Can. J. Math., Vol. XXVI, No. 3, 1974, pp. 600–607

PERMANENTS OF RANDOM DOUBLY STOCHASTIC MATRICES

R. C. GRIFFITHS

1. Introduction. The permanent of an $n \times n$ matrix $A = (a_{ij})$ is defined as

$$\operatorname{per}(A) = \sum_{\alpha \in S_n} \prod_{i=1}^{n} a_{i\alpha(i)},$$

where S_n is the symmetric group of order n. For a survey article on permanents the reader is referred to [**2**]. An unresolved conjecture due to van der Waerden states that if A is an $n \times n$ doubly stochastic matrix; then per $(A) \geqq n!/n^n$, with equality if and only if $A = J_n = (1/n)$. For an $n \times n$ matrix A define $P_0(A) = 1$ and $P_r(A)$, $r = 1, 2, \ldots, n$, as the average of the $\binom{n}{r}^2$ permanents of sub-matrices obtained by deleting $n - r$ rows and $n - r$ columns of A. A generalization of the van der Waerden conjecture is that if A is an $n \times n$ doubly stochastic matrix; then $P_r(A) \geqq r!/n^r$, $r = 2, 3, \ldots, n$, with equality if and only if $A = J_n$.

Suppose $Q_1, Q_2, \ldots, Q_{n!}$ is the set of $n \times n$ permutation matrices; then any $n \times n$ doubly stochastic matrix A has a decomposition

$$A = c_1 Q_{i_1} + c_2 Q_{i_2} + \ldots + c_t Q_{i_t},$$

where $t \leqq n^2 - n + 1$, $c_i > 0$, $i = 1, 2, \ldots, t$ and $\sum c_i = 1$ (see e.g. [**3**]). This note studies permanents of random doubly stochastic matrices of the form

$$(1) \qquad \Omega(\bar{c}) = c_1 \Gamma_1 + c_2 \Gamma_2 + \ldots + c_m \Gamma_m,$$

where $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ is a set of mutually independent, identically distributed random matrices such that

$$(2) \qquad \text{Prob. } (\Gamma_i = Q_j) = (n!)^{-1}, \quad j = 1, 2, \ldots, n!,$$

$c_i \geqq 0$ and $\sum c_i = 1$. Throughout Γ and Ω will refer to random matrices given by (1) and (2). E will denote the expected value operator.

2. Expected values of random permanents.

LEMMA. *If B is an $n \times n$ matrix and c a constant; then*

$$\mathrm{E}P_r(B + c\Gamma) = \sum_{k=0}^{r} \binom{r}{k}^2 c^k P_k(I) P_{r-k}(B), \qquad r = 0, 1, \ldots, n.$$

Received November 28, 1972.

https://doi.org/10.4153/CJM-1974-057-4 Published online by Cambridge University Press

Proof. Denote by Q_{rn} the set of all $\binom{n}{r}$ subsets of r distinct elements from $(1, 2, \ldots, n)$. For any $n \times n$ matrix B, $B[\alpha|\beta]$, $\alpha, \beta \in Q_{rn}$, will denote the submatrix obtained by deleting all rows other than those numbered in α and all columns other than those numbered in β. $B(\alpha|\beta)$ will denote the sub-matrix obtained by deleting those rows numbered in α and those columns numbered in β. By definition

$$P_r(B + c\Gamma) = \binom{n}{r}^{-2} \sum_{\alpha,\beta \in Q_{rn}} \operatorname{per}(B[\alpha|\beta] + c\Gamma[\alpha|\beta]).$$

A result needed is the expansion

$$\text{(3)} \qquad \operatorname{per}(C + D) = \sum_{k=0}^{r} \sum_{\gamma,\delta \in Q_{kr}} \operatorname{per}(C(\gamma|\delta)) \operatorname{per}(D[\gamma|\delta]),$$

for any $r \times r$ matrices C and D (see e.g. [**2**]). Placing $C = B[\alpha|\beta]$ and $D = c\Gamma[\alpha|\beta]$ in (3);

$$\text{(4)} \qquad \mathrm{E} \operatorname{per}(B[\alpha|\beta] + c\Gamma[\alpha|\beta])$$
$$= \sum_{k=0}^{r} \sum_{\gamma,\delta \in Q_{kr}} c^k \operatorname{per}(B[\alpha|\beta](\gamma|\delta))\mathrm{E} \operatorname{per}(\Gamma[\alpha|\beta][\gamma|\delta])$$
$$= \sum_{k=0}^{r} \sum_{\gamma,\delta \in Q_{kr}} c^k \operatorname{per}(B[\alpha - \gamma|\beta - \delta])\mathrm{E} \operatorname{per}(\Gamma[\gamma|\delta]).$$

Summation in (4) is taken over $\gamma \subset \alpha$, $\delta \subset \beta$.

$$\mathrm{E} \operatorname{per}(\Gamma[\gamma|\delta]) = (n!)^{-1} \sum_{j} \operatorname{per}(Q_j[\gamma|\delta])$$
$$= \binom{n}{k}^{-1} \sum_{\gamma \in Q_{kn}} \operatorname{per}(I[\gamma|\delta])$$
$$= \binom{n}{k}^{-1} \sum_{\delta \in Q_{kn}} \operatorname{per}(I[\gamma|\delta]),$$

because there are $k!(n-k)!$ permutations of the rows (columns) of I which leave per $(I[\gamma|\delta])$ unaltered.

$$\mathrm{E} \operatorname{per}(\Gamma[\gamma|\delta]) = P_k(I),$$

since it does not depend on γ, $\delta \in Q_{kn}$. For a fixed $\theta \in Q_{(r-k)n}$ there are $\binom{n-(r-k)}{k}$ pairs (α, γ), $\alpha \in Q_{rn}$, $\gamma \in Q_{kn}$, $\gamma \subset \alpha$ for which $\alpha - \gamma = \theta$. Averaging over $\alpha, \beta \in Q_{rn}$ in (4),

$$\mathrm{E}P_r(B + c\Gamma)$$
$$= \sum_{k=0}^{r} \binom{n}{r}^{-2} \binom{n-(r-k)}{k}^{2} \binom{n}{r-k}^{2} c^k P_{r-k}(B) P_r(I)$$
$$= \sum_{k=0}^{r} \binom{r}{k}^{2} c^k P_{r-k}(B) P_r(I).$$

THEOREM 1.

$$\text{(5)}\qquad \mathrm{E}P_r(\Omega) = r! \sum_{r_1+r_2+\ldots+r_m=r} (r!/r_1!\ldots r_m!)n_{(r_1)}{}^{-1}\ldots n_{(r_m)}{}^{-1}c_1{}^{r_1}\ldots c_m{}^{r_m},$$

where $n_{(k)} = n(n-1)\ldots(n-k+1)$.

Proof.

$$\text{(6)}\qquad \mathrm{E}(P_r(\Omega)|\Gamma_t,\ldots,\Gamma_m) = \sum_{r_1+r_2+\ldots+r_t=r} (r!/r_1!\ldots r_t!)^2 P_{r_1}(I)\ldots P_{r_{t-1}}(I)c_1{}^{r_1}\ldots c_{t-1}{}^{r_{t-1}} \cdot P_{r_t}(c_t\Gamma_t+\ldots+c_m\Gamma_m),$$

$t = 1, 2, \ldots, m$, will be proved by induction; then it follows that

$$\text{(7)}\qquad \mathrm{E}P_r(\Omega) = \mathrm{EE}(P_r(\Omega)|\Gamma_m) = \sum_{r_1+r_2+\ldots+r_m=r} (r!/r_1!\ldots r_m!)^2 P_{r_1}(I)\ldots P_{r_m}(I)c_1{}^{r_1}\ldots c_m{}^{r_m}.$$

If $t = 1$, (6) is trivially true, assume it true for $t = 1, 2, \ldots, q$.

$$\text{(8)}\qquad \mathrm{E}(P_r(\Omega)|\Gamma_{q+1},\ldots,\Gamma_m) = \mathrm{E}(\mathrm{E}(P_r(\Omega)|\Gamma_q,\ldots,\Gamma_m)|\Gamma_{q+1},\ldots,\Gamma_m)$$
$$= \sum_{r_1+r_2+\ldots+r_q=r} (r!/r_1!\ldots r_q!)^2 P_{r_1}(I)\ldots P_{r_{q-1}}(I)c_1{}^{r_1}\ldots c_{q-1}{}^{r_{q-1}} \cdot \mathrm{E}(P_{r_q}(c_q\Gamma_q+\ldots+c_m\Gamma_m)|\Gamma_{q+1},\ldots,\Gamma_m).$$

Using Lemma 1 with $B = c_{q+1}\Gamma_{q+1}+\ldots+c_m\Gamma_m$ and $\Gamma = \Gamma_q$,

$$\text{(9)}\qquad \mathrm{E}(P_{r_q}(c_q\Gamma_q+\ldots+c_m\Gamma_m)|\Gamma_{q+1},\ldots,\Gamma_m) = \sum_{k=0}^{r_q} \binom{r_q}{k}^2 c_q{}^k P_k(I)P_{r_q-k}(c_{q+1}\Gamma_{q+1}+\ldots+c_m\Gamma_m).$$

Substituting (9) in (8) completes the induction proof.

$P_r(I) = \binom{n}{r}^{-1}$ can be calculated by a combinatorial argument or by comparing the expansion obtained from (3);

$$\operatorname{per}(zI+J) = \sum_{r=0}^{n} \binom{n}{r}^2 z^r P_r(I)(n-r)!,$$

with the known expansion [**2**]

$$\operatorname{per}(zI+J) = n!\sum_{r=0}^{n} z^r/r!,$$

where $J = (1)$.

Placing $P_r(I) = \binom{n}{r}^{-1}$ in (7) completes the proof.

https://doi.org/10.4153/CJM-1974-057-4 Published online by Cambridge University Press

COROLLARY 1.

$$(r!/n^r)(1 + (\sum c_i^2)\tfrac{1}{2}r(r-1)/n) < \mathrm{E}P_r(\Omega) < (r!/n^r) \times (1 + (\sum c_i^2)e^{\frac{1}{2}r(r-1)}\tfrac{1}{2}r(r-1)).$$

Proof.

$$n_{(t)}^{-1} = n^{-t}\prod_{q=1}^{t-1}(1-q/n)^{-1}$$
$$> n^{-t}\left(1 + \sum_{q=1}^{t-1} q/n\right)$$
$$= n^{-t}(1 + \tfrac{1}{2}t(t-1)/n), \text{ so}$$

(10) $$n_{(r_1)}^{-1}\dots n_{(r_m)}^{-1} > n^{-r}(1 + \sum \tfrac{1}{2}r_i(r_i-1)/n).$$

$$n_{(t)}^{-1} < n^{-t}\exp\left(\sum_{q=1}^{t-1}(1-q/n)^{-1}q/n\right)$$
$$< n^{-t}\exp\left(\sum_{q=1}^{t-1}(1-(n-1)/n)^{-1}q/n\right)$$
$$= n^{-t}\exp(\tfrac{1}{2}t(t-1)), \text{ so}$$

(11) $$n_{(r_1)}^{-1}\dots n_{(r_m)}^{-1} < n^{-r}\exp(\tfrac{1}{2}\sum r_i(r_i-1)) < n^{-r}(1 + \tfrac{1}{2}\sum r_i(r_i-1)\exp(\tfrac{1}{2}r(r-1))).$$

Using the inequalities (10) and (11) in (5) and noting that

$$\sum_{r_1+r_2+\dots+r_m=r}(r!/r_1!\dots r_m!)\sum r_i(r_i-1)c_1^{r_1}\dots c_m^{r_m} = r(r-1)\sum c_i^2,$$

completes the proof.

Corollary 1 compares $\mathrm{E}P_r(\Omega)$ with $P_r(J_n)$; $\sum c_i^2$ is a measure of the variation of Ω from J_n. If $||A|| = (\sum a_{ij}^2)^{1/2}$ for any matrix A; then

$$\text{var. } ||\Omega - J_n||^2 = \sum_{ij}\text{var. }\omega_{ij}$$
$$= \sum_{ij}\sum_q c_q^2\text{ var. }\gamma_{ij}$$
$$= K\sum_q c_q^2,$$

where K is a positive constant.

COROLLARY 2. *If* $\Omega_1, \Omega_2, \dots$ *is a sequence of random doubly stochastic matrices such that* var. $||\Omega_i - J_n||^2 \to 0$ *as* $i \to \infty$, *then* $\mathrm{E}P_r(\Omega_i) \to r!/n^r$ *as* $i \to \infty$.

Actually Corollary 2 is a very weak result, if var. $||\Omega_i - J_n||^2 \to 0$ as $i \to \infty$; then the sequence $\Omega_1, \Omega_2, \dots$ converges in probability to J_n and $P_r(\Omega_i)$ converges in probability to $r!/n^r$.

COROLLARY 3. *If $W_1, W_2, \ldots, W_m$ are mutually independent, identically distributed random variables with a common probability density function*

$$(n!)^{-1}w^n e^{-w}, \quad w > 0,$$

and U is a random variable, independent of $W_1, W_2, \ldots, W_m$, with a probability density function

$$e^{-u}, \quad u > 0,$$

and

$$V = \sum c_i W_i^{-1},$$

then

$$\mathrm{E}P_r(\Omega) = \mathrm{E}(UV)^r, \quad r = 0, 1, \ldots, n.$$

Proof.

$$\begin{aligned}\mathrm{E}(UV)^r \\ &= \mathrm{E}U \sum_{r_1+r_2+\ldots+r_m=r} (r!/r_1!\ldots r_m!)c_1^{r_1}\ldots c_m^{r_m}\mathrm{E}W_1^{-r_1}\ldots \mathrm{E}W_m^{-r_m} \\ &= r! \sum_{r_1+r_2+\ldots+r_m=r} (r!/r_1!\ldots r_m!)c_1^{r_1}\ldots c_m^{r_m} n_{(r_1)}^{-1}\ldots n_{(r_m)}^{-1} \\ &= \mathrm{E}P_r(\Omega).\end{aligned}$$

COROLLARY 4. *$\{(n^r/r!)\mathrm{E}P_r(\Omega)\}^{1/r}$ is a strictly increasing function of $r = 1, \ldots, n$.*

Proof. Using Holder's inequality,

$$\begin{aligned}(n^r/r!)\mathrm{E}P_r(\Omega) &= \mathrm{E}V^r 1 \\ &< \{\mathrm{E}V^{r(r+1)/r}\}^{r/(r+1)}\{\mathrm{E}1\}^{1/(r+1)} \\ &= \{(n^{r+1}/(r+1)!)\mathrm{E}P_{r+1}(\Omega)\}^{r/(r+1)},\end{aligned}$$

where V is defined in Corollary 2.

COROLLARY 5. *$\mathrm{E}P_r(\Omega(\bar{c}))$ is a strictly convex function of $\bar{c}$, that is, if $0 < \lambda < 1$ and $\bar{c}_1 \neq \bar{c}_2$, then*

$$\mathrm{E}P_r(\Omega(\lambda\bar{c}_1 + (1-\lambda)\bar{c}_2)) < \lambda\mathrm{E}P_r(\Omega(\bar{c}_1)) + (1-\lambda)\mathrm{E}P_r(\Omega(\bar{c}_2)),$$

$r = 2, 3, \ldots, n.$

Proof. $\mathrm{E}(\sum c_i W_i^{-1})^r$ is a strictly convex function of $\bar{c}$, where $W_1, W_2, \ldots, W_m$ are defined in Corollary 3.

COROLLARY 6. *If* Ω_1 *and* Ω_2 *are independent and* $0 < \lambda < 1$, *then*

$$\mathrm{E}P_r(\lambda\Omega_1 + (1-\lambda)\Omega_2) < \lambda \mathrm{E}P_r(\Omega_1) + (1-\lambda)\mathrm{E}P_r(\Omega_2),$$

$r = 2, 3, \ldots, n$.

Proof. Represent

$$\Omega_1 = a_1\Gamma_1 + \ldots + a_p\Gamma_p, \; \Omega_2 = b_1\Gamma_{p+1} + \ldots + b_q\Gamma_{p+q},$$

where $\Gamma_1, \Gamma_2, \ldots, \Gamma_{p+q}$ are mutually independent. Corollary 6 is a particular case of Corollary 5 where $m = p + q$,

$$\bar{c}_1 = (a_1, a_2, \ldots, a_p, 0, \ldots, 0) \text{ and } \bar{c}_2 = (0, \ldots, 0, b_1, b_2, \ldots, b_q).$$

COROLLARY 7. *If* $\bar{\Omega} = m^{-1}\sum_1^m \Gamma_i$, *then*

$$\mathrm{E}P_r(\bar{\Omega}) \leqq \mathrm{E}P_r(\Omega), \quad r = 2, 3, \ldots, n,$$

with equality if and only if $\Omega = \bar{\Omega}$.

Proof. Denote by $\bar{c}_1, \ldots, \bar{c}_{m!}$ vectors formed from the different permutations of elements from $\bar{c}$, and $\bar{e} = (m^{-1}, \ldots, m^{-1})$.

$$\bar{e} = (m!)^{-1}\sum \bar{c}_i,$$

so from Corollary 5,

$$\mathrm{E}P_r(\bar{\Omega}) \leqslant (m!)^{-1}\sum \mathrm{E}P_r(\Omega(\bar{c}_i))$$
$$= \mathrm{E}P_r(\Omega).$$

The inequality is strict unless $\bar{c}_i = e$ for $i = 1, 2, \ldots, m!$, in which case $\Omega = \bar{\Omega}$.

COROLLARY 8. $\mathrm{E}P_r(\bar{\Omega})$ *is a strictly decreasing function of* m, $r = 2, 3, \ldots, n$.

Proof. Denote by $\bar{f}_j$ the vector with kth element $(1 - \delta_{jk})/(m-1)$, where δ_{jk} is the Kronecker delta. Since $\bar{e} = m^{-1}\sum \bar{f}_j$, Corollary 8 follows from Corollary 5.

3. A limit theorem. The multivariate central limit theorem gives that as $m \to \infty$, $m^{1/2}(\bar{\Omega} - J_n)$ converges in probability law to a matrix of normal random variables; this is used to prove a limit theorem for $\{m(P_r(\bar{\Omega}) - r!/n^r); r = 2, 3, \ldots, n\}$. $\xrightarrow{\mathscr{L}}$ will denote convergence in probability law.

THEOREM 2.

$$\{m(P_r(\bar{\Omega}) - r!/n^r)\,;\, r = 2, 3, \ldots, n\}$$
$$\xrightarrow{\mathscr{L}} \left\{\tfrac{1}{2}(n-1)^{-1}n^{-(r-2)}(r-2)!\binom{r}{2}^2\binom{n}{2}^{-2}X; r = 2, 3, \ldots, n\right\},$$

where X *has a chi-squared distribution with* $(n-1)^2$ *degrees of freedom.*

Proof. Using the expansion (3),

$$\begin{aligned} P_r(\bar{\Omega}) &= P_r(\bar{\Omega} - J_n + J_n) \\ &= r!/n^r + \binom{r}{2}^2 (r-2)!n^{-(r-2)} P_2(\bar{\Omega} - J_n) \\ &\qquad + \sum_{k=3}^{r} \binom{r}{k}^2 P_k(\bar{\Omega} - J_n) P_{r-k}(J_n). \end{aligned}$$

By the multivariate central limit theorem

$$m^{\frac{1}{2}}(\bar{\Omega} - J_n) \xrightarrow{\mathscr{L}} \Lambda,$$

where Λ is a matrix of normal random variables. Since

$$mP_k(\bar{\Omega} - J_n) \xrightarrow{\mathscr{L}} 0, \qquad k > 2,$$

it suffices to show

$$mP_2(\bar{\Omega} - J_n) \xrightarrow{\mathscr{L}} \tfrac{1}{2}(n-1)\binom{n}{2}^{-2} X.$$

$P_2(\bar{\Omega} - J_n) = \frac{1}{2}\binom{n}{2}^{-2} ||\bar{\Omega} - J_n||^2$ (the calculation is omitted), so

$$mP_2(\bar{\Omega} - J_n) \xrightarrow{\mathscr{L}} \frac{1}{2}\binom{n}{2}^{-2} ||\Lambda||^2.$$

To calculate the distribution of $||\Lambda||^2$ the covariance matrix of Λ needs to be found. The product of two different elements from Q_i is zero if they are in the same row or column, or 1 for $(n-2)!$ values of i otherwise; which gives

$$\begin{aligned} E\gamma_{ij}\gamma_{rs} &= (1 - \delta_{ir})(1 - \delta_{js})(n-2)!/n! + \delta_{ir}\delta_{js}/n. \\ \text{covariance } (\lambda_{ij}, \lambda_{rs}) &= \text{covariance } (\gamma_{ij}, \gamma_{rs}) \\ &= (n-1)^{-1}(\delta_{ir} - 1/n)(\delta_{js} - 1/n). \end{aligned}$$

A representation of Λ is given by

$$(12) \quad \lambda_{rs} = (n-1)^{-\frac{1}{2}} \sum_{p=2}^{n} \sum_{q=2}^{n} h_{rp} h_{sq} \phi_{(p-1)(q-1)},$$

where Φ is an $(n-1) \times (n-1)$ matrix of normal random variables with zero means and an identity covariance matrix, and H is an $n \times n$ orthogonal matrix with $h_{i1} = n^{-1/2}$. To prove (12) only the covariance matrix needs to be checked. $||\Lambda||^2 = (n-1)^{-1}||\Phi||^2$, which is distributed as $(n-1)^{-1}X$.

References

1. M. Marcus, *Inequalities for matrix functions of combinatorial interest*, SIAM J. Appl. Math. *17* (1969), 1023–1031.
2. M. Marcus and H. Minc, *Permanents*, Amer. Math. Monthly *72* (1965), 577–591.
3. H. R. Ryser, *Combinatorial mathematics*, No. 14 of the Carus Mathematical Monographs, the Mathematical Association of America, 1963.

Macquarie University,
N.S.W., Australia;
Monash University,
Clayton, 3168, Australia