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ALGEBRAS WITH A DIAGONABLE SUBSPACE WHOSE CENTRALIZER SATISFIES A POLYNOMIAL IDENTITY

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1. Introduction. The literature concerning rings with polynomial identity contains several theorems in which the existence of a polynomial identity on a subring implies the existence of such an identity on the ring itself. Belluce and Jain showed in 1968 that a prime ring will satisfy a polynomial identity provided it contains a right ideal with zero left annihilator which satisfies a polynomial identity [**2**]. This present paper was inspired by papers of Montgomery [**7**] and Smith [**10**] in which the P.I. subrings of interest were centralizers of certain elements in the ring. These authors have subsequently extended their work to the centralizers of separable subalgebras [**8**]; we extend to centralizers of certain subspaces.

In a previous work of this author [**3**], the notion of a *diagonable subspace* of an algebra over a field k was defined. This is a subspace L with the property that the linear transformations ad x: $a \mapsto (a, x) = ax - xa$ for $x \in L$ are simultaneously diagonalizable. Equivalently, the algebra A is the direct sum of the subspaces

$$A_\alpha = A_\alpha(L) = \{a \in A\colon (a, x) = \alpha(x)a \text{ for all } x \in L\}$$

the α's called *roots* of L in A being maps (easily seen to be linear) $L \to k$. The subspace L is *finitely diagonable* if the set Δ of roots is finite. Any idempotent or more generally any algebraic element whose minimal polynomial splits into distinct linear factors over k spans a finitely diagonable subspace as does any Cartan subalgebra of a finite dimensional simple Jordan algebra (with k algebraically closed and of characteristic 0) when embedded in the universal enveloping algebra. A Cartan subalgebra of a finite dimensional simple Lie algebra becomes a diagonable subspace of the universal enveloping algebra though not finitely diagonable. The centralizer of a diagonable subspace L plays an important role in the representation theory of the algebra. If one calls a module λ-*weighted*, λ a linear functional on L, if it contains a non-zero element which is annihilated by some power of $x - \lambda(x)$ for every x in L, then for any λ, there is a one-to-one correspondence between the irreducible λ-weighted modules of the algebra and those of the centralizer of L. In the case where L is spanned by a separable element of the kind described above (and this includes the aforementioned case of the Cartan subalgebra of a Jordan

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algebra) then any module is λ-weighted for some λ and this implies that the centralizer of L is actually "large" enough to distinguish between all irreducible representations of the algebra.

It is reasonable to expect that the algebraic structure of the centralizer of a diagonable subspace should be closely tied to that of the algebra itself, and in [**3**] we did give several results which substantiate this expectation. To these we here add another; namely, the existence of a polynomial identity for the centralizer of a finitely diagonable subspace implies the existence of a polynomial identity for the entire algebra.

THEOREM 1.1. *Let A be an algebra over a field k of characteristic* 0 *which possesses a finitely diagonable subspace L with no more than $2n - 1$ roots. Then if the centralizer of L satisfies a polynomial identity of degree m, A satisfies the standard identity S_{nm} if A is semi-prime, else some power of S_{nm}.*

Any finite set Ω of commuting algebraic elements each of whose minimal polynomials has distinct roots in the base field spans a finitely diagonable subspace (see 2.1 and 2.2 of [**3**]). Thus we obtain immediately the following corollary, which for the case card $\Omega = 1$ is a consequence of a theorem of Smith [**10**].

COROLLARY 1.2. *Let A be an algebra over a field k of characteristic* 0 *and suppose Ω is a finite set of commuting separable elements of A with all minimal polynomials splitting in k. Then if the centralizer of Ω satisfies a polynomial identity, A satisfies a standard identity if A is semi-prime, else some power of a standard identity.*

2. Central simple algebras. In this section, we characterize diagonable elements of central simple algebras (finite dimensional over a field) as linear combinations of orthogonal idempotents and use this fact to establish Theorem 1.1 in this special situation. By a (*finitely*) *diagonable element*, we simply mean an element x which spans a (finitely) diagonable subspace. In this case, we will always identify a root α with the scalar $\alpha(x)$. The following general result about algebraic diagonable elements is crucial.

LEMMA 2.1. *Let x be a diagonable algebraic element in an algebra A (with* 1) *over a field k of characteristic* 0. *Assume that the minimal polynomial of x is irreducible. Then x is central.*

Proof. Let q be the minimal polynomial of x and let A_α, $\alpha \in k$, be any (nonzero) root space. Since $A_\alpha q(x) = 0$, it is readily checked that $q(x + \alpha)A_\alpha = 0$. But $q(x)A_\alpha = 0$ and so the polynomials $q(t)$ and $q(t + \alpha)$ cannot be relatively prime. Assuming as we may that they are monic, they are equal because they are irreducible. Now comparing the terms in $q(t)$ and $q(t + \alpha)$ of degree $(\deg q) - 1$ we find $\alpha = 0$; i.e., $A = A_0$ and x is central.

PROPOSITION 2.2. *Suppose A is a finite dimensional central simple algebra over*

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a field k of characteristic 0. Then an element of A is diagonable if and only if it is a linear combination of orthogonal idempotents.

Proof. If $x = \sum_1^n \alpha_i e_i$ is a linear combination of the orthogonal idempotents $e_1, \ldots, e_n$ (which we assume have sum 1 with no loss of generality), every $a \in A$ can be written $a = \sum_{i,j} e_i a e_j$. Since $e_i a e_j \in A_{\alpha_j - \alpha_i}$, x is diagonable. For the converse, suppose the minimal polynomial for x is $\prod_1^s q_i^{n_i}$, where $q_1, \ldots, q_s$ are distinct irreducible monic polynomials. Then since A is semi-prime, so is A_0 [**3**, Theorem 5.4] and thus each $n_i = 1$; otherwise, $\prod_1^s q_i(x)$ would be a nilpotent element in the centre of A_0. Using a standard argument, we can write $1 = \sum_1^s e_i$, $e_1, \ldots, e_s$ being orthogonal idempotents which commute with x and are such that $q_i(x)e_i = 0$ for all i. It follows that q_i is the minimal polynomial of $x_i = xe_i$. Denote by $A_1(e_i)$ the algebra $e_i A e_i$. This is central simple over k because it is isomorphic to $\mathrm{End}_D\, Ve_i$, where A is isomorphic to $\mathrm{End}_D\, V$, the ring of linear transformations of the vector space V over division ring D. Since $A_1(e_i)$ ad $x = A_1(e_i)$ ad $x_i \subset A_1(e_i)$, we see as in [**3**, 2.2] that x_i is a diagonable element of $A_1(e_i)$ and hence $x_i = \alpha_i e_i$, $\alpha_i \in k$ by Lemma 2.1. Thus $x = x(\sum_i e_i) = \sum_i \alpha_i e_i$.

One feature of the root structure of semi-prime algebras that we will find very useful in what follows is this:

(1) If A is semi-prime with 1 and a_α is a non-zero element of a root space A_α, $\alpha \neq 0$, then $a_\alpha A_{-\alpha} \neq 0$ and $A_{-\alpha} a_\alpha \neq 0$.

To see this, simply notice that if $a_\alpha A_{-\alpha} = 0$, then $a_\alpha A$ is a right ideal of A not meeting A_0. Hence it is nilpotent by [**3**, Lemma 5.3], an impossibility. In particular, the remark (1) implies that in a semi-prime ring with 1, the number of roots, including 0, is always odd. We are now in a position to prove Theorem 1.1 for finite dimensional central simple algebras.

THEOREM 2.3. *Let A be a central simple algebra of finite dimension over a field k of characteristic 0. Suppose L is a finitely diagonable subspace of A with no more than $2n - 1$ roots whose centralizer satisfies a polynomial identity of degree m. Then A satisfies S_{nm} and $[A : k] \leqq \frac{1}{4}(nm)^2$.*

Proof. We have $A \simeq D_t$ for some division algebra D, central over k. Letting K be any maximal subfield of D containing k, $D \otimes_k K \simeq K_s$, $s = [D : k]$, using a result in Herstein [**4**, p. 96]. Thus $A \otimes_k K \simeq D_t \otimes_k K \simeq K_{st}$. Since K_{st} satisfies S_{2st}, so does A. Also $[A : k] = (st)^2$ so we may complete the proof by establishing $st \leqq \frac{1}{2}nm$. For this, we first note that the finite dimensionality of A implies that any collection of centralizers of diagonable elements in A has minimal elements (with respect to inclusion), and thus there is some $x \in L$ for which the centralizer of L is just the centralizer of x [**3**; Theorem 6.2]. In our notation, $A_0(L) = A_0(x)$. By Proposition 2.2,

(2) $$x = \sum_1^t \alpha_i e_i$$

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is a linear combination of orthogonal idempotents which we may assume are primitive with sum 1. Suppose that the number of distinct α_i appearing in (2) is l. Then some α_j occurs as a coefficient r times, $r \geqq t/l$. Thus D_r is a subring of $A_0(x)$ and consequently satisfies a polynomial identity of degree m. By Kaplansky's famous theorem (see for instance [**4**; p. 157]), $[D_r : k] \leqq [m/2]^2$; i.e. $rs \leqq [m/2]$ and so $st \leqq (t/r)[m/2]$. Since $r \geqq t/l$, $st \leqq \frac{1}{2}\, ml$. Now write $x = \sum_1^l \beta_i f_i$ where $\beta_1, \ldots, \beta_l$ are the distinct α_i and each f_j is the sum of all the e_i's which had the same coefficient in (2). Then the f_i's have sum 1 and so $A = \sum_{i,j} f_i A f_j$. Since A is prime, $0 \neq f_i A f_j \subset A_{\beta_j - \beta_i}(x)$ and so the non-zero roots of x are precisely the set of $\beta_j - \beta_i$, i and j running from 1 to l. We conclude the proof by establishing that this set is of cardinality at least $2l - 1$, and hence $l \leqq n$ and $st \leqq \frac{1}{2}\, nm$ as required.

Because its characteristic is 0, we can consider k to be a vector space over the rationals. Let $\gamma_1, \ldots, \gamma_p$ be a basis for the subspace U spanned by $\beta_1, \ldots, \beta_l$ and order U be declaring $\sum_1^p r_i \gamma_i > 0$ if the first non-zero r_i is positive. Assume that β_1 is the largest β_i with respect to this ordering. Then $\beta_1 - \beta_2$, $\beta_1 - \beta_3$, $\ldots$, $\beta_1 - \beta_l$ are $l - 1$ different positive elements of U yielding together with their negatives and 0, a total of $2(l - 1) + 1 = 2l - 1$ different elements $\beta_j - \beta_i$.

3. Prime algebras. We prove in this section two theorems on which our main Theorem 1.1 heavily depends.

THEOREM 3.1. *Let A be a prime algebra over a field k of characteristic* 0 *and L a finitely diagonable subspace. Then if the centralizer of L satisfies a polynomial identity, A satisfies a generalized polynomial identity.*

Proof. Denote by $\tilde{\Delta}$ the vector space over the rationals (the prime field of k) spanned by the set Δ of roots of L. Since Δ is finite, $\tilde{\Delta}$ has a finite basis $\alpha_1, \ldots, \alpha_n$ relative to which $\tilde{\Delta}$ may be ordered just as in the previous section. Let $\alpha \in \Delta$ be a maximal root in this ordering. Then for any positive root β, $\alpha + \beta$ cannot be a root and so $A_\alpha A_\beta = 0$ (because $A_\alpha A_\beta \subset A_{\alpha+\beta}$). Also $-\alpha$ will be a minimal root and so $A_\beta A_{-\alpha} = 0$ for any negative root β. We recall here the earlier remark (1) and its implication that γ is a root if and only if $-\gamma$ is also a root. Let a_α and $a_{-\alpha}$ be arbitrary (non-zero) elements of A_α and $A_{-\alpha}$ respectively such that $a_{-\alpha}a_\alpha \neq 0$. Then $f(u) = a_\alpha u a_{-\alpha} \in A_0$ for any $u \in A$. Now as in Herstein [**4**, p. 156-7], we may assume A_0 satisfies a multilinear homogeneous identity of the form

$$g(x_1, \ldots, x_t) = x_1 \ldots x_t + \sum_{1 \neq \sigma \in S_t} a_\sigma x_{\sigma(1)} \ldots x_{\sigma(t)}.$$

Then A certainly satisfies the generalized polynomial identity $g(f(y_1), \ldots, f(y_t))$. This is a non-trivial identity because the term involving $y_1, \ldots, y_t$ in this order is $a_\alpha y_1 a_{-\alpha} a_\alpha y_2 a_{-\alpha} \ldots a_\alpha y_t a_{-\alpha}$ which is non-zero because of the primeness of A and the fact that $a_{-\alpha}a_\alpha \neq 0$.

THEOREM 3.2. *Let A be a primitive algebra over a field k of characteristic 0 with non-zero socle. Suppose L is a finitely diagonable subspace whose centralizer satisfies a polynomial identity. Then A is the complete ring of linear transformations of a vector space finite dimensional over a division ring D.*

Proof. Let S denote the socle of A. Then as a two-sided ideal of A, S is homogeneous, $S = S_0 + \sum_{0\neq\alpha\in\Delta} S_\alpha$ relative to the collection Δ of roots of L in A. For if $s \in S$, we can write $s = \sum_{\alpha\in\Delta} u_\alpha$ with $u_\alpha \in A_\alpha$. Because S is two-sided, the commutator $(s, x) = \sum_{\alpha\neq 0} \alpha(x)u_\alpha$ is in S and hence so is $((s, x), x) = \sum_{\alpha\neq 0} \alpha(x)^2 u_\alpha$. Generally

$$(3)\quad \sum_{\alpha\neq 0} \alpha(x)^j u_\alpha = s_j$$

is in S for any positive integer j. Now $u_\alpha = 0$ except for α in some finite set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \Delta$ and since the α_i are linear functionals on L and k is infinite, there is some $x \in L$ for which the scalars $\alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x)$ are distinct (the union of the finitely many subspaces which are the kernels of $\alpha_i - \alpha_j$ cannot be all of L). Then letting j run from 1 to n, the matrix of coefficients of the system of linear equations given by (3) is an invertible Vandermonde matrix. Consequently each u_α with $\alpha \neq 0$ is in S, but then so is $u_0 = s - \sum_{\alpha\neq 0} u_\alpha$. We next recall that S is a von Neumann regular ring: for every $u \in S$, there is a $v \in S$ such that $u = uvu$, and the homogeneity of S implies that if $u \in S_\alpha$, we may assume $v \in S_{-\alpha}$. In particular, $S_\alpha S_{-\alpha} S_\alpha = S_\alpha$ for for all $\alpha \in \Delta$ and also S_0 is a von Neumann regular ring satisfying the same polynomial identity as A_0. A key point in proving this theorem is the observation that

(4) Any non-zero (two-sided) ideal of S_0 contains a central idempotent.

Firstly, as a von Neumann regular ring S_0 is semi-prime and so any non-zero ideal I at least contains some central element u by Rowen's result [**9**]. Then choosing $v \in S_0$ such that $uvu = u$, $e = uv$ is an idempotent in I which is central because $es_0 = uvs_0 = vs_0u = vs_0eu = uvs_0e = es_0e = es_0uv = eus_0v = us_0v = s_0uv = s_0e$ for any $s_0 \in S_0$.

Now define for each non-zero $\alpha \in \Delta$, $I_\alpha = S_\alpha S_{-\alpha}$. Amongst the minimal elements in the set U of non-zero intersections of these ideals, choose $I = \bigcap_{\alpha\in\Delta_1} I_\alpha$ with Δ_1 of maximal cardinality. Since S is simple, $S = SIS$, and so $S_0 = \sum_{\beta\in\Delta} S_\beta I S_{-\beta}$. Now for any $\beta \in \Delta$, $S_\beta I S_{-\beta} \subset \bigcap_{\alpha\in\Delta_1} S_\beta I_\alpha S_{-\beta}$, and $S_\beta I_\alpha S_{-\beta} \subset I_{\alpha+\beta}$ if $\alpha + \beta \neq 0$ (because $S_\alpha S_\beta \subset S_{\alpha+\beta}$) and $S_\beta I_\alpha S_{-\beta} = S_\beta S_{-\beta} S_\beta S_{-\beta} = S_\beta S_{-\beta} = I_\beta$ if $\alpha + \beta = 0$. As a consequence, $S_\beta I S_{-\beta}$ is contained in $\bigcap_{\alpha\in\Delta'} I_\alpha$ for some set $\Delta' \subset \Delta$ of the same cardinality as Δ_1. Thus if $S_\beta I S_{-\beta}$ is not zero, neither is $\bigcap_{\alpha\in\Delta'} I_\alpha$ which must then be a minimal ideal of U by the definition of I. We thus see that S_0 is a sum of ideals $A_1, \ldots, A_n$, minimal elements of U, each $A_i = \bigcap_{\alpha\in\Delta_i} I_\alpha$, all the Δ_i's of the same maximal cardinality. Assuming the A_i's are distinct, their intersections and hence products in pairs is 0, by minimality.

Moreover,

(5) $A_i S_\beta A_i = 0$ for any $\beta \neq 0$.

If this were not so for some particular β, we would have $0 \neq A_i S_\beta S_\alpha \subset A_i S_{\beta+\alpha}$ for any $\alpha \in \Delta_i$ and so $0 \neq A_i I_{\beta+\alpha} \subset A_i \cap I_{\beta+\alpha}$ by (1). This forces $\beta + \alpha \in \Delta_i$ because of the maximality of Δ_i. Repeating this argument with α replaced by $\beta + \alpha$, we see that $2\beta + \alpha \in \Delta_i$. It follows that $n\beta + \alpha \in \Delta_i$ for any $\alpha \in \Delta_i$ and integer n, a contradiction to the finiteness of Δ_i.

Now using (4), choose idempotents $e_1, \ldots, e_n$, central in S_0, each $e_i \in A_i$ (implying orthogonality). Since S is simple, $S = Se_iS$ for any i from which we see that

$$(6) \quad S_0 = e_i S_0 + \sum_{0 \neq \alpha \in \Delta} S_\alpha e_i S_{-\alpha}.$$

Certainly $e_i S_0 \subset A_i$, and for $\alpha \neq 0$, $S_\alpha e_i S_{-\alpha}$ is either zero or contained in a minimal element B of U, repeating a previous argument. This B must be one of the A_j, for $B \neq A_j$ implies $BA_j = 0$ and if this occurs for all j, $B = BS_0 = B(\sum A_j) = 0$. Also $B \neq A_i$, for otherwise $(S_\alpha e_i S_{-\alpha})^2 \subset (S_\alpha e_i S_{-\alpha})A_i = 0$ by (5). This cannot occur because S_0 contains no non-zero nilpotent ideals. Using the directness of the sum $S_0 = \sum A_j$ and (6) it follows readily that $e_i S_0 = A_i$. Thus $\sum_1^n e_i$ is an identity element for S_0 and hence for all of S because $S_\alpha = S_\alpha S_{-\alpha} S_\alpha \subset S_0 S_\alpha \cap S_\alpha S_0$. Now both the primitive algebra A and its socle S are dense rings of linear transformations of a vector space V over a division algebra D. To say that S has an identity element is to say that $[V : D] < \infty$, because every element of S has finite rank. Thus A is the complete ring of linear transformations of V.

4. Semi-prime algebras. We now prove Theorem 1.1. Suppose that L is a diagonable subspace of an algebra A over a field k of characteristic 0, that L possesses at most $2n - 1$ roots, and that the centralizer of L satisfies a polynomial identity of degree m. If $\bar{A}$ is any homomorphic image of A, then $\bar{L}$ is a diagonable subspace of $\bar{A}$, in fact with no more roots than L because $\bar{A}_\alpha(\bar{L}) = A_\alpha(L)$. In particular, the centralizer of $\bar{L}$ in $\bar{A}$ is a homomorphic image of the centralizer of L in A and hence satisfies the same polynomial identity as $A_0(L)$. For the sake of completeness, we include here an argument due to Amitsur [**1**] which allows us to assume that A is semi-prime. Assuming the truth of 1.1 for semi-prime algebras and supposing A to be an arbitrary algebra satisfying the hypotheses of this theorem, we let N be the lower nil radical of A and deduce that $S_d(a_1, \ldots, a_d) \in N$ for any choice of $a_1, \ldots, a_d \in A$, $d = nm$. Let $A' = \prod_{\lambda \in \Lambda} A_\lambda$, $\Lambda = \{(a_1, \ldots, a_d) : a_i \in A\}$, $A_\lambda = A$ for all λ and $L' = \{f_x : x \in L\}$, where for each $x \in L$, f_x is defined by $f_x(\lambda) = x$, for every λ. Then L' is diagonable in A' with the same roots as L in A because every $f \in A'$ can be written $f = \sum f_\alpha$ where $f_\alpha \in A_\alpha'(L')$ is defined by $f_\alpha(\lambda) =$ α-component in A of $f(\lambda)$. We see here that $A_\alpha'(L') = \{f : f(\lambda) \in A_\alpha \text{ for all } \lambda\}$ $= \prod_{\lambda \in \Lambda} A_\alpha(L)_\lambda$. Thus the centralizer of L' in A' satisfies the same polynomial identity as $A_0(L)$. As above, this means that $S_d(f_1, \ldots, f_d)$ is in the lower nil radical of A' and hence is nilpotent for every $f_1, \ldots, f_d \in A'$. Choosing f_i as that element of A' such that $f_i(\lambda) = a_i$ for $\lambda = (a_1, \ldots, a_d)$ we obtain $S_d(a_1, \ldots, a_d)^l = 0$ for some integer l and for all $\lambda = (a_1, \ldots, a_d)$. In other words, $S_{nm}{}^l$ is an identity for A.

So we now assume that A is semi-prime. In this case, A is the subdirect sum of the prime algebras A/P, P ranging over the prime ideals of A. As homomorphic images of A, each of the algebras A/P has finitely diagonable subspaces whose centralizers satisfy the same identity as $A_0(L)$. If each prime algebra A/P satisfies S_{nm}, certainly A does too. Thus we may assume that A is prime.

In this case, following Martindale [**6**], we let $C(\subset k)$ be the extended centroid and $S = AC$ the central closure of A. As a subspace of S, L is diagonable and $S_\alpha(L) = A_\alpha(L)C$ because C centralizes A; $A_\alpha(L)C \subset S_\alpha(L)$ is easily seen, and for the converse, if $s = \sum_i a_i c_i \in S$ and $a_i = \sum_{\alpha \in \Delta} a_{i\alpha}$ relative to L, then $s = \sum_\alpha (\sum_i a_{i\alpha} c_i)$ showing that $S = \sum_\alpha A_\alpha(L)C$ and $S_\alpha(L) = A_\alpha(L)C$. Also the centralizer of L in S, $S_0(L) = A_0(L)C$ satisfies the same polynomial identity as does $A_0(L)$ (see the proof of Theorem 1, p. 225 of [**5**]). Since S is a prime algebra over C, our Theorem 3.1 shows that S satisfies a generalized polynomial identity and hence is primitive with non-zero socle by Theorem 3 of [**6**]. Our Theorem 3.2 indicates that S is the complete ring of linear transformations of a vector space which is finite dimensional over a division ring D, and a theorem of Amitsur [**6**, Theorem 5] reveals that D is finite dimensional over its centre. Thus S is finite dimensional central simple and satisfies S_{nm} by Theorem 2.3. Since A is a subalgebra of S, A too satisfies S_{nm}.

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