THE DISTRIBUTION OF THE TIME TO RUIN IN THE CLASSICAL RISK MODEL

BY

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ABSTRACT

We study the distribution of the time to ruin in the classical risk model. We consider some methods of calculating this distribution, in particular by using algorithms to calculate finite time ruin probabilities. We also discuss calculation of the moments of this distribution.

1. INTRODUCTION

In recent years, research in ruin theory has focussed on moments of the time to ruin, particularly in the classical risk model. Lin and Willmot (1999 and 2000) develop ideas given in Gerber and Shiu (1998). They present methods from which explicit solutions for moments of the time to ruin can be found recursively for this model provided that an explicit solution exists for the ultimate ruin probability. Egídio dos Reis (2000) presents a recursion scheme to find the moments of the time to ruin for a discrete time risk model, and uses this to approximate moments of the time to ruin in the classical risk model, while Picard and Lefèvre (1998) consider the classical risk model with a discrete individual claim amount distribution. Cheng *et al* (2000) consider a discrete time risk model and find expressions for the moments of the time to ruin for this model. Cardoso and Egídio dos Reis (2002) study the shape of the density of the time to ruin. Further references can be found in these papers.

Our objective in this paper is to study aspects of the time to ruin in the classical risk model. In particular, we focus on the actual distribution of the time to ruin. By calculating values of both finite and infinite time ruin probabilities, we can construct numerically the conditional distribution of the time to ruin, and use this to create density functions. We also show how Lin and Willmot's (2000) results can be used to calculate approximate values for moments of the time to ruin when explicit solutions for the probability of ultimate ruin do not exist.

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The layout of this paper is as follows. In Section 2 we introduce notation. In Section 3 we summarise methods of approximating both finite and infinite time ruin probabilities, and of approximating the distribution of the time to ruin in the classical risk model. In Section 4 we illustrate how moments of the time to ruin can be calculated, and in Section 5 we give some illustrations of densities of the time to ruin, given that ruin occurs.

2. Notation

In the classical risk model, the insurer's surplus at time t, given an initial surplus u, is $U(t)$ where

$$U(t) = u + ct - S(t).$$

The aggregate claims process $\{S(t)\}_{t\geq 0}$ is a compound Poisson process, with Poisson parameter λ. We denote by P the distribution function of individual claim amounts, and assume that $P(0) = 0$. Let p_k denote the kth moment of this distribution. We assume that the insurer's premium income is received continuously at rate c per unit time, where $c = (1 + \theta)\lambda p_1$ and θ is the premium loading factor. Without loss of generality we can set both λ and p_1 to be 1 and these values will be assumed in all numerical illustrations in this paper.

The time to ruin is denoted T and defined by

$$T = \begin{cases} \inf(t: U(t) < 0) \\ \infty \text{ if } U(t) \geq 0 \text{ for all } t > 0. \end{cases}$$

The probability of ultimate ruin from initial surplus u is denoted $\psi(u)$ and defined by $\psi(u) = \Pr(T < \infty)$. We write $\delta(u) = 1 - \psi(u)$ and denote by T_c the random variable $T | T < \infty$. The aggregate loss process $\{L(t)\}_{t\geq 0}$ is defined by $L(t) = S(t) - ct$. We denote by L the maximum of the aggregate loss process so that $\psi(u) = \Pr(L > u)$. It is straightforward to show that:

$$E[L] = \int_0^\infty \psi(x)dx = \frac{p_2}{2\theta p_1} \tag{2.1}$$

$$E\left[L^2\right] = 2\int_0^\infty x\psi(x)dx = \frac{p_3}{3\theta p_1} + \frac{1}{2}\left(\frac{p_2}{\theta p_1}\right)^2 \tag{2.2}$$

$$E\left[L^3\right] = 3\int_0^\infty x^2\psi(x)dx = \frac{p_4}{4\theta p_1} + \frac{3}{4}\left(\frac{p_2}{\theta p_1}\right)^3 + \frac{p_2 p_3}{(\theta p_1)^2}$$

See, for example, Gerber (1979).

The probability of ruin by time t from initial surplus u is denoted $\psi(u, t)$ and given by $\psi(u, t) = \Pr(T \leq t)$ so that

$$\Pr(T_c \leq t) = \Pr(T \leq t \mid T < \infty) = \psi(u, t) / \psi(u)$$

is the distribution function of the time to ruin given that ruin occurs.

3. Algorithms, approximations and asymptotic results

In this section we give a brief description of some approaches to approximating the distribution of the time to ruin.

3.1. Algorithms

Our calculations in Sections 4 and 5 are based on calculated values of $\psi(u)$ and $\psi(u, t)$. Values of $\psi(u)$ have been calculated using the stable recursive algorithm described in Dickson *et al* (1995). Values of $\psi(u, t)$ have been calculated using the algorithm described in Dickson and Waters (1991, Section 8).

Each of these algorithms is based on a rescaling and a discretisation of the classical surplus process described in Section 2. Values of ruin probabilities are calculated in a recursive manner for a discrete time risk model, and are used to approximate probabilities for the classical model. In general, the scaling factor, denoted β in these papers, determines the quality of the approximations. The larger the value of β, the better the approximations are.

3.2. Segerdahl's asymptotic result

Segerdahl (1955) showed that asymptotically as $u \to \infty$, the distribution of T_c is normal provided that the moment generating function of the individual claim amount distribution is finite for some positive value of the argument. Asmussen (1984) suggests conditions under which Segerdahl's result gives a reasonable approximation to the distribution of T_c. We mention this result as it is well known in the literature. However, we will not apply it in our examples in Section 5. It will be apparent from our calculation of the coefficient of skewness of T_c in Section 4 and our graphical illustrations in Section 5 that it would be unreasonable to approximate the densities we plot there by normal densities.

3.3. Diffusion and Inverse Gaussian approximations

We can approximate the surplus process $\{U(t)\}$ by a diffusion process. Letting $\tilde{U}(t) = u + W(t)$ where $W(t) \sim N(\theta\lambda p_1 t, \lambda p_2 t)$ for all $t > 0$, we have the well known result that for $u > 0$ the conditional distribution of the time to ruin, given that ruin occurs, for the process $\{\tilde{U}(t)\}$ is Inverse Gaussian with density

$$f(t) = \frac{u}{\sqrt{2\pi\lambda p_2}} t^{-3/2} \exp\left\{-\frac{(u - \theta\lambda t p_1)^2}{2\lambda t p_2}\right\}. \tag{3.1}$$

See, for example, Klugman *et al* (1998). The moments of this distribution can be regarded as approximations to the moments of T_c; we illustrate this idea in Section 4. In Section 5, we use f as an approximation to the density of T_c.

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Based on this exact result for the diffusion surplus process, we also test the idea in Section 5 that the distribution of T_c can be approximated by an Inverse Gaussian distribution, with parameters determined by the first two moments of T_c.

3.4. Translated gamma approximation

Dickson and Waters (1993) show that $\psi(u, t)$ for a classical surplus process for which the premium loading factor is θ can be approximated by the ruin probability $\psi_{SG}(\beta u, \alpha t)$ for a standardised gamma process for which the premium loading factor is $\hat{\theta} = \theta(1 + k\beta/\alpha)$ where the parameters α, β and k are given by

$$\alpha = 4\lambda p_2^3 / p_3^2 \quad \beta = 2p_2 / p_3 \quad k = \lambda\left(p_1 - 2p_2^2 / p_3\right).$$

Formulae to calculate values of $\psi_{SG}(u, t)$ are given by Dickson and Waters (1993, Section 2). Dufresne *et al* (1991) explain how values of

$$\psi_{SG}(u) = \lim_{t \to \infty} \psi_{SG}(u, t)$$

can be calculated. Thus, we can use the methods of these papers to compute $\psi_{SG}(\beta u, \alpha t) / \psi_{SG}(\beta u)$ as an approximation to the distribution of T_c.

The numerical illustrations in Dickson and Waters (1993) suggest that this approach should give reasonably good approximations, except for small values of u (relative to p_1). The main advantage of this approach is that, for large values of t, the calculation of a finite time ruin probability is fairly quick as it involves numerical integration rather than a recursive calculation.

3.5. Other approaches

Seal (1978) describes methods for calculating or approximating finite time ruin probabilities. In particular, when the individual claim amount distribution is exponential, a formula exists from which values of $\psi(u, t)$ can be calculated. (See also Asmussen (2000).) As the algorithms described in Section 3.1 give excellent approximations to both finite and infinite time ruin probabilities, we will not employ the techniques described by Seal, although we acknowledge that these provide alternative methods of approximation.

Similarly, in the case when $u = 0$, a formula exists from which finite time ruin probabilities can be calculated:

$$\psi(0, t) = 1 - \frac{1}{ct} \int_0^{ct} G(x, t) dx$$

where, for a fixed value of t, $G(x, t) = \Pr(S(t) \le x)$. In this special case, given that ruin occurs, the distribution of the time to ruin is the same as the distribution of the time to recovery to surplus level 0, and Dickson and Egídio dos

Reis (1996, Figure 1) illustrate this density in the case of exponential individual claim amounts. In this case, the distribution of T_c has a strong positive skew, a feature that will be evident in the examples in Sections 4 and 5.

4. Moments of the time to ruin

In this section we illustrate how the first three moments of T_c can be calculated and approximated. We note that Delbaen (1988) proved that the kth moment of T_c exists only if the $(k+1)$th moment of the individual claim amount distribution exists. In the following subsection we assume that p_4 exists and that we can calculate values of $\psi(x)$ for $x = 0, h, 2h, ..., u$, where u is an integer multiple of the constant h, using the algorithm described in Section 3.1. The ideas presented here can be extended to higher moments.

4.1. Formulae for moments

Lin and Willmot (2000, formula (6.21)) show that $E(T_c) = \psi_1(u) / \psi(u)$ where

$$\psi_1(u) = \frac{1}{\lambda p_1 \theta}\left(\int_0^u \psi(u-x)\psi(x)dx + \int_u^\infty \psi(x)dx - \frac{p_2}{2\theta p_1}\psi(u)\right). \tag{4.1}$$

Using (2.1), we can rewrite (4.1) as

$$\begin{aligned}\psi_1(u) &= \frac{1}{\lambda p_1 \theta}\left(\int_0^u \psi(u-x)\psi(x)dx + E(L)\delta(u) - \int_0^u \psi(x)dx\right)\\ &= \frac{1}{\lambda p_1 \theta}\left(E(L)\delta(u) - \int_0^u \psi(x)\delta(u-x)dx\right)\end{aligned} \tag{4.2}$$

so that we can evaluate $\psi_1(u)$ using numerical integration.

Similarly, Lin and Willmot (2000, Theorem 6.3 and formula (6.29)) show that

$$E\left(T_c^k\right) = \psi_k(u) / \psi(u),$$

where

$$\psi_k(u) = \frac{k}{\lambda p_1 \theta}\left(\int_0^u \psi(u-x)\psi_{k-1}(x)dx + \delta(u)\int_0^\infty \psi_{k-1}(x)dx - \int_0^u \psi_{k-1}(x)dx\right). \tag{4.3}$$

This formula involves integration over an infinite range and so cannot in general be used directly to calculate $\psi_2(u)$ and $\psi_3(u)$.

For $k = 2$ the first and third terms on the right hand side of formula (4.3) can be combined and evaluated by numerical integration. To evaluate the middle term, we proceed as follows:

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$$\lambda p_1\theta\int_0^\infty \psi_1(x)dx=\int_0^\infty\left(\int_0^x \psi(x-y)\psi(y)dy+\int_x^\infty \psi(y)dy-E(L)\psi(x)\right)dx$$
$$=\int_0^\infty\int_y^\infty \psi(x-y)dx\psi(y)dy+\int_0^\infty\int_0^y dx\psi(y)dy-E(L)^2$$
$$=E(L)^2+\int_0^\infty y\psi(y)dy-E(L)^2$$
$$=\tfrac{1}{2}E(L^2),$$

using (2.2). Thus, we can write $\psi_2(u)$ as

$$\psi_2(u)=\frac{2}{\lambda p_1\theta}\left(\frac{E(L^2)\delta(u)}{2\lambda p_1\theta}-\int_0^u \psi_1(x)\delta(u-x)dx\right). \tag{4.4}$$

Similarly, we can write $\psi_3(u)$ as

$$\psi_3(u)=\frac{3}{\lambda p_1\theta}\left\{\delta(u)\int_0^\infty \psi_2(x)dx-\int_0^u \delta(u-x)\psi_2(x)dx\right\}$$

The second integral on the right hand side can be evaluated by numerical integration. Consider the first integral. Using (4.3), we can write this as:

$$\int_0^\infty \psi_2(u)du=\frac{2}{\lambda p_1\theta}\left\{\int_0^\infty\int_0^u \psi(u-x)\psi_1(x)dxdu\right.$$
$$+\int_0^\infty\int_u^\infty \psi_1(x)dxdu$$
$$\left.-\int_0^\infty \psi(u)\int_0^\infty \psi_1(x)dxdu\right\}$$

We consider the evaluation of this expression term by term below. First:

$$\int_0^\infty\int_0^u \psi(u-x)\psi_1(x)dxdu=\frac{E[L]E\left[L^2\right]}{2\lambda p_1\theta}$$

Next:

$$\int_0^\infty\int_u^\infty \psi_1(x)dxdu=\int_0^\infty u\psi_1(u)du$$
$$=\frac{1}{\lambda p_1\theta}\left\{\tfrac{1}{2}E[L]E\left[L^2\right]+\tfrac{1}{6}E\left[L^3\right]\right\}$$

Finally:

$$\int_0^\infty \psi(u)\int_0^\infty \psi_1(x)dxdu=\int_0^\infty \psi(u)\frac{E\left[L^2\right]}{2\lambda p_1\theta}du=\frac{E[L]E\left[L^2\right]}{2\lambda p_1\theta}$$

Putting all these pieces together, we have:

$$\psi_3(u) = \frac{3\delta(u)E[L]E\left[L^2\right]}{(\lambda p_1\theta)^3} + \frac{\delta(u)E\left[L^3\right]}{(\lambda p_1\theta)^3} - \frac{3}{\lambda p_1\theta}\int_0^u \delta(u-x)\psi_2(x)dx. \qquad (4.5)$$

4.2. Approximate moments

In Section 3.3 we noted that the time to ruin, given that ruin occurs, for a diffusion process has an Inverse Gaussian distribution. By choosing the parameters of the diffusion process appropriately, as in Section 3.3, we can regard the moments of the Inverse Gaussian distribution as approximations to the moments of T_c for values of u greater than 0. Hence, we can write for $u > 0$:

$$E[T_c] \approx \frac{u}{\lambda\theta p_1}; \quad V[T_c] \approx \frac{up_2}{\lambda^2\theta^3 p_1^3}; \quad Sk[T_c] \approx 3\left(\frac{p_2}{\theta p_1 u}\right)^{1/2} \qquad (4.6)$$

where $Sk(T_c)$ denotes the coefficient of skewness of T_c.

Note that these approximations do not depend on any moments of the individual claim size distribution above the second. This is because the surplus process is being approximated by a diffusion process matched through the first two moments. However, it should be remembered that if, for example, p_4 does not exist, then the third moment, and hence the coefficient of skewness, of T_c does not exist. The advantage of these formulae is that they are simple and depend on the various parameters in a transparent way.

4.3. Numerical illustrations

In Examples 4.1 and 4.3 below, approximate values of $E(T_c^k)$ for $k = 1,2,3$ were calculated using (4.2), (4.4) and (4.5) respectively, with numerical integration by the trapezoidal rule. These values are labelled "App" in Tables 4.1 to 4.6. Values of δ were calculated using the stable recursive algorithm of Dickson *et al* (1995), with a scaling factor of 1000. This means that (approximate) values of $\psi(w)$ were calculated for $w = 0, 0.001, 0.002, \dots$, so that each trapezium had a base of 0.001. Similarly, values of $\psi_1(w)$ and $\psi_2(w)$ were calculated for the same values of w, using exactly the same method of numerical integration. A second set of approximate values for the first three moments of T_c was calculated using (4.6). These values are labelled "Dif" in the Tables below. In Example 4.2, only the first two moments of T_c are shown since the fourth moment of the individual claim amount distribution does not exist.

Example 4.1 *Let the individual claim amount distribution be exponential (with mean 1). Tables 4.1 and 4.2 show exact and approximate values of the mean, standard deviation and coefficient of skewness of T_c when $\theta = 10\%$ and when*

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TABLE 4.1

MEAN, STANDARD DEVIATION AND COEFFICIENT OF SKEWNESS OF T_c, EXPONENTIAL CLAIMS, $\theta = 10\%$.

	Mean			St. Dev.			Skewness		
u	Exact	App	Dif	Exact	App	Dif	Exact	App	Dif
0	10.00	10.00	–	45.83	45.83	–	17.737	17.737	–
10	100.91	100.91	100	148.66	148.66	141.42	4.238	4.238	4.243
20	191.82	191.82	200	205.18	205.18	200.00	3.070	3.070	3.000
30	282.73	282.73	300	249.20	249.20	244.95	2.528	2.528	2.449
40	373.64	373.64	400	286.53	286.53	282.84	2.199	2.199	2.121
50	464.55	464.55	500	319.53	319.53	316.23	1.972	1.972	1.897

TABLE 4.2

MEAN, STANDARD DEVIATION AND COEFFICIENT OF SKEWNESS OF T_c, EXPONENTIAL CLAIMS, $\theta = 25\%$.

	Mean			St. Dev.			Skewness		
u	Exact	App	Dif	Exact	App	Dif	Exact	App	Dif
0	4	4	–	12.00	12.00	–	8.963	8.963	–
10	36	36	40	37.74	37.74	35.78	2.861	2.861	2.683
20	68	68	80	52.00	52.00	50.60	2.076	2.076	1.897
30	100	100	120	63.12	63.12	61.97	1.711	1.711	1.549
40	132	132	160	72.55	72.57	71.55	1.488	1.486	1.342
50	164	163.98	200	80.90	81.01	80.00	1.335	1.313	1.200

$\theta = 25\%$, respectively. The exact values are calculated from formulae (4.2), (4.4) and (4.5). When $\theta = 10\%$, over the range of values of u in Table 4.1, the smallest value of $\psi(u)$ is 0.0097 (when $u = 50$).

Example 4.2 *Let the individual claim amount distribution be Pareto with distribution function $P(x) = 1 - (3/(3 + x))^4$. Table 4.3 shows approximate values of the mean and standard deviation of T_c when $\theta = 10\%$ and when $\theta = 25\%$. In this case it is not possible to compare these approximations with exact values. When $\theta = 10\%$, over the range of values of u in Table 4.3, the smallest calculated value of $\psi(u)$ is 0.0102 (when $u = 80$).*

Example 4.3 *We now extend the previous example by introducing excess of loss reinsurance, with retention level M. In this case all moments of the individual claim size distribution, and hence of T_c exist. Tables 4.4, 4.5 and 4.6 show approximate values of the mean, standard deviation and coefficient of skewness of T_c when $\theta = 10\%$ and when the reinsurance premium is calculated by the expected value principle with a loading $\xi = 25\%$, for three different values of M.*

TABLE 4.3

MEAN AND STANDARD DEVIATION OF T_C, PARETO CLAIMS.

	$\theta = 10\%$				$\theta = 25\%$			
	Mean		St. Dev.		Mean		St. Dev.	
u	App	Dif	App	Dif	App	Dif	App	Dif
0	15.00	–	71.94	–	6.00	–	19.90	–
20	203.77	200	271.39	244.95	70.49	80	75.50	61.97
40	372.13	400	373.14	346.41	119.00	160	113.74	87.64
60	531.90	600	456.49	424.26	155.88	240	164.94	107.33
80	681.88	800	535.33	489.90	186.27	320	233.05	123.94

TABLE 4.4

MEAN, STANDARD DEVIATION AND COEFFICIENT OF SKEWNESS OF T_C,
PARETO CLAIMS AND EXCESS OF LOSS REINSURANCE, $M = 2$.

	Mean		St. Dev.		Skewness	
u	App	Dif	App	Dif	App	Dif
0	14.64	–	86.25	–	17.765	–
20	426.94	434.78	472.16	465.81	3.246	3.214
40	842.32	869.57	663.27	658.76	2.311	2.273
60	1257.70	1304.35	810.51	806.81	1.891	1.856
80	1673.07	1739.13	934.89	931.62	1.639	1.607

Comments

In each of the above examples, we have taken a fairly large scaling factor in our algorithm to calculate δ. With the smaller scaling factor of 100, approximations in Example 4.1 are poorer than those given by Egídio dos Reis (2000) who also considered this example. As his algorithms are based on the same model we use to calculate values of δ, the role of the scaling factor is identical in each method. Our method is perhaps a little more transparent than his, and does not appear to suffer from problems of numerical stability. Interestingly, choosing a more sophisticated method of numerical integration such as Simpson's rule does not materially improve the quality of our approximations in Example 4.1 with a scaling factor of 100. In Example 4.1 at least we can see that the integrand in formula (4.2) is an exponentially decreasing function (using the well known formula $\psi(u) = \exp\{-\theta u / (1 + \theta)\} / (1 + \theta)$) whereas our numerical integration technique effectively assumes it is a linearly decreasing function. In each of the above examples, the choice of a large scaling factor did not result in lengthy computer run times.

TABLE 4.5

MEAN, STANDARD DEVIATION AND COEFFICIENT OF SKEWNESS OF T_c, PARETO CLAIMS AND EXCESS OF LOSS REINSURANCE, $M = 4$.

	Mean		St. Dev.		Skewness	
u	**App**	**Dif**	**App**	**Dif**	**App**	**Dif**
0	12.29	–	59.98	–	14.666	–
20	241.73	249.00	271.16	264.98	3.247	3.193
40	472.32	498.00	379.14	374.74	2.322	2.257
60	702.90	747.01	462.56	458.97	1.903	1.843
80	933.48	996.01	533.10	529.97	1.651	1.596

TABLE 4.6

MEAN, STANDARD DEVIATION AND COEFFICIENT OF SKEWNESS OF T_c, PARETO CLAIMS AND EXCESS OF LOSS REINSURANCE, $M = 6$.

	Mean		St. Dev.		Skewness	
u	**App**	**Dif**	**App**	**Dif**	**App**	**Dif**
0	12.72	–	60.05	–	14.128	–
20	213.93	220.41	251.36	243.90	3.379	3.320
40	414.91	440.82	350.24	344.92	2.425	2.347
60	615.89	661.22	426.80	422.44	1.990	1.917
80	816.87	881.63	491.57	487.79	1.727	1.660

Given that a diffusion process has continuous sample paths, we would expect a diffusion approximation to give better results if claim sizes are small relative to the initial surplus. It can be seen that the diffusion approximations are better in Tables 4.1 and 4.2 (exponential claims) than in Table 4.3 (Pareto claims) and that the approximations improve as the excess loss retention level, M, decreases (*cf.* Tables 4.4 to 4.6).

A feature of Examples 4.1 and 4.3 is the large positive value for each of the coefficients of skewness. This indicates that in each case the distributions of T_c are far from normal. This feature will be illustrated in the examples in Section 5. Formula (4.6) indicates that

$$\lim_{u \to \infty} Sk\left[T_c\right] = 0$$

as Segerdahl's (1955) asymptotic result shows it must for these examples since in the limit the distribution of T_c is normal. We can use formula (4.6) for the coefficient of skewness of T_c to gain some insight into when the distribution of T_c is approximately normal. For example, consider Example 4.1, for which

$p_1 = 1$, $p_2 = 2$ and $\theta = 10\%$. Formula (4.6) indicates that to obtain a coefficient of skewness as low as 0.5, u must be about 720 and for the coefficient to be as low as 0.25, u should be about 2880. However for these two values of u, the probabilities of ultimate ruin are 3.4×10^{-29} and 1.34×10^{-114}, respectively, way beyond any area of practical interest. (We note that for these two values of u the exact values of the coefficient of skewness are 0.525 and 0.262 respectively.)

We remark that the quality of the approximations denoted "App" in Example 4.1 is excellent.

5. The density of T_c

5.1 Calculation methods

In this section our aim is to illustrate the shape of the density of T_c. In each of the examples in this section, four different methods of calculating/approximating this density were used. The following methods were used to produce graphs of density functions.

1. **Algorithms:** For a given value of u and a fixed value of t, the algorithm to approximate finite time ruin probabilities described in Section 3.1 provided approximate values of $\psi(u, \tau)$ for $\tau = j / [(1+\theta)\beta]$, $j = 1, 2, \ldots, (1+\theta)\beta t$. Dividing these by the value of $\psi(u)$ calculated from the infinite time algorithm of Section 3.1 provides values of the distribution function, say $H(\tau) = \Pr(T_c \leq \tau)$. From these, we estimated the density at $\tau = j / [(1+\theta)\beta]$ as

$$(1+\theta)\,\beta\left[H\left(\frac{j}{(1+\theta)\,\beta}\right) - H\left(\frac{j-1}{(1+\theta)\,\beta}\right)\right]$$

for $j = 1, 2, 3, \ldots$,
We regard this as the "true" density and measure the three approximations below against it. In the calculations in Examples 5.1, 5.2 and 5.3 we have set $\beta = 20$. Illustrations in Dickson and Waters (1991) suggest this value is sufficient to calculate accurate approximations to both finite and infinite time ruin probabilities. A larger value of β will give better approximations, but such extra accuracy is of limited value to us in what follows as our aim is to illustrate the shape of the density of T_c.

2. **Diffusion approximation:** We have calculated this approximation directly from formula (3.1) given the Poisson parameter λ, the moments $p_1 (= 1)$ and p_2, the initial surplus u, and the loading θ.

3. **Inverse Gaussian approximation:** We have calculated the first two moments of T_c using formulae (4.2) and (4.4). We then matched the first two moments of an Inverse Gaussian distribution to these, and calculated values of the density directly, using the formulation in Klugman *et al* (1998, p. 583).

4. **Translated gamma approximation:** For the same τ values as under Method 1 above, we approximated values of $\psi(u, \tau)$ and, having divided these by our approximation to $\psi(u)$ under this method, we estimated the density in the same way as under Method 1.

5.2. Illustrations

Example 5.1 *Let the individual claim amount distribution be exponential. Figure 1 shows densities calculated by each method for* $\theta = 10\%$ *and* $u = 40$. *We have chosen this value of u as it provides an ultimate ruin probability in the range of practical interest. (In fact* $\psi(40) = 0.024$.*) Using the exact values of the mean and standard deviation from Table 4.1, we can calculate the parameters of our approximating Inverse Gaussian density as* 373.64 *and* 635.36 *(in the parameterisation used by Klugman et al (1998)). In Figure 1, the densities calculated by Methods 1 and 4 are virtually indistinguishable from each other, whilst the approximations under Methods 2 and 3 are reasonably close to the true density. A clear feature of Figure 1 is that the distribution is positively skewed, as indicated by the value of the coefficient of skewness in Table 4.1. Figure 2 shows the densities when* $\theta = 25\%$ *and* $u = 20$ *(so that* $\psi(u) = 0.015$*). It has exactly the same features as Figure 1.*

Example 5.2 *Let the individual claim amount distribution be Pareto as in Example 4.2, let* $u = 80$ *and let* $\theta = 10\%$ *(so that* $\psi(80) = 0.010$*). Figure 3 shows the same densities as Figures 1 and 2. In this example, we have used the "App" values from Table 4.3 to find the parameters of the approximating Inverse Gaussian density. We observe that Method 4 again provides the best approximation to the true density and that Method 3 provides a better approximation than Method 2.*

Example 5.3 *We extend the previous example to include the effect of excess of loss reinsurance. Figure 4 shows the density of* T_c *when the retention level is 6, 10 and 14, and when the reinsurance premium is calculated with a loading of*

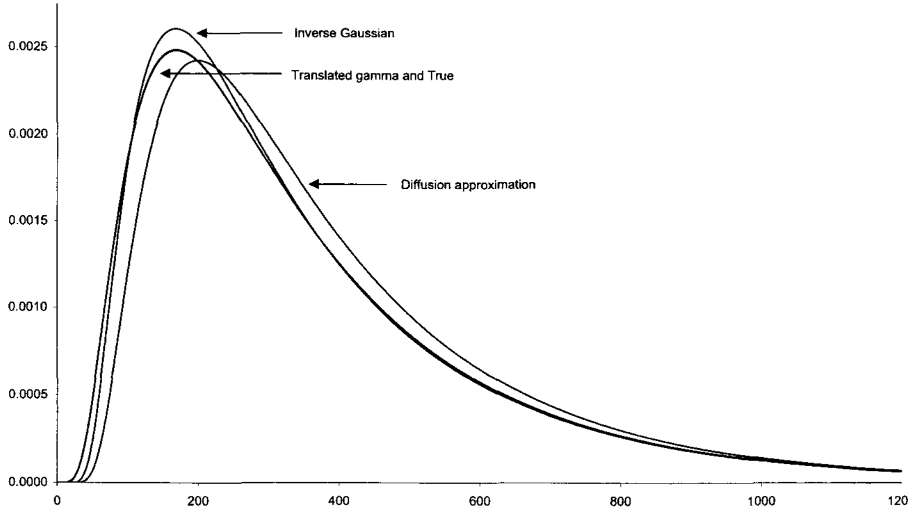

Figure 1: Exponential claims, $u = 40$, 10% loading

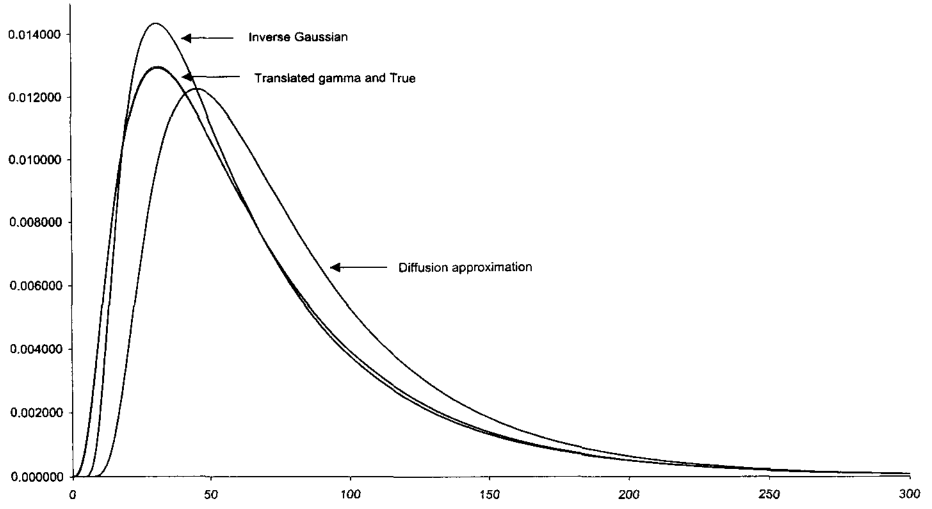

Figure 2: Exponential claims, $u = 20$, 25% loading

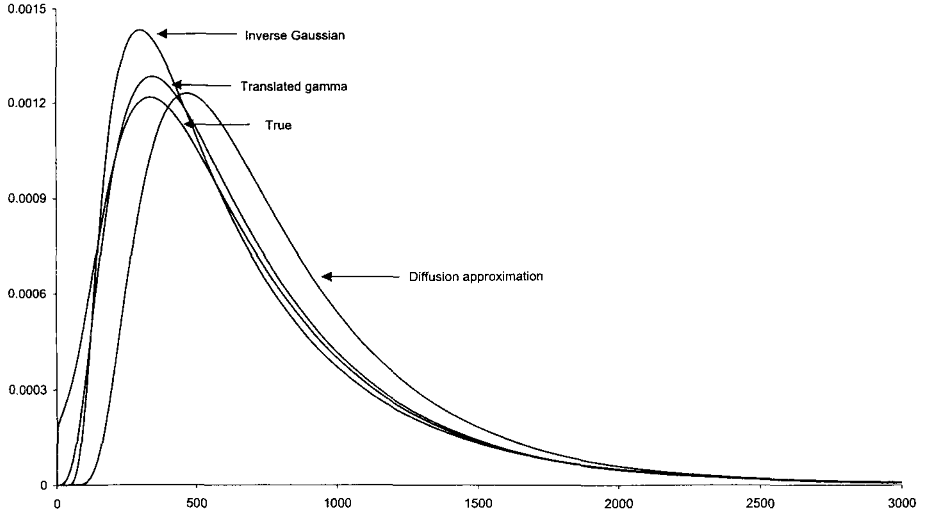

Figure 3: Pareto(4,3) claims, $u = 80$, 10% loading

50%. These densities have been calculated using Method 1. We observe that the common feature of each of these densities is a strong positive skew.

It is clear from Figures 1 to 3 that the translated gamma approximation performs better than both the other two approximations and performs particularly well for the lighter-tailed exponential claims distribution (Figures 1 and 2)

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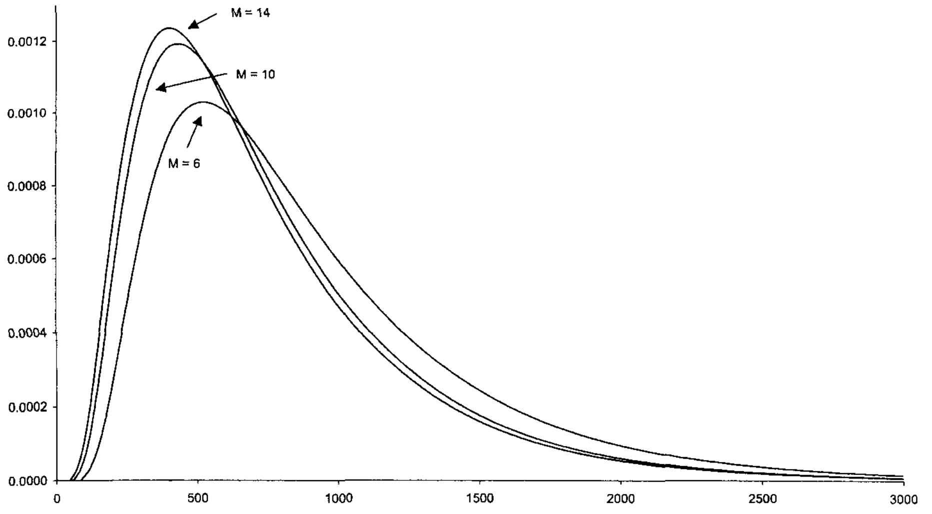

Figure 4: Pareto claims and excess of loss reinsurance

compared to the heavier-tailed Pareto claims distribution (Figure 3). It is not surprising that the translated gamma approximation, which is based on matching three moments, performs better than methods based on matching just two moments.

In each of the above examples, the consistent feature is that the true density is positively skewed, and this feature was even more apparent in other densities that we plotted for the same individual claim amount distributions, but for smaller values of u. This is consistent with the numerical examples in Cardoso and Egídio dos Reis (2002). Based on the numerical illustrations in Dickson and Waters (1993), we are not surprised by the fact that Method 4 produces good approximations to the density of T_c.

One feature that is apparent from our figures is that for the range of parameter values and individual claim amount distributions that we considered, the distribution of T_c is not normal. The straightforward approach of Methods 2 and 3 provides much better approximations than a normal distribution does, particularly in Example 5.2.

6. Concluding Remarks

Our aim has been to calculate moments of T_c, and to investigate the shape of its density. A simple numerical integration procedure suffices for the former provided we can accurately calculate values of the ultimate ruin probability. Our examples in Sections 4 and 5 indicate that the distribution of T_c is positively skewed, and that simple approximations based on Inverse Gaussian densities can give reasonable results, whereas a normal approximation would be inappropriate.

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