Stochastic stability of hyperbolic attractors

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Abstract. We study the effects of small random errors on the asymptotic distribution of points in the basin of a hyperbolic attractor.

Leaving precise statements and generalities for later, we give an indication of the type of result in this paper. Let $U \subset \mathbb{R}^n$ be a neighbourhood of a compact attracting set, i.e. $f\bar{U} \subset U$ for some map f. Suppose that as we apply f to any point $x \in U$, instead of obtaining the precise value fx, the possible outcomes are given by the uniform distribution on an ε-disk centred at fx. Now we start this process at some arbitrary point and let μ_ε be the asymptotic distribution. Our result says that if

$$\Lambda = \bigcap_{n \geq 0} f^n U$$

is a hyperbolic attractor, then μ_ε tends to the so-called Bowen–Ruelle–Sinai measure on Λ as $\varepsilon \to 0$.

Recall that, according to the theory of Bowen, Ruelle and Sinai ([**B**], [**BR**], [**Si**]), if $\Lambda \subset U$ is a hyperbolic or Axiom A attractor, then

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x}$$

tends to the B–R–S measure for Lebesgue-a.e. $x \in U$. Our result then can be interpreted as a statement of stochastic stability for hyperbolic attractors, in the sense that the asymptotic distributions of points in these systems are not drastically altered by the cumulative effects of small random errors.

Results very similar to ours were first obtained by Y. Kifer [**K1**], [**K2**]. His technical hypotheses do not coincide with ours, but more importantly, our perspectives are quite different. We view this process as the random composition of maps nearby f (see [**F**]), and our goal is to demonstrate that under the constraints of our hypotheses this type of stochastic stability is a direct consequence of the persistence of hyperbolic structures and their equilibrium state measures.

1. *Preliminaries and statement of result*
Let $f: M \circlearrowleft$ be a C^2 diffeomorphism of a finite dimensional Riemannian manifold. Let $U \subset M$ be an open subset with compact closure and suppose that $f\bar{U} \subset U$. All the processes we consider take place inside $\bar{U}$. We assume that the errors that may

occur as we apply f to x depend entirely on the point x. This puts us into the context of a Markov chain with stationary transition probabilities.

For $x \in \bar{U}$, let $p(\cdot | x)$ be a probability on the Borel subsets of $\bar{U}$. We think of $p(A|x)$ as the probability of sending x into the set A. Thus for instance the probability of starting at x_0 and being in $A_1, A_2, \ldots, A_n$ successively is

$$\int_{A_{n-1}} \int_{A_{n-2}} \cdots \int_{A_1} p(A_n | x_{n-1}) p(dx_{n-1} | x_{n-2}) \cdots p(dx_1 | x_0).$$

Definition (1.1). Given transition probabilities $p(\cdot | x)$, a Borel probability measure μ on $\bar{U}$ is said to be *invariant* if

$$\mu(A) = \int p(A|x) \, d\mu(x)$$

for every Borel set A.

Let $\mathcal{M}(\bar{U})$ denote the space of Borel probability measures on $\bar{U}$, always endowed with the weak topology. The dirac measure at x is denoted by δ_x.

PROPOSITION (1.2). (1) *If $x \mapsto p(\cdot | x)$ is continuous, then an invariant measure exists.*

(2) *If $p(\cdot | x)$ is absolutely continuous with respect to Lebesgue measure (written $p(\cdot | x) \ll$ Leb.) for every x, then so are the invariant measures.*

(3) *Suppose for $\varepsilon > 0$, μ_ε is an invariant measure corresponding to transition probabilities p_ε, and $p_\varepsilon(\cdot | x) \to \delta_{fx}$ uniformly in x as $\varepsilon \to 0$, then all limit points of μ_ε are f-invariant.*

Proof. We prove (1), leaving the rest as a straightforward exercise for the reader. Define $\mathcal{L} : \mathcal{M}(\bar{U}) \leftrightarrows$ by

$$\int \phi \, d(\mathcal{L}\mu) = \int \int \phi(y) p(dy | x) \, d\mu(x),$$

for every continuous function $\phi : \bar{U} \to \mathbb{R}$. Suppose $\mu_n \to \mu$. Since $x \mapsto \int \phi(y) \, p(dy | x)$ is continuous, it follows that

$$\int \phi \, d(\mathcal{L}\mu_n) \to \int \phi \, d(\mathcal{L}\mu),$$

which verifies the continuity of $\mathcal{L}$. Now pick any $\mu_0 \in \mathcal{M}(\bar{U})$. Limit points of the sequence $(1/n) \sum_{i=0}^{n-1} \mathcal{L}^i \mu_0$ are the fixed points of $\mathcal{L}$, which are easily seen to satisfy definition 1.1. $\qquad \square$

We return briefly to the uniform diffusion example mentioned in the introduction. One way of describing it is as follows: Let $D_\varepsilon = \{\alpha \in \mathbb{R}^n : \|\alpha\| \le \varepsilon\}$ and write $f_\alpha = f + \alpha$. Then

$$p_\varepsilon(A|x) = \nu_\varepsilon\{\alpha \in D_\varepsilon : f_\alpha x \in A\},$$

where ν_ε is normalized Lebesgue measure on D_ε. Keeping this example in mind we study the following class of Markov chains:

Let Ω be a set of maps from $\bar{U}$ into itself. We consider transition probabilities of the form

$$p(A|x) = \nu\{g \in \Omega : gx \in A\},$$

where ν is a probability on Ω. It is easy to verify that as long as $\Omega \subset C^0(\bar{U})$, the space of continuous maps from $\bar{U}$ into itself endowed with the C^0-topology, and ν is a Borel measure with compact support, then $x \mapsto p(\,\cdot\,|x)$ is continuous and hence this process has an invariant measure.

Next we turn to the conditions we wish to impose on $f: U \hookleftarrow$.

Definition (1.3). A compact f-invariant set $\Lambda \subset M$ is said to be *uniformly hyperbolic* or simply *hyperbolic* if there is a continuous splitting of the tangent bundle over Λ into a direct sum of two invariant subbundles $E^u \oplus E^s$ and that there are numbers $c > 0$ and $\lambda > 1$ s.t. for all $x \in \Lambda$ and $n \geq 0$,

$$v \in E^s(x) \Rightarrow \|Df_x^n v\| \leq c\lambda^{-n} \|v\|,$$
$$v \in E^u(x) \Rightarrow \|Df_x^{-n} v\| \leq c\lambda^{-n} \|v\|.$$

Via a standard change of metric we may – and will – assume that $c = 1$.

Definition (1.4). An invariant set $\Lambda \subset M$ is called a *hyperbolic attractor* with basin U if:

(1) U is an open set s.t. $f\bar{U} \subset U$ and

$$\bigcap_{n \geq 0} f^n U = \Lambda;$$

(2) Λ is a uniformly hyperbolic set; and

(3) $f|\Lambda$ has a dense orbit.

By the spectral decomposition theorem [**B**] every hyperbolic attractor Λ can be decomposed into finitely many components $\Lambda_1, \ldots, \Lambda_k$ with $f\Lambda_1 = \Lambda_2$, $f\Lambda_2 = \Lambda_3, \ldots, f\Lambda_k = \Lambda_1$. For simplicity we shall consider $f^k|\Lambda_1$ and call it f.

Definition (1.5). Let $\Lambda \subset U$ be a hyperbolic attractor with basin U. The *Bowen-Ruelle-Sinai measure* (written 'B-R-S measure' from here on) is the unique measure μ with the property that for Lebesgue-a.e. $x \in U$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x} \to \mu.$$

For further information regarding the theory of hyperbolic attractors see [**B**] or [**Sh**].

We now state the main result of this paper. Let $L(\,\cdot\,)$ denote the Lipschitz constant of a map. For $f \in C^2(\bar{U})$ and $\alpha, \beta > 0$, let

$$\Omega_{\alpha,\beta}(f) = \{g \in C^1(\bar{U}) : \|f - g\|_{C^1} \leq \alpha \text{ and } L(Dg) \leq \beta\}.$$

We put the C^1 metric on $\Omega_{\alpha,\beta}(f)$ (which makes it compact).

THEOREM (1.6). *Let f be a C^2 diffeomorphism of M and suppose that $\Lambda \subset M$ is a hyperbolic attractor with basin U. Let $\alpha > 0$ be sufficiently small and let $\beta > L(Df)$. For each $\varepsilon > 0$, let ν_ε be a Borel probability measure on $\Omega_{\alpha,\beta}(f)$ with $\nu_\varepsilon \to \delta_f$ as $\varepsilon \to 0$ and let μ_ε be an invariant measure for the process defined by*

$$p_\varepsilon(A|x) = \nu_\varepsilon\{g : gx \in A\}.$$

If $p_\varepsilon(\,\cdot\,|x) \ll$ Leb. for every $\varepsilon > 0$ and $x \in \bar{U}$, then μ_ε tends to the B–R–S measure as $\varepsilon \to 0$.

We thank Y. Kifer for a suggestion in the formulation of this theorem.

2. Skew product representation

We shall represent the type of Markov chains under consideration as skew products. Part A of this section concerns the measure-theoretic aspects of this representation while part B is about the topological-differential structure of the resulting object.

(A) This subsection has nothing to do with hyperbolic attractors. Let X be a compact metric space and let Ω be a compact subset of $C^0(X)$. Let $\Omega^{\mathbf{Z}}$ denote the bi-infinite product of Ω with itself and define $F:\Lambda^{\mathbf{Z}}\times X \circlearrowleft$ by

$$F(g, x) = (\sigma g, g_0 x)$$

where $g = (\cdots g_{-1}g_0g_1\cdots)$ and $\sigma:\Omega^{\mathbf{Z}}\circlearrowleft$ is the shift transformation, i.e. $(\sigma g)_n = (g)_{n+1}$. If ν is a measure on Ω, let $\nu^{\mathbf{Z}}$ denote the product measure of ν with itself on $\Omega^{\mathbf{Z}}$. Finally, let $p_1:\Omega^{\mathbf{Z}}\times X \to \Omega^{\mathbf{Z}}$ and $p_2:\Omega^{\mathbf{Z}}\times X \to X$ be projections onto the first and second factors respectively.

PROPOSITION (2.1). *Let ν be a Borel probability measure on Ω and suppose that μ is an invariant measure for the process defined by*

$$p(A\,|\,x) = \nu\{g: gx \in A\}.$$

Then $F^n(\nu^{\mathbf{Z}}\times\mu)$ converges weakly to an F-invariant measure μ^* on $\Omega^{\mathbf{Z}}\times X$. Moreover, $p_1\mu^* = \nu^{\mathbf{Z}}$, $p_2\mu^* = \mu$.

Here $F^n(\nu^{\mathbf{Z}}\times\mu)(E)$ is defined to be $(\nu^{\mathbf{Z}}\times\mu)(F^{-n}E)$.

Proof. First we observe that once weak convergence is established, μ^* is automatically F-invariant. Also, $p_1\mu^* = \nu^{\mathbf{Z}}$ is obvious. To prove weak convergence let $\mathcal{B}_n$ be the σ-algebra on $\Omega^{\mathbf{Z}}\times X$ generated by sets of the form $E = \tilde{A}\times B$ where $B\subset X$ and $A\subset \prod_{-n}^{\infty}\Omega$ are Borel sets and $\tilde{A} = (\prod_{-\infty}^{-(n+1)}\Omega)\times A$. We write $m_n = F^n(\nu^{\mathbf{Z}}\times\mu)$ and will show that $m_{n+1}|\mathcal{B}_n = m_n|\mathcal{B}_n$ for all $n\geq 0$ by verifying this for $E\in\mathcal{B}_n$.

Let $m_n(\cdot\,|g)$ denote the conditional probability of m_n on the fibre above g. Then

$$m_n(E\,|\,g) = \begin{cases} \mu(g_{-n}^{-1}\cdots g_{-1}^{-1}E) & \text{if } g\in\tilde{A}, \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$m_n(E) = \int_A \mu(g_{-n}^{-1}\cdots g_{-1}^{-1}E)\, d\left(\prod_{-n}^{\infty}\nu\right).$$

Now

$$m_{n+1}(E) = \int_{\Omega\times A} \mu(g_{-(n+1)}^{-1}\cdots g_{-1}^{-1}E)\, d\left(\prod_{-(n+1)}^{\infty}\nu\right)$$

$$= \int_A \left\{\int_\Omega \mu(g^{-1}(g_{-n}^{-1}\cdots g_{-1}^{-1}E))\, d\nu(g)\right\} d\left(\prod_{-n}^{\infty}\nu\right).$$

To see that the inner integral equals $\mu(g_{-n}^{-1}\cdots g_{-1}^{-1}E)$, observe that for $C\subset X$,

$$\int_\Omega \mu(g^{-1}C)\, d\nu(g) = \int_X \nu\{g: gx\in C\}\, d\mu(x) = \mu C.$$

That $p_2\mu^* = \mu$ follows easily.   □

(B) We now return to the situation where $f: U \circlearrowleft$ contains a hyperbolic attractor. Let $\beta > L(Df)$ be fixed and for small α write $\Omega_\alpha = \Omega_{\alpha,\beta}(f)$ (see the end of §1 for definitions).

Consider the skew product $F: \Omega_\alpha^Z \times \bar{U} \circlearrowleft$ as in §2(A). For $g \in \Omega_\alpha^Z$, let U_g denote the U-fibre over g and write the fibre map as $F_g: U_g \to U_{\sigma g}$. Then F is a homeomorphism of $\Omega_\alpha^Z \times \bar{U}$ onto a subset of itself. Each fibre map F_g is C^1 with $L(DF_g) \leq \beta$ and $(g, x) \mapsto DF_g(x)$ is continuous. Let $\tilde{\Lambda} = \bigcap_{n \geq 0} F^n(\Omega_\alpha^Z \times U)$. Then $\tilde{\Lambda}$ is a compact F-invariant set, fibre-wise attracting.

By the 'tangent space' at (g, x) we refer always to the tangent space to U_g at x. Likewise $T\tilde{\Lambda} = \bigcup_{(g, x) \in \tilde{\Lambda}} T_{(g,x)} U_g$.

PROPOSITION (2.2). *For sufficiently small* $\alpha > 0$, *there is a continuous splitting of* $T\tilde{\Lambda}$ *into* $\tilde{E}^u \oplus \tilde{E}^s$ *and there exists* $\tilde{\lambda} > 1$ *s.t. for all* $(g, x) \in \tilde{\Lambda}$ *and* $n \geq 0$,

$$v \in \tilde{E}^u(g, x) \Rightarrow \|DF_g(x)v\| \geq \tilde{\lambda}\|v\|,$$
$$v \in \tilde{E}^s(g, x) \Rightarrow \|DF_g(x)v\| \leq \tilde{\lambda}^{-1}\|v\|.$$

To prove this proposition we invoke the 'invariant cones argument', the standard argument used to prove the persistence of hyperbolic structures. We refer the reader to [**Sh**, chapters 7 & 8] for a detailed proof.

From here on we fix α small enough for the conclusion of proposition (2.2) to hold and write $\Omega = \Omega_\alpha$. For $(g, x) \in \tilde{\Lambda}$, we define

$$\tilde{W}^u(g, x) = \{(g, y) \in \Omega^Z \times \bar{U}: d(F^{-n}(g, x), F^{-n}(g, y)) \to 0 \text{ as } n \to \infty\}.$$

The same proofs that work in the single transformation case show that for every $(g, x) \in \tilde{\Lambda}$, $\tilde{W}^u(g, x)$ is an immersed submanifold of U_g and that $\tilde{W}^u(g, x) \subset \tilde{\Lambda}$. We shall call

$$\tilde{W}^u(\tilde{\Lambda}) = \bigcup_{(g, x) \in \tilde{\Lambda}} \tilde{W}^u(g, x)$$

the 'unstable manifolds of $\tilde{\Lambda}$'. These unstable manifolds form a continuous lamination of the compact set $\tilde{\Lambda}$.

We remark that the result of §2(A) is clearly not affected by our using the C^1 metric on Ω.

3. *Measure absolutely continuous on unstable manifolds and proof of theorem*

Our proof of theorem (1.6) relies heavily on the Radon–Nikodym derivatives of certain measures along $\tilde{W}^u$-manifolds. First we give precise definitions.

Consider the hyperbolic attractor $f: \Lambda \circlearrowleft$. If $L \subset W^u(x)$ is open in the leaf topology, let d^u denote the induced Riemannian metric on L and m^u the corresponding Riemannian measure. Let V be a Borel subset of Λ. Suppose V is the disjoint union of a continuous family of open W^u-disks and μ is a Borel probability measure on Λ with $\mu V > 0$. For $x \in V$, let V_x denote the W^u-disk in V containing x and let μ_x denote the conditional probability measure of μ on V_x.

Definition (3.1.) (1) We say that μ has *absolutely continuous conditional measures on* W^u-*manifolds* if for every $V \subset \Lambda$ above, $\mu_x \ll m^u$ for μ-a.e. x.

(2) Suppose μ has absolutely continuous conditional measures on W^u. We say that the *densities of μ on W^u have bounded ratio* if there exist B, $\delta > 0$ s.t. for every V above, $\rho = d\mu_x/dm^u$ satisfies

$$\frac{1}{B} \leq \frac{\rho(z_1)}{\rho(z_2)} \leq B,$$

for m^u-a.e. z_1, $z_2 \in V_x$ with $d^u(z_1, z_2) < \delta$, μ-a.e. x.

We omit the corresponding definitions for measures on $\tilde{\Lambda}$. The generalization is quite obvious.

Recall the following characterization of B–R–S measures:

THEOREM ([B], [Si]). *The B–R–S measure is the unique f-invariant Borel probability measure on Λ with absolutely continuous conditional measures on W^u.*

Recall also that the B–R–S measure can be realized as the weak limit as $n \to \infty$ of $f^n m$, where $m = $ *normalized Lebesgue measure on U.*

Proof of theorem 1.6. Let $\{\nu_\varepsilon, \varepsilon > 0\}$ be Borel probability measures on Ω converging weakly to δ_f as $\varepsilon \to 0$, and let $\{\mu_\varepsilon\}$ be invariant measures of the corresponding processes. In § 4 we prove the following lemma.

LEMMA (3.2). *Let $\alpha > 0$ be sufficiently small. Then for each $\varepsilon > 0$, there is a unique F-invariant Borel probability measure m_ε on $\tilde{\Lambda}$ satisfying:*
   (1) $p_1 m_\varepsilon = \nu_\varepsilon^Z$; and
   (2) m_ε has absolutely continuous conditional measures on $\tilde{W}^u$-manifolds.
Moreover, the densities of all the m_ε on $\tilde{W}^u$ have bounded ratios with a common bound depending only on F.

Now it follows from the hypothesis on ν_ε and proposition 1.2 (2) that each $\mu_\varepsilon \ll$ Leb. on U. Let $\mu_\varepsilon^* = \lim_{n\to\infty} F^n(\nu_\varepsilon^Z \times \mu_\varepsilon)$, as guaranteed to exist by proposition 2.1.

LEMMA (3.3). $\mu_\varepsilon^* = m_\varepsilon$.
This lemma is also proved in § 4. Let m^* be a limit point of m_ε as $\varepsilon \to 0$. The bounded ratio property of the densities of m_ε on $\tilde{W}^u$ is passed on to m^*. Since m^* is supported on the single fibre over g with $g_n = f$ for all n, $p_2 m^*$ is f-invariant and so it must be the B–R–S measure on Λ. Quoting proposition 2.1 again we see that

$$\mu_\varepsilon = p_2 m_\varepsilon \to p_2 m^*,$$

which is the desired conclusion. □

We remark that for our purposes it is not really necessary to prove lemma 3.2. One could show directly that μ_ε^* has absolutely continuous conditional measures on $\tilde{W}^u$-manifolds. We leave that as an exercise.

4. *Proofs of lemmas*
Before giving the proofs of lemmas 3.2 and 3.3 we establish the following preliminary result.

LEMMA 4.1. *Let L be an open disk in $\tilde{W}^u(g, x)$ for some $(g, x) \in \tilde{\Lambda}$ and let m_L be the normalized Riemannian measure on L. Then any limit point μ of*

$$\frac{1}{n} \sum_{i=0}^{n-1} F^i m_L$$

has absolutely continuous conditional measures on $\tilde{W}^u$-manifolds. Moreover, the densities of μ on $\tilde{W}^u$ have bounded ratio with B and δ depending only on F.

The proof of this is almost identical to the corresponding one for a single hyperbolic attractor. We include a sketch for completeness.

Proof. For $\delta > 0$, let $\tilde{W}^u_\delta(g, x) = \{(g, y) \in \tilde{W}^u(g, x) : \tilde{d}^u(x, y) \le \delta\}$ where $\tilde{d}^u$ and $\tilde{m}^u$ are analogous to d^u and m^u in $\Omega^\mathbb{Z} \times \bar{U}$. First there exist A, $\delta > 0$ s.t. for every $(g, x) \in \tilde{\Lambda}$, if $z_1, z_2 \in \tilde{W}^u_\delta(g, x)$ then

$$|J(DF_g | \tilde{E}^u) z_1 - J(DF_g | \tilde{E}^u) z_2| \le A \tilde{d}^u(z_1, z_2).$$

This follows from the uniform Lipschitzness of the $\tilde{E}^u$-bundle *along* $\tilde{W}^u$-manifolds. (See e.g. § 6.1 of [HP]. A uniform bound on $L(Dg)$ for all $g \in \Omega$ is needed here.) From this and the fact that $\tilde{d}^u(F^{-n} z_1, F^{-n} z_2) \le \tilde{\lambda}^{-n} \tilde{d}^u(z_1, z_2)$, for all $z_1, z_2 \in \tilde{W}^u(g, x)$ we obtain the following important estimate: there exists $B > 0$ such that for all $(g, x) \in \tilde{\Lambda}$, and all $z_1, z_2 \in \tilde{W}^u_\delta(g, x)$ and $n \ge 0$,

$$\frac{1}{B} \le \frac{|J(Dg_{-n} \circ \cdots \circ g_{-1} | \tilde{E}^u) z_1|}{|J(Dg_{-n} \circ \cdots \circ g_{-1} | \tilde{E}^u) z_2|} \le B.$$

Suppose $(1/n_k) \sum_{i=0}^{n_k - 1} F^i m_L \to \mu$ for some subsequence $\{n_k\}$. We construct certain canonical neighbourhoods of $\tilde{\Lambda}$. Let T be an embedded disk in U having the same dimension as the bundle dimension of $\tilde{E}^s$ and with the property that for all $g \in \Omega^\mathbb{Z}$, T is transverse to $\tilde{W}^u(\tilde{\Lambda}) | U_g$. Let $\Sigma = (\Omega^\mathbb{Z} \times T) \cap \tilde{\Lambda}$. Then assuming δ is small enough, $V = \bigcup_{(g,x) \in \Sigma} \tilde{W}^u_\delta(g, x)$ is the disjoint union of a continuous family of $\tilde{W}^u$-disks. We may assume that $\mu V > 0$ and $\mu(\partial V) = 0$ (by shrinking Σ or δ if necessary).

For each n, let $L_n = \{z \in L : F^n z \in \tilde{W}^u_\delta(g, x) \text{ for some } (g, x) \in \Sigma \text{ but } F^n L \not\supset \tilde{W}^u_\delta(g, x)\}$. Since F expands distances uniformly along $\tilde{W}^u$, $m_L(L_n) \to 0$ as $n \to \infty$, so we can realize μ as $\lim_{n_k \to \infty} (1/n_k) \sum F^i(m_L | (L - L_i))$.

Suppose $F^n(L - L_n) \supset \tilde{W}^u_\delta(g, x)$, $(g, x) \in \Sigma$ and

$$\rho_n = \frac{dF^n(m_L | (L - L_n))}{d\tilde{m}^u},$$

on $\tilde{W}^u_\delta(g, x)$. Then

$$\frac{1}{B} \le \frac{\rho_n(z_1)}{\rho_n(z_2)} \le B,$$

for all $z_1, z_2 \in \tilde{W}^u_\delta(g, x)$. These inequalities continue to hold for the conditional measures of the limit μ. $\qquad \square$

Proof of lemma 3.2. Pick a $\nu_\varepsilon^\mathbb{Z}$-generic point g. Let $L \subset \tilde{W}^u(g, x)$ for some x, let μ be a measure constructed as in the last lemma and let m_ε be an ergodic component of μ. Since ergodic components of a measure contain entire $\tilde{W}^u$-leaves, m_ε has absolutely continuous conditional measures on $\tilde{W}^u$ as well. Also, $\nu_\varepsilon^\mathbb{Z}$ is ergodic with respect to the shift transformation, so we may assume $p_1 m_\varepsilon = \nu_\varepsilon^\mathbb{Z}$.

To prove uniqueness first consider $f: \Lambda \circlearrowleft$. Let $x, y \in \Lambda$ be nearby points. Define $[x, y]$ to be the unique point in $W^u_\delta(x) \cap W^s_\delta(y)$. Suppose δ is small enough that $N_\delta(x) = \{[z_1, z_2]: z_1 \in W^u_\delta(x), z_2 \in W^s_\delta(x)\}$ is well defined for every $x \in \Lambda$. Then there exist $x_1, \ldots, x_k$ s.t. $\bigcup_{i=1}^k N_{\delta/4}(x_i) \supset \Lambda$. By the topological mixing property of f ([**B**, p. 72]), there exists N s.t. for all $i, j, 1 \le i, j \le k, f^N N_{\delta/4}(x_i)$ 'crosses' $N_\delta(x_j)$, meaning that $f^N N_{\delta/4}(x_i)$ contains an entire W^u-leaf in $N_\delta(x_j)$.

In the skew product $F: \Omega^{\mathbf{Z}}_\alpha \times \bar{U} \circlearrowleft, [(g, x), (g, y)]$ and $\tilde{N}_\delta(g, x)$ have the obvious meanings. If α is small enough, then the Hausdorff distance between Λ and $U_g \cap \tilde{\Lambda}$ is small and f^N is near $F^N | U_g$. So for every $g \in \Omega^{\mathbf{Z}}_\alpha$, there exist $x_1, \ldots, x_k \in U_g \cap \tilde{\Lambda}$ and $y_1, \ldots, y_k \in U_{\sigma^N g} \cap \tilde{\Lambda}$ such that $\bigcup_{i=1}^k \tilde{N}_{\delta/2}(g, x_i) \supset U_g \cap \tilde{\Lambda}, \bigcup_{i=1}^k \tilde{N}_{\delta/2}(\sigma^N g, y_i) \supset U_{\sigma^N g} \cap \tilde{\Lambda}$, and $F^N \tilde{N}_{\delta/2}(g, x_i)$ 'crosses' $\tilde{N}_{\delta/2}(\sigma^N g, y_j)$ for every i, j.

Now let $m_\varepsilon | g$ be the conditional probability of m_ε on U_g for some $\nu^{\mathbf{Z}}_\varepsilon$-generic g and suppose $(m_\varepsilon | g) N_{\delta/2}(g, x_i) > 0$ for some i. Because the densities of m_ε on $\tilde{W}^u$ are strictly positive wherever they are defined, there is a $\tilde{W}^u$-disk L_0 in $\tilde{N}_{\delta/2}(g, x_i)$ with the property that $\tilde{m}^u$-a.e. $z \in L_0$ is m_ε-generic. The absolute continuity property of the $\tilde{W}^s$-foliation restricted to U_g ([**PS**, p. 4]) then tells us that every $\tilde{W}^u$-disk L in $\tilde{N}_{\delta/2}(g, x_i)$ has the property that $\tilde{m}^u$-a.e. $z \in L$ is m_ε-generic. It follows from this and our choice of the x_i's and y_j's that $\tilde{m}^u$-a.e. z in every $\tilde{W}^u$-leaf in $U_{\sigma^N g} \cap \tilde{\Lambda}$ is m_ε-generic, which proves the uniqueness of m_ε with respect to (1) and (2) in the statement of lemma 3.1. $\square$

Proof of lemma 3.3. First there is an open set $\tilde{V} \subset \Omega^{\mathbf{Z}} \times U$ containing $\tilde{\Lambda}$ s.t. for every $g \in \Omega^{\mathbf{Z}}, U_g \cap \tilde{V}$ is foliated by $\tilde{W}^s(\tilde{\Lambda})$. It follows from lemma 3.2 and the absolute continuity of this $\tilde{W}^s$-foliation on each U_g that $(\nu^{\mathbf{Z}}_\varepsilon \times \text{Leb.})$-a.e. (g, x) in $\tilde{V}$ is m_ε-generic. The same is true for $(\nu^{\mathbf{Z}}_\varepsilon \times \text{Leb.})$-a.e. (g, x) in $\Omega^{\mathbf{Z}} \times U$ since $F^n(\Omega^{\mathbf{Z}} \times U) \subset \tilde{V}$ for all n greater than some n_0. This together with the fact that $\mu_\varepsilon \ll \text{Leb.}$ implies that $(\nu^{\mathbf{Z}}_\varepsilon \times \mu_\varepsilon)$-almost every point is m_ε-generic.

To complete the proof, let $\phi: \Lambda^{\mathbf{Z}} \times \bar{U} \to \mathbb{R}$ be a continuous function. Then

$$\int \phi \, dm_\varepsilon = \int \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi \circ F^i d(\nu^{\mathbf{Z}}_\varepsilon \times \mu_\varepsilon)$$

$$= \lim_{n \to \infty} \int \phi \, d\left(\frac{1}{n} \sum_{i=0}^{n-1} F^i(\nu^{\mathbf{Z}}_\varepsilon \times \mu_\varepsilon)\right)$$

$$= \int \phi \, d\mu^*_\varepsilon.$$

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