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FINE SPECTRA AND LIMIT LAWS, II FIRST-ORDER 0–1 LAWS

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ABSTRACT. Using Feferman-Vaught techniques a condition on the fine spectrum of an admissible class of structures is found which leads to a first-order 0–1 law. The condition presented is best possible in the sense that if it is violated then one can find an admissible class with the same fine spectrum which does not have a first-order 0–1 law.

If the condition is satisfied (and hence we have a first-order 0–1 law) we give a natural model of the limit law theory; and show that the limit law theory is decidable if the theory of the directly indecomposables is decidable. Using asymptotic methods from the partition calculus a useful test is derived to show several admissible classes have a first-order 0–1 law.

1. **Front-loaded classes.**

We will continue using the notation of Part I, the first paper [1] of this sequel. First we study, in an abstract setting, the key property of fine spectra which suffices to prove 0–1 laws exist. In this section a subscripted lower case letter is used for members of a series, *e.g.*, (a_n), and the corresponding upper case letter for the partial sum function, *e.g.*, $A(x) = \sum_{n \le x} a_n$.

LEMMA 1.1. *For (a_n) a sequence of non-negative integers the following are equivalent:*

(a) $\lim_{t\to\infty} \frac{A(tx)}{A(t)} = 1$ *for all [some]* $x > 1$.

(b) $\lim_{n\to\infty} \frac{A(nx)}{A(n)} = 1$ *for all [some]* $x > 1$.

(c) $\lim_{n\to\infty} \frac{A(x^{n+1})}{A(x^n)} = 1$ *for all [some]* $x > 1$.

We also obtain further equivalent statements by replacing tx by t/x in (a)*, and nx by n/x in* (b)*.*

PROOF. Regarding the '*for all x*' versions one has (a) $\Rightarrow$ (b), (c). Likewise for the '*for some x*' versions. Also, in each case the '*for all x*' version implies the '*for some x*' version. Thus for the equivalences (a)–(c) it suffices to show that the '*for some x*' versions of (b), (c) each imply the '*for all x*' version of (a).

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First suppose the '*for some*' version of (b) holds. Choose $u > 1$ such that

$$\lim_{n\to\infty} \frac{A(nu)}{A(n)} = 1.$$

For n sufficiently large we have $un > n + 1$, and consequently

$$1 \le \frac{A(n+1)}{A(n)} \le \frac{A(nu)}{A(n)}.$$

Thus

$$\lim_{n\to\infty} \frac{A(n+1)}{A(n)} = 1.$$

Then

$$1 \le \frac{A(tu)}{A(t)} \le \frac{A\big((\lfloor t \rfloor + 1)u\big)}{A(\lfloor t \rfloor)} = \frac{A\big((\lfloor t \rfloor + 1)u\big)}{A(\lfloor t \rfloor + 1)} \cdot \frac{A(\lfloor t \rfloor + 1)}{A(\lfloor t \rfloor)}.$$

So

$$\lim_{t\to\infty} \frac{A(tu)}{A(t)} = 1.$$

Then for any positive integer s we have

$$\lim_{t\to\infty} \frac{A(tu^s)}{A(t)} = 1.$$

Given any $x > 1$ choose a positive integer s such that $1 < x < u^s$. Then

$$1 \le \frac{A(tx)}{A(t)} \le \frac{A(tu^s)}{A(t)}$$

implies

$$\lim_{t\to\infty} \frac{A(tx)}{A(t)} = 1.$$

Next suppose the '*for some*' version of (c) holds. Choose $u > 1$ such that

$$\lim_{n\to\infty} \frac{A(u^{n+1})}{A(u^n)} = 1.$$

Then, for $u^n \le t \le u^{n+1}$, we have $u^{n+1} \le tu \le u^{n+2}$, and then

$$\frac{A(u^{n+2})}{A(u^n)} \ge \frac{A(tu)}{A(t)} \ge 1,$$

so

$$\lim_{t\to\infty} \frac{A(tu)}{A(t)} = 1.$$

Now, as in the previous case, we have, for any $x > 1$,

$$\lim_{t\to\infty} \frac{A(tx)}{A(t)} = 1.$$

To see that one can replace nx by n/x in (b) it suffices to note the following:

$$\frac{A(2\lfloor nx\rfloor)}{A\left(\frac{1}{2x}(2\lfloor nx\rfloor)\right)} \geq \frac{A(nx)}{A(n)} \geq 1,$$

and for $n > x$

$$\frac{A(\lfloor n/x\rfloor)}{A(2x\lfloor n/x\rfloor)} \leq \frac{A(n/x)}{A(n)} \leq 1.$$

The same argument shows that one can replace tx by t/x in (a). ■

DEFINITION 1.2. A sequence of non-negative integers (a_n) is said to be *front-loaded* if $A(x)$ is slowly varying, *i.e.*, for all $x > 0$,

$$\lim_{t\longrightarrow\infty} \frac{A(tx)}{A(t)} = 1.$$

A class K of finite structures is *front-loaded* if its fine spectrum is front-loaded.

THEOREM 1.3. *The Dirichlet convolution product of finitely many front-loaded sequences is front-loaded.*

PROOF. It suffices to consider two front-loaded sequences, say (a_n) and (b_n). We want to show that the sequence (c_n) defined by $c_n = \sum_{m|n} a_m b_{n/m}$ is front-loaded. Now

$$C(x) = \sum_{k\leq x} a_k \cdot B(x/k).$$

We have to prove, for $x > 1$ and $\delta > 0$, that there is a $t_0(x,\delta)$ such that

$$C(tx) \leq (1+\delta)\cdot C(t) \text{ for } t > t_0(x,\delta).$$

Since the b-sequence is front-loaded,

$$B(tx) \leq (1+\delta/2)\cdot B(t) \text{ for } t > t_1(x,\delta),$$

and we assume $t_1 > x$. Then

$$\begin{aligned} C(tx) &= \sum_{k\leq tx} a_k \cdot B(tx/k) \\ &\leq \Big(\sum_{k\leq t/t_1} a_k \cdot B(tx/k)\Big) + B(t_1 x)\cdot \big(A(tx) - A(t/t_1)\big) \\ &\leq (1+\delta/2)\cdot \Big(\sum_{k\leq t}(a_k \cdot B(t/k)\Big) + B(t_1 x)\cdot \big(A(tx) - A(t/t_1)\big) \\ &= (1+\delta/2)\cdot C(t) + o\big(A(t)\big) \end{aligned}$$

since the a-sequence is front-loaded, which completes the proof. ■

Using this result one can slightly simplify the proof of Theorem 5.15 in Part I, namely it suffices to prove the theorem for the case that the bound U on the multiplicities is 1. The next item is closely related to Corollary 4.4 of Part I.

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LEMMA 1.4. *Let* K *be an admissible class. Then the following are equivalent:*
(a) K *is front-loaded.*
(b) $\mathrm{Prob}_{\mathsf{K}}($ *is divisible by* $\mathbf{A}) = 1$ *for all* $\mathbf{A} \in \mathsf{K}$.
(c) $\mathrm{Prob}_{\mathsf{K}}($ *is divisible by* $\mathbf{A}) = 1$ *for some nontrivial* $\mathbf{A} \in \mathsf{K}$.

PROOF. Observe that

$$\begin{aligned}\mathrm{Prob}_{\mathsf{K}}(\text{is divisible by }\mathbf{A}) &= \lim_{n\to\infty} \frac{\tau_{\mathsf{K}}(n \mid \textit{is divisible by } \mathbf{A})}{\tau_{\mathsf{K}}(n)} \\ &= \lim_{n\to\infty} \frac{\tau_{\mathsf{K}}(n/d)}{\tau_{\mathsf{K}}(n)}\end{aligned}$$

where d is the size of $\mathbf{A}$. Then apply Lemma 1.1. ■

LEMMA 1.5. *An admissible front-loaded class* K *is loaded.*

PROOF. Let $\mathsf{F}_1, \ldots, \mathsf{F}_k$ be a partition of F, and let $r_1, \ldots, r_k$ be a sequence of non-negative integers. Choose any algebra $\mathbf{A}$ with at least r_i factors from each F_i. Then

$$\frac{\tau_{\mathsf{K}}(n \mid \text{is divisible by }\mathbf{A})}{\tau_{\mathsf{K}(n)}} \leq \frac{\tau_{\mathsf{K}}(n \mid \text{is in } \mathsf{F}_1^{\geq r_1} \cdots \mathsf{F}_k^{\geq r_k})}{\tau_{\mathsf{K}(n)}} \leq 1.$$

Thus, by Lemma 1.4, $\mathrm{Prob}_{\mathsf{K}}($ is in $\mathsf{F}_1^{\geq r_1} \cdots \mathsf{F}_k^{\geq r_k}) = 1$, so K is loaded. ■

2. **Logical Aspects.**

THEOREM 2.1. *Suppose that* K *is admissible. If* K *is front-loaded then we have the following:*
(a) K *has a first-order 0–1 law.*
(b) Let R *be a selection of representatives from the isomorphism equivalence classes of* F, *and let* $\mathbf{T} = (\prod \mathsf{R})^{\omega}$. *Then, for* ϕ *a first-order sentence,* $\mathrm{Prob}_{\mathsf{K}}(\phi) = 1$ iff $\mathbf{T} \models \phi$.
(c) If the first-order theory of F *is decidable then so is the limit law theory of* K, *i.e, the set of first-order* ϕ *with* $\mathrm{Prob}_{\mathsf{K}}(\phi) = 1$.
If, on the other hand, K *is not front-loaded, then there is an admissible* K' *with the same fine spectrum as* K, *and* K' *does not have a first-order 0–1 law.*

PROOF. (a) Examining the proof of part (a) of Theorem 3.4 in Part I we see in the front-loaded case that $p_{j_0,\ldots,j_{\ell-1}} = 0$ if any $j_i < c$. Thus at most one nonzero term survives in the formula for the cumulative probability of ϕ, namely $p_{c,\ldots,c}$, and this term has the value 1.

(b) Given a first-order sentence ϕ let Feferman-Vaught sequences be determined as in the proof of part (a) of Theorem 3.4 in Part I, and also the F_i. By regrouping the factors of $\mathbf{T}$ by 'members of the same F_i', we have

$$\mathbf{T} \cong \mathbf{T}_0 \times \cdots \times \mathbf{T}_{\ell-1},$$

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where $\mathbf{T}_i = \left(\prod(\mathsf{R}\cap\mathsf{F}_i)\right)^{\omega}$. $\mathbf{T}$ will satisfy ϕ iff the structures from K with at least c factors from each F_i satisfy ϕ (by Lemma 3.1 in Part I), and the latter holds iff ϕ is in the limit law theory.

(c) Suppose Th(F), the first order theory of F, is decidable. Given a first-order sentence ϕ we now show how to effectively determine if $\mathbf{T} \models \phi$, *i.e.*, how to determine if ϕ is in the limit law theory. First we use [3] to effectively find the Feferman-Vaught sequences $\langle \Phi, \phi_1, \ldots, \phi_k \rangle$, $\langle \Phi_i, \phi_{i,1}, \ldots, \phi_{i,k_i} \rangle$ $(1 \leq i \leq k)$ in the proof of part (a). Now we define a *constituent* of ϕ to be any conjunction γ of the $\phi_{i,j}$'s and their negations such that for each (i,j) precisely one of $\phi_{i,j}$ and $\neg\phi_{i,j}$ appears in the conjunction.

Suppose γ is such a constituent. Then either γ has no model in F or γ defines one of the classes F_i, *i.e.*, $\mathsf{F}_i = \{\mathbf{D} \in \mathsf{F} : \mathbf{D} \models \gamma\}$. Note that, up to ordering of the conjuncts, each F_i is determined by a unique constituent, say by γ_i.

Thus we can determine the ℓ in the proof of part (a) by determining the constituents which have models in F. And we can do this by using the decidability of Th(F), namely a constituent γ has a model in F iff $\neg\gamma \notin$ Th(F).

Now that we have ℓ, we want to determine the $[\![\phi_i]\!]$ in $\mathbf{2}^{\ell}$. This is because $\mathbf{T} \models \phi$ iff $\mathbf{2}^{\ell} \models \Phi([\![\phi_1]\!], \ldots, [\![\phi_k]\!])$. To determine $[\![\phi_i]\!]$ we will find the set S_i of j such that $\mathbf{T}_j \models \phi_i$. $[\![\phi_i]\!]$ is just the characteristic function of S_i (in the set $\ell = \{0, \ldots, \ell-1\}$). So we look at the Feferman-Vaught sequence for ϕ_i, namely $\langle \Phi_i, \phi_{i,1}, \ldots, \phi_{i,k_i} \rangle$. As $\mathbf{T}_j$ is a countably infinite product of members of F_i, say $\mathbf{T}_j = \prod_{n<\omega} \mathbf{D}_n$, we have

$$\prod_{n<\omega} \mathbf{D}_n \models \phi_i \quad \text{iff} \quad \mathbf{2}^{\omega} \models \Phi([\![\phi_{i,1}]\!], \ldots, [\![\phi_{i,k_i}]\!]).$$

As the $\mathbf{D}_n \models \gamma_j$, and each $\phi_{i,r}$ or its negation appears as a conjunct of γ_j, we know that

$$[\![\phi_{i,r}]\!] = 1 \quad \text{if } \phi_{i,r} \text{ appears in } \gamma_j$$
$$[\![\phi_{i,r}]\!] = 0 \quad \text{if } \neg\phi_{i,r} \text{ appears in } \gamma_j.$$

Thus we can effectively find the $[\![\phi_{i,r}]\!]$'s. Having determined $\Phi_i([\![\phi_{i,1}]\!], \ldots, ([\![\phi_{i,k_i}]\!])$, a sentence in the language of Boolean algebras, we use Skolem's result that Th($\mathbf{2}^{\omega}$) is decidable to determine if $\Phi_i([\![\phi_{i,1}]\!], \ldots, [\![\phi_{i,k_i}]\!]) \in$ Th($\mathbf{2}^{\omega}$), and thus if $\mathbf{T}_j \models \phi_i$.

Now we have all the information needed to determine the S_i's, and hence the $[\![\phi_i]\!]$'s, so we can effectively find $\Phi([\![\phi_1]\!], \ldots, [\![\phi_k]\!])$. Finally we determine if $\mathbf{2}^{\ell} \models \Phi([\![\phi_1]\!], \ldots, [\![\phi_k]\!])$; this is clearly decidable as $\mathbf{2}^{\ell}$ is a finite algebra. This finishes the proof of (c).

Now let us suppose that K is not front-loaded. Let F be the class of K-indecomposables. Let F^t be the expansion of F by the ternary discriminator t, as in Part I. Let F' be an expansion of F^t by two constants a, b, *i.e.*, for each member $\mathbf{D}$ of F^t we create one structure $\mathbf{D}'$ by interpreting the constant symbols a, b.

CASE 1. $\mathrm{Prob}_{\mathsf{K}'}(\phi_{\mathrm{ind}})$ does not exist. In this case K' does not have a first-order law.

CASE 2. $\mathrm{Prob}_{\mathsf{K}'}(\phi_{\mathrm{ind}}) = c > 0$. In this case we have an infinite number of indecomposables. Choose positive integers $n_1 < n_2 < \cdots$ such that

$$\tau_{\mathsf{F}'}(n_k) < \frac{1}{6}\tau_{\mathsf{F}'}(n_{k+1}).$$

and

$$\left|\frac{\tau_{\mathsf{F}'}(n_k)}{\tau_{\mathsf{K}'}(n_k)} - c\right| < \frac{c}{5}.$$

We now assume the interpretation of the constants a, b in each member $\mathbf{D}$ of F^t is as follows: if the size of $\mathbf{D}$ is in (n_{k-1}, n_k) with k even, put $a = b$; otherwise put $a \neq b$. Then

$$\frac{\tau_{\mathsf{K}'}(n_{2k} \mid a = b \wedge \phi_{\text{ind}})}{\tau_{\mathsf{K}'}(n_{2k})} > \frac{2}{3}c$$

and

$$\frac{\tau_{\mathsf{K}'}(n_{2k+1} \mid a = b \wedge \phi_{\text{ind}})}{\tau_{\mathsf{K}'}(n_{2k+1})} < \frac{1}{3}c.$$

Thus $\text{Prob}_{\mathsf{K}'}(a = b \wedge \phi_{\text{ind}})$ does not exist, so K' does not have a first-order law.

CASE 3. $\text{Prob}_{\mathsf{K}'}(\phi_{\text{ind}}) = 0$. Without loss of generality regarding the fine spectrum being considered we can assume that

($\star$) for every relation symbol r of the language there is a corresponding function symbol f_r such that for each nontrivial $\mathbf{A} \in \mathsf{K}'$ we have $r(a_1, \ldots, a_n)$ holds iff $f_r(a_1, \ldots, a_n) = a_1$ holds, where $a_i \in \mathbf{A}$.

Given a member $\mathbf{A}$ of K' one can use the ternary discriminator to find a first-order sentence $\phi_{\mathbf{A}}$ which, for members of K', says "$\mathbf{A}$ is a factor".

If for some $\mathbf{A} \in \mathsf{K}'$ the cumulative probability $\text{Prob}_{\mathsf{K}'}(\phi_{\mathbf{A}})$ is not defined then K' does not have a first-order law, and we are finished. So we assume that $\text{Prob}_{\mathsf{K}'}(\phi_{\mathbf{A}})$ exists for all $\mathbf{A} \in \mathsf{K}'$.

CASE 3A. $\text{Prob}_{\mathsf{K}'}(\phi_{\mathbf{A}}) = 0$ for every nontrivial $\mathbf{A} \in \mathsf{K}'$. The number of structures, up to isomorphism, in F' must be infinite; for otherwise we could use Theorem 1.3 to show K is front-loaded.

For k a positive integer let $\phi_{<k}$ be a first-order sentence which, for members of K', says "there is a non-trivial factor of size less than k". From our assumptions follows $\text{Prob}_{\mathsf{K}}(\phi_{<k}) = 0$. Choose positive integers $n_1 < n_2 < \cdots$ such that

$$\tau_{\mathsf{K}'}(n_{k+1} \mid \phi_{<n_k}) < \frac{1}{3}\tau_{\mathsf{K}'}(n_{k+1}).$$

We again assume the interpretation of the constants a, b in each member $\mathbf{D}$ of F^t is as follows: if the size of $\mathbf{D}$ is in $(n_{k-1}, n_k]$ with k even, put $a = b$; otherwise put $a \neq b$. Let $\phi_{a,b}$ be a sentence expressing '*has a nontrivial factor in which* $a = b$'. Then

$$\frac{\tau_{\mathsf{K}'}(n_{2k} \mid \phi_{a,b})}{\tau_{\mathsf{K}'}(n_{2k})} > \frac{2}{3}$$

and

$$\frac{\tau_{\mathsf{K}'}(n_{2k+1} \mid \phi_{a,b})}{\tau_{\mathsf{K}'}(n_{2k+1})} < \frac{1}{3}.$$

Thus $\text{Prob}_{\mathsf{K}'}(\phi_{a,b})$ does not exist, and again K' does not have a first-order law.

CASE 3B. $\text{Prob}_{\mathsf{K}'}(\phi_{\mathbf{A}}) > 0$ for some nontrivial $\mathbf{A} \in \mathsf{K}'$. $\text{Prob}_{\mathsf{K}'}(\phi_{\mathbf{A}}) < 1$ for every nontrivial $\mathbf{A} \in \mathsf{K}'$ by Lemma 1.4 as K' is not front-loaded. But then K' does not have a 0–1 law. ∎

Thus we see that, among the admissible classes K, those for which knowledge of the fine spectrum alone is sufficient to conclude a first-order 0–1 law are precisely those which are front-loaded. An example of an admissible K where more information is needed is the class of finite sets. We already mentioned that it is loaded, and thus has a first-order law; however it is well-known that it has a first-order 0–1 law. This K is clearly not front-loaded, so more information than that given by the fine spectrum is required to deduce the 0–1 law.

PROPOSITION 2.2. *Suppose* K_i *is admissible and front-loaded, for* $1 \le i \le m$. *Let* F_i *be the* K_i*-indecomposables. Suppose the* F_i *are pairwise disjoint. Let* $\mathsf{K} = \mathsf{K}_1 \cdots \mathsf{K}_m$. *If* K *has unique factorization then* K *has a first-order 0–1 law.*

PROOF. The hypotheses ensure that K is admissible, and that the Dirichlet convolution product of the fine spectra $\sigma_{\mathsf{K}_1}, \ldots, \sigma_{\mathsf{K}_m}$ is the fine spectrum $\sigma\mathsf{K}$. Now apply Theorems 1.3 and 2.1. ∎

REMARK 2.3. We can apply the above to show

$$\mathsf{K}^\star = \bigcup_{S \subseteq \{1,\ldots,m\}} \prod_{i \in S} \mathsf{K}_i$$

has a 0–1 law if it has unique factorization by observing that

- adding/deleting one-element structures that act as multiplicative units with respect to direct products from a class K does not affect either the admissibility of K or the fact that K is front-loaded.

COROLLARY 2.4. *Suppose* K *is admissible, and that the set* F *of* K*-indecomposables is the disjoint union of* $\mathsf{F}_1, \ldots, \mathsf{F}_m$*, where each* F_i *is closed under isomorphism. Let* $\mathsf{K}_i = IP_{\text{fin}}(\mathsf{F}_i)$*. If each* K_i *is front-loaded then* K *has a first-order 0–1 law.*

PROOF. Each K_i is admissible, and $\mathsf{K} = \mathsf{K}^\star$ where $\mathsf{K}^\star$ is as in Remark 2.3. Thus by Proposition 2.2 and Remark 2.3 we arrive at the desired conclusion. ∎

3. **Asymptotics.** Let K be admissible, and let F be the class of K-indecomposables. To estimate $\tau(n \mid P)$ and $\tau(n)$ we shall consider Dirichlet generating functions. Chapter XVII of [4] contains an excellent introduction for our purposes to Dirichlet generating functions. Perhaps noting that K and F correspond to the integers and primes respectively and that

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1}, \ p \text{ a prime},$$

will motivate what follows. If m runs through the integers which are not divisible by the prime q then

$$\sum_{m=1}^{\infty} m^{-s} = \prod_{p \ne q} (1 - p^{-s})^{-1}, \ p \text{ a prime}.$$

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Now suppose we are given a fixed positive integer M. Let $b_n = \sigma_{\mathsf{K}}(n)$. Let $\mathbf{D}_1, \mathbf{D}_2, \ldots$ be a listing, up to isomorphism, of the members of F, and let β_n be the size of $\mathbf{D}_n$. Let a_n denote the number of structures of size n in K which have no copies of $\mathbf{D}_M$ in their F-factorization. Then it is not difficult to see that

$$\sum b_n n^{-s} = \prod_{m=1}^{\infty} (1 - \beta_m^{-s})^{-1},$$

and

$$\sum a_n n^{-s} = \prod_{\substack{m=1 \\ m \neq M}}^{\infty} (1 - \beta_m^{-s})^{-1}.$$

Furthermore

$$\mathrm{Prob}_{\mathsf{K}}(\text{is not divisible by } \mathbf{D}_M) = \lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n},$$

provided this limit exists.

THEOREM 3.1. *Let* (β_m), $0 < \beta_1 < \beta_2 < \cdots$, *be a sequence of real numbers and*

$$\sum b_n n^{-s} = \prod_{m=1}^{\infty} (1 - \beta_m^{-s})^{-1}$$

$$\sum a_n n^{-s} = \beta_M^{-s} \prod_{\substack{m=1 \\ m \neq M}}^{\infty} (1 - \beta_m^{-s})^{-1}$$

where M *is a positive integer. If*

$$\log \beta_m \sim cm, \quad c > 0 \ \text{a constant,}$$

then

$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} = O\big((\log n)^{-\frac{1}{2}}\big).$$

PROOF. We will use Theorem 2.2 of [5] to derive our result. We begin with some notation and definitions used in [5]. Let $\Lambda = (\lambda_m)$, $0 < \lambda_1 < \lambda_2 < \cdots$, be an infinite sequence of real numbers without a finite limit point. Let $N(u)$ be defined by

$$N(u) = \sum_{\lambda_m \leq u} 1$$

and suppose that for each $\epsilon > 0$ there exists a constant $C = C(\epsilon)$ such that

$$N(u) \leq C(\epsilon) \exp(\epsilon u).$$

Then the infinite product

$$g(s) = \prod_{m=1}^{\infty} \big(1 - \exp(-\lambda_m s)\big)^{-1}$$

converges for all complex s with Re $s > 0$. Let ℓ_m run through the monotone increasing sequence of linear combinations of the λ_m with non-negative integral coefficients; then

$$g(s) = \sum_m p(\ell_m)e^{-\ell_m s},$$

where $p(\ell_m)$ is the number of partitions of ℓ_m into summands from $\{\lambda_m\}$. Let

$$P(u) = \sum_{\ell<u} p(\ell).$$

REMARK 3.2. If $\lambda_m = \log \beta_m$ then

$$\sum_{m\leq n} b_m = \sum_{\ell<\log n} p(\ell) = P(\log n).$$

Now let $\alpha = \alpha(u)$ be determined (uniquely for large u as demonstrated in [5]) from

$$u = \sum_m \lambda_m (e^{\alpha\lambda_m} - 1)^{-1} - 2\alpha^{-1}$$

and define $B_2 = B_2(u)$ by

$$B_2 = \sum_m \frac{\lambda_m^2 e^{\alpha\lambda_m}}{(e^{\alpha\lambda_m} - 1)^2} - 4\alpha^{-2}.$$

Of course u is defined by a very complicated equation; however Roth and Szekeres [6] show that if $\lambda_m \sim cm$ then

$$\alpha \sim \frac{\pi}{\sqrt{6cu}}$$

$$B_2(\alpha) \sim \frac{\pi^2}{3c}\alpha^{-3} \sim \frac{2\sqrt{6c}}{\pi}u^{\frac{3}{2}}.$$

If $\lambda_m \sim cm$ then Λ has properties I and II of Theorem 2.2 of [5] (see conditions (ii) on page 375 of [5]). Finally, for any positive constants C_1, C_2 and δ ($\delta < \frac{1}{6}$) there is a λ_N such that

$$C_1\alpha^{-\frac{1}{3}} \leq \lambda_N \leq C_2\alpha^{-\frac{1}{3}-\delta}$$

for all sufficiently small α (or large u) since this is equivalent to there being a λ_N such that

$$C_3 u^{\frac{1}{6}} \leq \lambda_N \leq C_4 u^{\frac{1}{6}+\delta},$$

and this is true since $\lambda_N \sim cN$. Finally

$$\alpha^{\frac{8}{3}-\delta} B_2^{\frac{1}{2}} = O(\alpha^{\frac{8}{3}-\frac{3}{2}-\delta}) = O(\alpha^{\frac{7}{6}-\delta}) = o(1)$$

and

$$\alpha^{\frac{5}{3}-\delta} B_2^{\frac{1}{2}} = O(\alpha^{\frac{1}{6}-\delta}) = o(1)$$

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Hence all the hypotheses of Theorem 2.2, part 6, of [5] are satisfied (note that $\alpha^{\frac{1}{3}-\delta}B_2^{\frac{1}{2}} = o(1)$ should read $\alpha^{\frac{8}{3}-\delta}B_2^{\frac{1}{2}} = o(1)$; see Lemma 2.4), and

$$P(u) \sim (2\pi B_2)^{-\frac{1}{2}}\alpha^{-1}\exp\{\alpha u - \sum_{m=1}^{\infty}\log(1-e^{-\alpha\lambda_m})\}. \tag{1}$$

REMARK 3.3. We cannot express the asymptotic behaviour of the exp term in (1) in terms of elementary functions, but as Roth and Szekeres [6] showed, this is not necessary for the proof of Theorem 3.1.

Roth and Szekeres were interested in proving that certain partition functions are monotonic. They did this by working out the asymptotic behaviour of a partition function analogous to our $P(u+1)-P(u)$, noting that this corresponded to multiplying their generating function by $1-e^{-\alpha s}$. They showed that this alteration in the generating function alters α by so little that the asymptotic behaviour of their function can be obtained by adding the term $\log(1-e^{-\alpha})$ to the exp term in (1). Their arguments can be seen to apply here.

The generating function for the a_i is $1-\beta_M^{-s}$ times the generating function for the b_i (here we denote β_i by λ_i). Thus, in the notation just above, the coefficient of $\exp(-\ell_m s)$ when λ_M is not used is $p(\ell_m)-p(\ell_m-\lambda_M)$. That is, $p(\ell_m-\lambda_M)$ counts the representations of ℓ_m as linear combinations using λ_M. Lemma 2.2 of [5] gives that $\Delta\alpha$, the change in α, is $O(B_2^{-1}) = O(\alpha^3)$ when $\lambda_m \sim cm$ as we have seen. In [5] and [6] it is shown that when α is changed by $\Delta\alpha$ in a sum involving α the sum changes by its derivative times $\Delta\alpha$. Thus α^{-1} will change by $O(\alpha^{-2}\Delta\alpha) = O(\alpha)$ and $\beta_2^{-1/2}$ will change by $O(\alpha^{-5/2}\Delta\alpha) = O(\alpha^{1/2})$. Finally the sum in the definition of α just after Remark 3.2 is, by [6], asymptotic to $\pi^2\alpha^{-2}/6$ so

$$\Delta\sum\log(1-e^{-\alpha\lambda_m}) \sim \Delta\alpha\sum\lambda_m e^{\alpha\lambda_m}(e^{\alpha\lambda_m}-1)^{-2} \sim \Delta\alpha\pi^2\alpha^{-2}/6 = O(\alpha).$$

Thus we can obtain the asymptotic behaviour of $\sum a_i$ simply by omitting the λ_M term or by multiplying that of $\sum b_i$ by $1-\exp(-\alpha\lambda_M) = O(\alpha)$. Remembering Remark 3.2 we therefore have

$$\sum_{\ell\le n} a_\ell \sim \frac{\pi\log\beta_M}{\sqrt{6c}}(\log n)^{-\frac{1}{2}}\sum_{\ell\le n} b_\ell \;=\; O\Big((\log n)^{-\frac{1}{2}}\sum_{\ell\le n} b_\ell\Big),$$

so we have Theorem 3.1. ∎

Note that we do not have to estimate the difference of functions asymptotically equal, so we have a simpler problem than Roth and Szekeres did. Next we summarize the cases for which our methods are known to apply and give a 0–1 law.

DEFINITION 3.4. A class F of finite structures has *approximately exponential growth* if one can, up to isomorphism, enumerate the structures $\mathbf{D}_n$ of F by strictly increasing size, and there is a constant c such that

$$\log(d_n) \sim cn,$$

where d_n is the size of $\mathbf{D}_n$.

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THEOREM 3.5. *Suppose* K *is admissible, and* F *is the set of* K*-indecomposables. If* F *is the disjoint union of finitely many* F_i*, where each* F_i *is closed under isomorphism and is either finite or has approximately exponential growth, then* K *has a first-order 0–1 law.*

PROOF. Let K_i be the closure of F_i under finite direct products and isomorphism. (1) If F_i has, up to isomorphism, only one member then clearly K_i is front-loaded. (2) If the members of F_i show approximately exponential growth then one can apply Theorem 3.1 and Lemma 1.4 to show that K_i is front-loaded.

Now, in the general case of the theorem we have subclasses K_i of K that belong to these two cases, so Corollary 2.4 gives the conclusion. ∎

EXAMPLE 3.6. Let V be the variety of monadic algebras $(B, \vee, \wedge, c, ', 0, 1)$ studied in algebraic logic, namely, one has Boolean algebras $(B, \vee, \wedge, ', 0, 1)$ augmented by a suitable closure operator c (see, *e.g.*, [2]). This is a congruence distributive variety, so unique factorization holds. Let K be the finite members of V. The directly indecomposables of V are precisely the Boolean algebras which satisfy $x > 0 \longrightarrow c(x) = 1$. Thus the sizes of the finite directly indecomposables of V form the sequence (2^n). By Theorem 3.5, K has a first-order 0–1 law.

From Skolem's work we know that the theory of finite Boolean algebras is decidable; and using this one can give a straightforward proof that the theory of the finite directly indecomposables of V is decidable. Thus, by Theorem 2.1(c), the limit law theory of K is decidable.

EXAMPLE 3.7. Let V be the variety of Heyting algebras generated by the three element chain. Again we have a congruence distributive variety, and thus unique factorization. Let K be the finite members of V. The directly indecomposables of V are precisely Boolean algebras with a new 0 adjoined. Thus the sizes of the finite directly indecomposables of V form the sequence $(2^n + 1)$. By Theorem 3.5, K has a first-order 0–1 law.

Again Skolem's work leads to a straightforward proof that the theory of the finite directly indecomposables of V is decidable. By Theorem 2.1(c) the limit law theory of K is decidable.

EXAMPLE 3.8. Let $p_1, \ldots, p_\ell$ be a set of prime numbers. Let K be the set of finite abelian groups whose exponent divides some power of $p_1 \cdots p_\ell$. Then the directly indecomposables fall into ℓ classes with the growth of the i-th class being the exponential sequence (p_i^n). Consequently K has a first-order 0–1 law by Theorem 3.5.

By Theorem 2.1(b) one has $\mathrm{Prob}_{\mathsf{K}}(\phi) = 1$ iff ϕ is true of the abelian group

$$\mathbf{G} = \prod_{i=1}^{\ell} \prod_{n=1}^{\infty} (\mathbf{Z}_{p_i^n})^{\omega}.$$

Referring to the work of Szmielew [7] we see that (i) the exponent of $\mathbf{G}$ is ∞, (ii) all elementary invariants of $\mathbf{G}$ which involve $p_1, \ldots, p_\ell$ are ∞, and (iii) all elementary invariants of $\mathbf{G}$ which involve other primes are 0. Thus the set of basic sentences which are

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true of **G** is recursive, and consequently the first-order theory of **G** is decidable. Consequently the limit law theory of K is decidable.

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