THEORIES OF RESONANT SATELLITE PAIRS IN SATURN'S SYSTEM

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INTRODUCTION

Upcoming missions to the outer planets have made evident the need for better theories of their satellites. We have been involved in an effort to provide better satellite theories through new observations (Abbot, Mulholland and Shelus 1975; Mulholland, Shelus and Abbot 1976; Mulholland and Shelus 1977; and Benedict, Shelus and Mulholland 1978), and new analytical theories (Jefferys and Ries 1975; Jefferys 1976. Hereinafter these are denoted Paper I and Paper II, respectively). In this paper we report on our incorporation of new and old observations into our theories, and on our progress on the theoretical front.

REVIEW OF EARLIER WORK

Our goals for the theoretical work have been to calculate new analytical theories for the three resonant pairs in Saturn's system: Enceladus-Dione, Titan-Hyperion and Mimas-Tethys, using the same software for all theories. We are attempting to calculate all terms down to the level of a few kilometers--potentially observable from space, and a great improvement in the accuracy of currently used theories (Struve 1930, 1933; Woltjer 1928). For example, observations of the satellites of Saturn with the Space Telescope can be expected to have errors of under 10 km.

We have employed the algebraic manipulation language TRIGMAN (Jefferys 1970, 1972) to calculate our theories, using the canonically invariant Hori-Lie theory (Hori 1966) in noncanonical variables. Because of the advantages of the Hori-Lie theory, it is possible to calculate the perturbations in any desired quantity (e.g., longitude, latitude, radius vector) directly, and it is not necessary to employ canonical variables, as long as routines for calculating Poisson brackets are available.

Because the theories are being developed automatically, they will

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be made available in the form of Fortran subroutines for direct calculation of the perturbations (and their partial derivatives) by machine. Since the constants of integration appear in literal form, it can be expected that the theories will be useful for a substantial period of time.

As explained in Paper I, we have developed the Hamiltonian in the variables

$$\begin{aligned} a &= \bar{a} + f_A A \\ e &= \bar{e} + f_E E \\ i &= \bar{i} + f_I I \end{aligned} \qquad (1)$$

where $\bar{a}$, $\bar{e}$, $\bar{i}$ are nominal values of the semimajor axis, eccentricity and inclination, respectively; f_A, f_E, f_I are numerical factors for controlling truncation; and A, E and I are the variables carried in the series expansions. Similar expressions involving truncation factors are used to express other quantities, e.g., the satellite masses and the dynamical form factors of Saturn.

By making use of the relations between the Delaunay variables and the usual elliptic variables, one can easily write down expressions for the partials of the variables A, E, and I with respect to the Delaunay variables, and hence of the Poisson bracket of any two functions expressed in terms of A, E, I and the Delaunay angles.

NEW THEORETICAL WORK

In Paper I a two-step method of eliminating first short-period and then long-period (i.e., resonant) arguments was described. Paper II showed how the elimination of the resonant arguments could be simplified by the introduction of a novel set of arguments, at the cost of an increase in the number of degrees of freedom of the system. This new set of arguments makes possible the treatment of the resonance completely independently of all other variables, and does not require the introduction of unnatural and awkward combinations of variables.

Our most recent work has simplified this procedure even more. The substitution of variables given in Paper II, in fact, can be made at any time, even prior to the elimination of the short-period terms. (Indeed, if there are two resonances, the same transformation can be applied to each, increasing the number of degrees of freedom by two. Under certain circumstances, such as small oscillations, the resulting equations can be solved. This approach may well provide a method for handling two simultaneous critical arguments under some circumstances). As a result, we have elected to make this transformation at the outset, and to eliminate both short and long with a single canonical transformation.

It has also become evident, since the publications of Papers I and II, that there are important terms at the 1-100 km level in the theories of these satellites when second order terms are computed. For example, the combination of the critical argument with short-periodic terms in the oblateness will, in the second-order theory of Enceladus and Dione produce terms in $(\ell \pm q)$, where q is the libration of the critical argument, having amplitudes of nearly 100 kilometers (about 0ʺ01 when observed from Earth). In addition, there are significant terms in the mean motions, such as those found by Kozai (1957), although whether they appear in first or second order depends upon how the calculation is done.

The theory of Mimas-Tethys, which we have computed in nearly final form (except for a few terms arising from the resonance) provides yet another reason to go to higher than the first order. In this resonant pair, the libration argument is quite large (amounting to over 43° in the case of Mimas' longitude). This in turn means that the small-oscillation approximation which could be applied for Enceladus and Dione is no longer valid. Other workers (e.g. Kozai 1957) have used the exact solution in elliptic functions, but this has two drawbacks in the present theory; first, TRIGMAN being a Poisson series processor, there is no easy way to incorporate elliptic functions; moreover, the possibility of second-order contributions from other terms in combination with the libration needs to be taken into account.

Our approach has been to develop the solution of the large-amplitude oscillation as a power series in the amplitude parameter, using the Hori-Lie method to obtain as many terms as are needed. According to the prescription of Paper II for handling the critical argument, the relevant part of the Hamiltonian is

$$F = F_o + n_o\Theta + \frac{A}{2}\Theta^2 - B\cos\theta, \qquad (2)$$

where $n_o \equiv 0$ is the constraint imposed at the resonance by the procedure of Paper II, although the partials of n_o do not vanish; A and B are constants depending on the initial conditions; and (Θ,θ) are canonically conjugate variables (θ being the libration argument).

Expanding the cosine in powers of θ yields in the lowest order a harmonic oscillator:

$$F = F_o + n_o\Theta - B + \frac{1}{2}(A\Theta^2 + B\theta^2) - \frac{B}{24}\theta^4 + \ldots \qquad (3)$$

By the substitution of the canonical pair

$$\Theta = (2p\gamma)^{1/2}\cos q, \qquad \theta = (2p/\gamma)^{1/2}\sin q, \qquad (4)$$

with $\gamma = (B/A)^{1/2}$ and

$$n_q = (AB)^{1/2}$$

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this can be brought into the form

$$F = F_o + n_o(2p\gamma)^{1/2}\cos q \quad - \quad B + n_q(p - \frac{1}{16\gamma}p^2 + \ldots)$$
$$+Ap^2[\frac{1}{12}\cos 2q - \frac{1}{48}\cos 4q] + \ldots$$

By considering $n_q(p - \frac{1}{16\gamma}p^2 + \ldots)$ to be the zero-order Hamiltonian insofar as (p,q) are concerned, the Hori-Lie procedure allows the elimination of the variable q through the use of a determining function S(p,q,...), where the ellipsis represents other variables.

If we consider now the effect of the libration on the longitude, the leading term in S is found to be

$$-\frac{n_o}{n_q}(2p\gamma)^{1/2}\sin q,$$

which contributes to the longitude (in first order) the term

$$\{\ell, S\} = -(2p\gamma)^{1/2}/n_q\{\ell, n_o\}\sin q$$
$$= (2p\gamma)^{1/2}/n_q \frac{\partial n_o}{\partial L}\sin q$$

Note that since $n_o \equiv 0$ because of the constraint condition, other partials do not contribute.

In second order, the elimination of terms involving p^2 yields terms in S arising from the Poisson brackets of $(2p\gamma)^{1/2}\cos q$ and $(2p\gamma)^{1/2}\sin q$ with terms in 2q and 4q, which are of the form

$$n_o p^{3/2}\sin q \quad \text{and} \quad n_o p^{3/2}\sin 3q.$$

These in turn give rise to terms in the longitude in q and in 3q, factored by $p^{3/2}$. Similarly, the third order theory provides terms in q, 3q and 5q, factored by $p^{5/2}$, and so on.

The theory of Mimas-Tethys is being extended to the level of accuracy which is the goal of this work. Judging by the ease with which the Enceladus-Dione programs were modified for the Mimas-Tethys case, we anticipate no great difficulties in this, nor in the extension to the Titan-Hyperion case.

COMPARISON OF THEORY AND OBSERVATION

We have applied our theory of Enceladus and Dione to the observations, including new observations made at McDonald Observatory over the past few years. Rather than to fit observations in each coordinate or quantity separately, we have attempted to make a single least-

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squares solution for all parameters. This has involved some difficulties and delays, partly because of the treatment of the critical argument that is described in Paper II. In that method, the number of degrees of freedom in the problem is increased by one through the introduction of the variables (Θ,θ), or equivalently, (p,q). This in turn means that two conditions of constraint must be imposed and consistently applied throughout the solution. There are methods of applying such constraints which are particularly elegant from a mathematical point of view (e.g., Brown, 1955). However, since these procedures were not available to us as programs, and other Least-Squares programs were available, it was decided to adapt the equations to the available programs rather than vice versa. In retrospect, this may not have been the best choice, although we have finally obtained satisfactory results.

The values of the parameters obtained from the theory do not differ greatly from those of Kozai (1957), although we have chosen to work in the equatorial system of 1950.0 rather than Struve's ecliptic system of 1889.25. The advantages of the present theory, therefore, lie not so much in the improvements that can be obtained from groundbased observations, but rather on their potential for improvement from space observations. Nevertheless, we do find that the recent observations made at McDonald Observatory are quite good, yielding mean residuals on the order of about 1 arcsecond or less in both right ascension and declination. The older observations are of variable quality, some yielding residuals as much as 5-10 times as large. Others among the older observations are of excellent quality.

We find for the forced librations of Enceladus and Dione the coefficients $12\overset{\circ}{.}18$ and $0\overset{\circ}{.}66$, respectively, and for the free librations $15\overset{\circ}{.}54$ and $0\overset{\circ}{.}84$. The value of J_2 that we obtain is $+0.01666 \pm 0.00001$; however, we have not been able to solve for J_4 separately, and have therefore adopted the value -0.001 in our solutions.

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DISCUSSION

Garfinkel: Does not your F_o, expanded in powers of the momentum θ, correspond to the Ideal Resonance Problem, rather than the Simple Pendulum?

Jefferys: Yes, if we take into account the powers of θ beyond the second. We have not done that in our first approximation.

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