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THE INDIVIDUAL WEIGHTED ERGODIC THEOREM FOR BOUNDED BESICOVITCH SEQUENCES

BY

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ABSTRACT. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, p fixed, $1<p<\infty$, T a linear operator of $L_p(X, \mathcal{F}, \mu)$, $\{\alpha_i\}$ a sequence of complex numbers. If

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \alpha_i T^i f$$

exists and is finite a.e. we say the individual weighted ergodic theorem holds for T with the weights $\{\alpha_i\}$.

In this paper we show that if $\{\alpha_i\}$ is a bounded Besicovitch sequence and T is a Dunford-Schwartz operator (i.e.: $\|T\|_1 \le 1$, $\|T\|_\infty \le 1$) then the individual weighted ergodic theorem holds for T with the weights $\{\alpha_i\}$.

1. Introduction. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, p fixed, $1<p<\infty$, $T_1 \cdots T_m$ linear operators of $L_p(X, \mathcal{F}, \mu)$. If T_i is simultaneously a contraction of $L_1(X, \mathcal{F}, \mu)$ and $L_\infty(X, \mathcal{F}, \mu)$ (and hence a contraction of $L_p(X, \mathcal{F}, \mu)$), then we say that T_i is a *Dunford-Schwartz* operator. If $\{\alpha_i^k\}_{i=0}^\infty$, $k=1, \ldots, m$, are sequences of complex numbers, we say the individual weighted ergodic theorem holds for $T_1, \ldots, T_m$ with the weights $\{\alpha_i^k\}_{i=0}^\infty$, $k=1,\ldots,m$, if

$$\lim_{n_1,\ldots,n_m\to\infty} \frac{1}{n_1 \cdots n_m} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_m=0}^{n_m-1} \alpha_{k_1}^1 \cdots \alpha_{k_m}^m T_1^{k_1} \cdots T_m^{k_m} f$$

exists and is finite a.e. for every $f \in L_p(X, \mathcal{F}, \mu)$. Here, the limit is taken in the sense that $n_1, \ldots, n_m$ tend to infinity independent of each other. In this paper we show that if $\{\alpha_i^k\}_{i=0}^\infty$, $k=1,\ldots,m$ are bounded Besicovitch sequences (see Section 2 for definition), and $T_1, \ldots, T_m$ are *Dunford-Schwartz* operators, then the individual weighted ergodic theorem holds for T with the weights $\{\alpha_i^k\}_{i=0}^\infty$.

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2. Preliminaries. We first define the term bounded Besicovitch sequence.

DEFINITION. Let $\{\alpha_i\}_{i=0}^{\infty}$ be a sequence of complex numbers. We say that $\{\alpha_i\}_{i=0}^{\infty}$ is a bounded Besicovitch sequence if

(1) There exists α such that $|\alpha_i| < \alpha$ for every i.

(2) For every $\varepsilon > 0$ there exists a trigonometric polynomial w_ε such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\alpha_i - w_\varepsilon(i)| < \varepsilon.$$

We will also need the following facts about Dunford-Schwartz operators.

LEMMA 2.1. *Let T be a Dunford-Schwartz operator. Then there exists a positive Dunford-Schwartz operator S ($f \geq 0$ implies $Sf \geq 0$) such that $|T^n f| \leq S^n |f|$ a.e. for all $f \in L_p(X, \mathcal{F}, \mu)$.*

Proof. See (1, Lemma VIII. 6.4).

We will also need the Dunford-Schwartz ergodic theorem [1, Theorem VIII 6.9].

THEOREM 2.1. *Let $T_1 \cdots T_m$ be Dunford-Schwartz operators, then*

$$\lim_{n_1,\ldots,n_m\to\infty} \frac{1}{n_1 \cdots n_m} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_m=0}^{n_{m-1}} T_1^{k_1} \cdots T_m^{k_m} f$$

exists and is finite a.e. for all $f \in L_p(X, \mathcal{F}, \mu)$.

3. Main Result. We now state and prove our main result.

THEOREM 3.1. *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, $T_1, \ldots, T_m$ Dunford-Schwartz operators, and $\{\alpha_i^k\}_{i=0}^{\infty}$, $k = 1, \ldots, m$ bounded Besicovitch sequences. Then*

$$\lim_{n_1,\ldots,n_m\to\infty} \frac{1}{n_1 \cdots n_m} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_m=0}^{n_m-1} \alpha_{k_1}^1 \cdots \alpha_{k_m}^m T_1^{k_1} \cdots T_m^{k_m} f$$

exists and is finite a.e. for all $f \in L_p(X, \mathcal{F}, \mu)$ and all p, $1 < p < \infty$.

Proof. First, let $\theta_1 \cdots \theta_m$ be complex numbers. Then the operators T_k^1 defined by $T_k^1 f = e^{i\theta_k} T_k f$ are Dunford-Schwartz operators, and the theorem follows in the case that $\alpha_n^k = e^{i\theta_k n}$ from Theorem 2.1 and by linearity of converging sequences,

$$\lim_{n_1,\ldots,n_m\to\infty} \frac{1}{n_1 \cdots n_m} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_m=0}^{n_m-1} \alpha^1(k_1) \cdots \alpha^m(k_m) T_1^{k_1} \cdots T_m^{k_m} f$$

exists and is finite a.e. for any trigonometric polynomials α^i, $i = 1, \ldots, m$.

Now fix p, $1 < p < \infty$, and let $f \in L_\infty(X, \mathcal{F}, \mu) \cap L_p(X, \mathcal{F}, \mu)$. Let $\varepsilon > 0$ and

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choose trigonometric polynomials $w^1 \cdots w^m$ as follows. Let w^1 be such that

$$\overline{\lim} \left| \frac{1}{n} \sum_{k=0}^{n-1} \alpha_k^1 - w^1(k) \right| < \frac{\varepsilon}{m \cdot \alpha_2 \cdots \alpha_m}$$

where $|\alpha_k^i| \le \alpha_i$ for all k. w^1, being a trigonometric polynomial, is bounded on $\mathbb{R}$, so there exists $m_1 < \infty$ such that $|w^1(k)| \le m_1$ for all k. Now choose w^2 such that

$$\overline{\lim} \left| \frac{1}{n} \sum_{k=0}^{n-1} \alpha_k^2 - w^2(k) \right| < \frac{\varepsilon}{m m_1 \alpha_3 \cdots \alpha_m}$$

Continuing by induction, choose w_l such that

$$\overline{\lim} \left| \frac{1}{n} \sum_{k=0}^{n-1} \alpha_k^l - w^l(k) \right| < \frac{\varepsilon}{m m_1 \cdots m_{l-1} \alpha_{l+1} \cdots \alpha_m}$$

where $|w_i(k)| \le m_i$ for all k, $1 \le i \le l-1$. Now we have

$$\begin{aligned}
&\overline{\lim} \left| \frac{1}{n_1 \cdots n_m} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_m=0}^{n_m-1} \alpha_{k_1}^1 \cdots \alpha_{k_m}^m T_1^k \cdots T_k^m f \right. \\
&\qquad \left. - \frac{1}{n_1 \cdots n_m} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_m=0}^{n_m-1} w^1(k_1) \cdots w^m(k_m) T_1^{k_1} \cdots T_m^{k_m} f \right| \\
&\le \overline{\lim} \frac{1}{n_1 \cdots n_m} \sum_{k_1=0}^{n_1-1} \sum_{k_m=0}^{n_m-1} \sum_{l=0}^{m-1} |w^1(k_1) \cdots w^l(k_l) \alpha_{k_{l+1}}^{l+1} \cdots \alpha_{k_m}^m \\
&\qquad - w^1(k_1) \cdots w^l(k_l) w^{l+1}(k_{l+1}) \alpha_{k_{l+2}}^{l+2} \cdots \alpha_{k_m}^m| \, \|f\|_\infty. \\
&\le \overline{\lim} \frac{1}{n_1 \cdots n_m} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_m=0}^{n_m-1} \sum_{l=0}^{m-1} m_1 \cdots m_l \alpha_{l+2} \cdots \alpha_m \left| \alpha_{k_{l+1}}^{l+1} \right. \\
&\qquad \left. - w^{l+1}(k_{l+1}) \right| \|f\|_\infty < \varepsilon \, \|f\|_\infty
\end{aligned}$$

From this we conclude that the theorem is true is $f \in L_p(X, \mathcal{F}, \mu) \cap L_\infty(X, \mathcal{F}, \mu)$. For arbitrary $f \in L_p(X, \mathcal{F}, \mu)$,

$$\begin{aligned}
&\left| \frac{1}{n_1 \cdots n_m} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_m=0}^{n_m-1} \alpha_{k_1}^1 \cdots \alpha_{k_m}^m T_1^{k_1} \cdots T_m^{k_m} f \right| \\
&\qquad \le \frac{1}{n_1 \cdots n_m} \alpha_1 \cdots \alpha_m \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_m=0}^{n_m-1} |T_1^{k_1} \cdots T_m^{k_m} f| \\
&\qquad \le \frac{\alpha_1 \cdots \alpha_m}{n_1 \cdots n_m} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_m=0}^{n_m-1} S_1^{k_1} \cdots S_m^{k_m} |f|
\end{aligned}$$

by Lemma 2.1, where

$$|T_i f| = S_i \, |f|.$$

Applying Theorem 2.1, to the right hand side of the above inequality, we now have the situation where

$$\lim_{n_1 \cdots n_m \to \infty} \frac{1}{n_1 \cdots n_m} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_m=0}^{n_m=1} \alpha^1_{k_1} \cdots \alpha^m_{k_m} \cdot T_1^{k_1} \cdots T_m^{k_m} f$$

converges a.e. on a dense set in $L_p(X, \mathcal{F}, \mu)$ and

$$\sup_{n_1 \cdots n_m} \left| \frac{1}{n_1 \cdots n_m} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_m=0}^{n_m-1} \alpha^1_{k_1} \cdots \alpha^m_{k_m} T_1^{k_1} \cdots T_m^{k_m} f \right| < \infty$$

for all f in $L_p(X, \mathcal{F}, \mu)$, which, using theorem IV: 11.3, p. 332, of [1], will imply convergence a.e. for all $f \in L_p(X, \mathcal{F}, \mu)$. (See [1, proof of theorem VIII, 6.9, p. 679]). This concludes the proof of the theorem.

We remark in closing that since a uniform sequence is a bounded Besicovitch sequence [see 3], and that the operators considered by De La Torre [see 5] are Dunford-Schwartz [see 4], this result includes the author's previous result for uniform sequences and contractions with fixed points [2]. The author wishes to express his gratitude to Prof. R. Sato of Okayama University for letting him see preprints of related results. The author also wishes to express his gratitude to the referee for suggesting that theorem 3.1 is true in this generality.

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