{P}^{\mathrm{op}}}$ be the exact full embedding constructed above. Each of the functors P in $\mathcal{P}$ can be extended to a left exact α-flat functor on $\mathcal{P}$. Thus Φ has an extension to a left exact, α-flat functor we will also denote $\Phi : \mathcal{A} \longrightarrow \mathbf{Set}^{\mathcal{P}^{\mathrm{op}}}$. We want to show that this extended Φ is full, faithful and preserves regular epis.

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LEMMA 4.2. *If C is an object of $\mathcal{C}$ and A is an object of $\mathcal{A}$, then every natural transformation $\phi : \Phi(C) \longrightarrow \Phi(A)$ is induced by a unique morphism $C \longrightarrow A$.*

PROOF. In this proof, we will be thinking of $\mathcal{C}$ as embedded in $\mathbf{FL}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}}$. Choose a resolution

$$P_1 \underset{d^1}{\overset{d^0}{\rightrightarrows}} P_0 \xrightarrow{c_0} C$$

for C. Let $A = \operatorname{colim} C_i$, the colimit taken over an α-filtered diagram in $\mathcal{C}$. Then c_0 is an element of $\Phi(C)(P_0)$ and so $\phi(P_0)(c_0) \in \Phi(A)(P_0)$. This element is, by the α-flatness of P, represented by a morphism $g : P_0 \longrightarrow C_i$ for some i. Since

$$\phi(P_0)(c_0) \circ d^0 = \phi(P_1 \circ d^0) = \phi(P_1)(c_0 \circ d^1) = \phi(P_0)(c_0) \circ d^1,$$

$g \circ d^0$ and $g \circ d^1$ represent the same element of $\Phi(P_1)(A)$. But the α-flatness implies that by an appropriate choice of i in the colimit diagram, we may suppose that $g \circ d^0 = g \circ d^1$. (In fact the α-flatness implies that any set of fewer than α such equalities could be simultaneously satisfied.)

But then there is a unique map $f : C \longrightarrow C_i$ such that $g = f \circ c_0$. If for some other index j, there were another map $h : C \longrightarrow C_j$ for which $g = f \circ c_0$ represented the same element of $\Phi(A)(P_0)$ as $h \circ c_0$, we could find an index k later than both i and j and transition arrows $t_{ki} : C_i \longrightarrow C_k$ and $t_{kj} : C_j \longrightarrow C_k$ for which $t_{ki} \circ f \circ c_0 = t_{kj} \circ h \circ c_0$. But with c_0 epi, this would imply that $t_{ki} \circ f = t_{kj} \circ h$ which would mean that f and h represent the same map from $C \longrightarrow A$ and thus must be already the same.

So far we have seen that there is a unique map $C \longrightarrow A$ such that $\phi(P_0)(c_0) = f \circ c_0$. Now let $c : P \longrightarrow C$ be an arbitrary element of $\Phi(C)(P)$. As above we have the square

$$\begin{array}{ccc} Q & \xrightarrow{p} & P \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle c} \\ P_0 & \xrightarrow[c_0]{} & C \end{array}$$

Then we have, from naturality,

$$\phi(c) \circ p = \phi(c \circ p) = \phi(c_0 \circ q) = \phi(c_0) \circ q = f \circ c_0 \circ q = f \circ c \circ p.$$

Since the filtered colimit in a set-valued functor category of monos is mono, it follows from the fact that p is epi, that each of the terms in

$$\operatorname{colim} \operatorname{Hom}(P, C_i) \longrightarrow \operatorname{colim} \operatorname{Hom}(Q, C_i)$$

induced by p is mono and hence the colimit is. Thus we have that $\phi(c) = f \circ c$ for a unique f. □

To finish the proof that Φ is full and faithful, let B be another object of $\mathcal{A}$ and let $B = \operatorname{colim} D_j$. Then we have

$$\operatorname{Hom}(\Phi(B), \Phi(A)) \cong \operatorname{Hom}(\operatorname{colim} \Phi(D_j), \Phi(A)) \cong \lim \operatorname{Hom}(\Phi(D_j), \Phi(A)) \cong \lim(\operatorname{Hom}(D_j, A) \cong \operatorname{Hom}(\operatorname{colim} D_j, A) \cong \operatorname{Hom}(B, A). \qquad \square$$

In order to see that Φ preserves regular epis, we need:

LEMMA 4.3. *Let $A \twoheadrightarrow C$ be a regular epi. Then there is a $C_i \to A$ for which the composite $C_i \to A \to C$ is a regular epi.*

PROOF. For each $C_i \to A$, let C_i' denote the image of $C_i \to A \to C$. Since an α-filtered colimit of monos is a mono, the colimit of the C_i' is a subobject of C. Since it includes the image of each of the $C_i \to A \to C$, it includes the image of $A \to C$ and hence is all of C. Since $\operatorname{Hom}(C, -)$ commutes with α-filtered colimits, we have

$$\operatorname{Hom}(C, C) = \operatorname{Hom}(C, \operatorname{colim} C_i') = \operatorname{colim} \operatorname{Hom}(C, C_i').$$

This implies that the identity map factors through one of the C_i', which means that C is one of the C_i' and that $C_i \twoheadrightarrow C$. $\square$

We now have that the composite $\Phi(C_i) \to \Phi(A) \to \Phi(C)$ is epi, which means that the second factor is. $\square$

5. **Intersections.** The embeddings constructed in [Barr, 1971, 1986] could be taken to preserve intersections. We show here that the same is true in this case. The way not to prove it is to represent a family of subobjects by families of subobjects of α-presentables mapping to it. This approach would necessitate a commutation of intersection with the colimit, which is false in general.

PROPOSITION 5.1. *Suppose $\Phi : \mathcal{C} \to$ **Set** preserves arbitrary intersection. Then its unique α-flat extension to $\mathcal{A}$ also preserves arbitrary intersections.*

PROOF. As shown in the proof of Proposition 4.1, the Kan extension of a functor which is a colimit of representables is the colimit of the same diagram of representables. In [Barr, 1986], Theorem 16, it is shown that a set-valued functor on a regular category preserves arbitrary intersection if and only if it is a colimit of representables, say $\operatorname{colim} \operatorname{Hom}(A_i, -)$ in which each transition arrow $\operatorname{Hom}(A_i, -) \to \operatorname{Hom}(A_j, -)$ in the diagram is induced by a regular epimorphism $A_j \twoheadrightarrow A_i$. (More generally, a strong epi would be necessary and sufficient.) But then if Φ preserves intersections on $\mathcal{C}$, it is such a colimit and then so is its left Kan extension. $\square$

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