THE MAXIMAL CO-RATIONAL EXTENSION BY A MODULE

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1. Introduction: Definitions. Modules are S-modules where S is an arbitrary ring with or without a unit element. We consider a projective module P having a submodule K such that $K + Y = P$ implies that the submodule Y is P (P, then, is a projective cover of P/K (Definition 4 in this section)) and we define the submodule X of P by

$$X = \sum_{f \in W} f(P), \qquad W = \mathrm{Hom}_S(P, K).$$

Our main result states that up to isomorphism P/X is the maximal co-rational extension over P/K (by P/K, in the more precise wording of the title). We introduce and define co-rationality by

Definition 1. The module A is *co-rational* over its factor module A/A_1 if and only if $\mathrm{Hom}_S(A, Y) = 0$ for every factor module Y of A_1. We then write $\mathrm{Hom}_S(A, A_1/*) = 0$. A is co-rational over a module M if and only if A is co-rational over a factor module A/A_1 S-isomorphic with M.

The existence theorem that we have mentioned for modules having a projective cover is set forth in §2. It is well known that the projective cover of a module M may fail to exist (**1**, p. 467, Theorem P). Thus it is consistent with our main result that some modules may fail to have maximal co-rational extensions, and we show in §3 that that is the case. Uniqueness up to isomorphism does hold for the maximal co-rational extension if it exists (Theorem 1.1). We note that our main result is analogous in form to one on maximal rational extensions (**3**, p. 168): Let $\hat{M}$ be the injective hull of the module M and let

$$V = \{f \in \mathrm{Hom}_S(\hat{M}, \hat{M}) | f(M) = 0\}.$$

Then the maximal rational extension of M exists and is isomorphic with $\bigcap_{f \in V} (\ker f)$. One definition of a rational extension (**3**, p. 167) is

Definition 2. The module A is a *rational extension* of its submodule A_1 if and only if $\mathrm{Hom}_S(X/A_1, A) = 0$ for every submodule X containing A_1.

We have symmetric concepts in definitions (1) and (2) in the following sense: the monomorphism (epimorphism) in the exact sequence

$$0 \to A_1 \to A \to A/A_1 \to 0$$

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expresses a rational (co-rational) extension if and only if no non-zero isomorphism exists between a factor module of $(A_1 + A_2)/A_1$ and a submodule of A (between a factor module of A and a submodule of $A_1/(A_1 \cap A_2)$) as A_2 runs through the submodules of A.

Definition 3. The module A is an *essential extension* of its submodule A_1 (we also say that A_1 is large in A) if and only if $A_1 \cap A_2 = 0$ implies that the submodule A_2 is 0. In this situation, if A is injective, it is called the *injective hull* of A_1. The injective hull always exists uniquely (**2**, pp. 75–78).

Definition 4. The module A is *co-essential* over its factor module A/A_1 (we also say that A_1 is small in A) if and only if $A = A_1 + A_2$ implies that the submodule A_2 is equal to A. In this situation, if the module A is projective, it is called the *projective cover* of any module S-isomorphic with A/A_1. The projective cover, if it exists, is in a sense unique (**1**, pp. 467, 472).

Definition 5. If a co-rational extension is given by the exact sequence

$$(\alpha) \qquad N \to N/L \to 0,$$

then N is a *maximal co-rational extension* over $N/L \cong M$ if and only if every co-rational extension over M

$$(\beta) \qquad N' \to N'/L' \to 0, \qquad N'/L' \cong M,$$

satisfies the following two conditions:
(5A) A homomorphism σ of N onto N' exists such that $\sigma(x) \in L'$ if and only if $x \in L$.
(5B) If g is a homomorphism of N' onto N such that $g(x) \in L$ if and only if $x \in L'$, then g is necessarily an isomorphism.

1.1. Theorem (Uniqueness). *If the co-rational extension* (α) *of Definition* 5 *is maximal, then there is an isomorphism between N and N^* if a co-rational extension given by*

$$(\gamma) \qquad N^* \to N^*/L^* \to 0, \qquad N^*/L^* \cong N/L,$$

satisfies either (5A) *or* (5B).

Proof. Let (5A) hold for the co-rational extension (γ). Thus there is a homomorphism g of N^* onto N such that L^* and L correspond under g. (5B), applied to the extension (α), requires that g be an isomorphism. Now assume (5B) for the co-rational extension (γ) and let g be the homomorphism of N onto N^* which exists by (5A) applied to (α) and is such that L and L^* correspond under g. (5B) asserts that g is an isomorphism.

Assuming the existence of a maximal co-rational extension for a given module M, Theorem 1.1 implies that no extension of M can satisfy only one of the requirements in Definition 5. It is an open question whether, in general, a module can have an extension that satisfies one of these requirements without the other.

2. Existence and uniqueness.

2.1. *Notation.* If $g \in Hom_S(X, Y)$ and if $W \subseteq Y$, we write

$$g^{-1}(W) = \{x \in X | g(x) \in W\}.$$

2.2. *Notation.* If $g \in \mathrm{Hom}_S(M, N/G)$ and if f is the canonical homomorphism $N \to N/G$, then

$$\overline{\mathrm{Im}\, g} = \{x \in N | f(x) \in g(M)\}.$$

2.3. Theorem. *A co-rational extension $N \to N/L \to 0$ is co-essential.*

Proof. If, on the contrary, $L + B = N, B \neq N$,

$$\text{then } 0 \neq N/B \cong L/(L \cap B)$$

contradicts the hypothesized co-rationality.

Throughout the remainder of this section we assume that the S-module M has a projective cover P. There exists, then, an exact sequence

$$0 \to K \to P \to M \to 0$$

where P is projective and K is small in P. We define

$$X = \sum f(P)$$

where the sum is taken over all $f \in \mathrm{Hom}_S(P, K)$. Proving the co-rationality of P/X over P/K and the two requirements in Definition 5, which constitute maximal co-rationality, occupies the remainder of this section. S, M, P, K, and X will have the meanings here assigned.

2.4. Theorem. *P/X is co-rational over P/K.*

We prove instead the more general

2.5. Proposition. *Let $X \subseteq X' \subseteq K' \subseteq K$. Then P/X' is co-rational over P/K'.*

Proof. Assume, on the contrary, that there is a non-zero element σ of $\mathrm{Hom}_S(P/X', (H/X')/(G/X'))$, where $X' \subseteq G \subseteq H \subseteq K'$. Define

$$\sigma^* \in \mathrm{Hom}_S(P, H/G)$$

as follows: if $\sigma(t + X') = w + (G/X')$, then $\sigma^*(t) = w + G$. Clearly $\sigma^* \neq 0$. Since P is projective, we can complete the diagram

$$\begin{array}{ccc} & P & \\ & & \searrow \sigma^* \\ \overline{\mathrm{Im}\, \sigma^*} & \xrightarrow[\text{nat}]{} & \overline{\mathrm{Im}\, \sigma^*}/G \to 0 \end{array}$$

to a commutative one by a homomorphism $h: P \to \overline{\mathrm{Im}\, \sigma^*}$. Then

$$\overline{\mathrm{Im}\, h}/(\overline{\mathrm{Im}\, h} \cap G) \cong \overline{\mathrm{Im}\, \sigma^*}/G \neq 0.$$

To prove the proposition, we obtain the contradiction: $\overline{\mathrm{Im}\, h} = \overline{\mathrm{Im}\, h} \cap G$. But $\overline{\mathrm{Im}\, h} \subseteq X \subseteq G$ is implied by $\overline{\mathrm{Im}\, h} \subseteq H \subseteq K$ and the definition of X, completing the proof.

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2.6. LEMMA. *In the commutative diagram of exact sequences*

$$\begin{array}{ccccc} & & A & & \\ & {}^{h}\swarrow & & \searrow & \\ B & \xrightarrow[\alpha]{} & C & \to & 0 \\ & & & \searrow & \\ & & & & 0 \end{array}$$

$B = h(A) + (\ker\alpha)$.

Proof. Let $b \in B$. Since $A \to C$ is onto, $\alpha(b) = \alpha h(a)$ for some $a \in A$. We have $b - h(a) \in (\ker\alpha)$, proving the lemma.

2.7. THEOREM. *The extension P/X over P/K satisfies* (5A).

Proof. Let $0 \to L \to N \to M \to 0$ be an exact sequence that expresses the co-rationality of N over M ($\cong P/K \cong N/L$). We are required to find a map α of P/X onto N such that $\alpha^{-1}(L) = K/X$.

Let ϕ be the map of P onto N/L whose kernel is K. Since P is projective, a map $g: P \to N$ exists which makes commutative the diagram

$$\begin{array}{ccccc} & & P & & \\ & {}^{g}\swarrow & & \searrow^{\phi} & \\ N & \xrightarrow[\text{nat}]{} & N/L & \to & 0 \\ & & & \searrow & \\ & & & & 0 \end{array}$$

By 2.6, $\operatorname{Im} g + L = N$ and we have $\operatorname{Im} g = N$, since L is small in N by 2.3. Since K is the kernel of ϕ, $g(K) = L$ and $g^{-1}(L) = K$. Thus g induces $\alpha: P/X \to N$, as required, provided $g(X) = 0$. If $T = (\ker g)$, then

$$\text{(A)} \qquad T \subseteq K$$

and, considering $g(X) \subseteq g(K) = L$,

$$\text{(B)} \qquad g(T + X) \subseteq L.$$

In order to prove that $g(X) = 0$, we assume the contrary: $T \cap X$ is properly contained in X. From the definition of X, there is a non-zero element σ of $\operatorname{Hom}_S(P, K)$ $(=\operatorname{Hom}_S(P, X))$ such that $T \cap X$ does not contain $\sigma(P)$. σ followed by $X \to X/(T \cap X)$ yields a non-zero element of

$$\operatorname{Hom}_S(P, (X/(T \cap X))).$$

With the aid of a familiar isomorphism we obtain a map $\sigma_1 \neq 0$ of P into $(T + X)/T$. Set $H = \sigma_1^{-1}(T)$; thus $H \neq P$ since $\sigma_1 \neq 0$. Since $T \subseteq K$ is small in P, $T + H \neq P$. Thus σ_1 induces the isomorphism σ^*:

$$\text{(C)} \qquad 0 \neq P/(T + H) \underset{\sigma^*}{\cong} Y \subseteq ((T + X)/(\sigma_1(T) + T)).$$

Using the composition $P/T \to P/(T + H) \cong Y$, we obtain a map

$$f\colon P/T \to Y \subseteq ((T + X)/(\sigma_1(T) + T))$$

and observe that (C) implies $f \neq 0$. Now we consider the isomorphism g^* of P/T onto N induced by g and the isomorphism g^{*-1}. Evidently, $g^*(\overline{\mathrm{Im}\, f}) \neq 0$, since Y is a non-zero submodule of a factor module of P/T, so that g^*fg^{*-1} is a non-zero element of $\mathrm{Hom}_S(N, N/^*)$. Considering (B), $\overline{\mathrm{Im}\, g^*fg^{*-1}} \subseteq L$ and g^*fg^{*-1} is a non-zero element of $\mathrm{Hom}_S(N, L/^*)$. We have arrived at a contradiction of the co-rationality of N over N/L. Hence $g(X) = 0$ and g^* induces the required map which takes P/X onto N.

2.8. Proposition. *Let H be a submodule of P such that an isomorphism σ of P/H onto P/X exists with $\sigma((X + H)/H) \subseteq K/X$. Then $H \supseteq X$.*

Proof. If, on the contrary, H does not contain X, let $t \notin H$ belong to X. By definition of X, $t \in f(P)$ for some element f of $\mathrm{Hom}_S(P, K)$ $(= \mathrm{Hom}_S(P, X))$. Since $f(P)$ is not contained in H, there is a non-zero map j obtained from f:

$$j\colon P \xrightarrow{\text{onto}} (f(P) + H)/H.$$

Since X is small in P and $(\ker j) \neq P$, $X + (\ker j) \neq P$. Consequently j induces a non-zero isomorphism j^* of $P/(X + (\ker j))$ onto a factor module of $(f(P) + H)/H$. Since σ is an isomorphism and since

$$\sigma((f(P) + H)/H) \subseteq \sigma((X + H)/H) \subseteq K/X,$$

σj^* is a non-zero element of $\mathrm{Hom}_S(P/X, K/^*)$ in contradiction of the co-rationality of P/X over P/K. We have proved that $H \supseteq X$.

2.9. Proposition. *Let H be a submodule of P such that P/X and P/H are isomorphic and that $X \subseteq H \subseteq K$. Then $X = H$.*

Proof. Let f be the map of P onto P/H obtained by composition of the natural map $P \to P/X$ with the isomorphism of the hypothesis. Clearly $(\ker f) = X$. Considering the projectivity of P, the following diagram:

$$\begin{array}{ccc} & P & \\ & & \searrow{\scriptstyle f} \\ P/X & \xrightarrow[\alpha]{} & P/H \to 0 \end{array}$$

in which α is the natural map, is completed to a commutative one by a homomorphism h: $P \to P/X$. Then

$$X' = (\ker h) \subseteq (\ker f) = X.$$

We prove now that

$$P/X' \underset{h}{\cong} P/X.$$

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It is sufficient to prove that $h(P) = P/X$. Since f is onto, 2.6 implies that

$$P/X = h(P) + (\ker\alpha) = h(P) + H/X.$$

Since P/X is co-rational over P/K by 2.5, 2.3 implies that K/X is small in P/X. We have H/X small, $h(P) = P/X, P/X' \cong P/X$.

From $\alpha h(X) = f(X) = H/H$ (commutativity of the diagram), we obtain $h(X) = H/X \subseteq K/X$. Thus the canonical isomorphism $\bar{h}$, obtained from h, satisfies $\bar{h}((X + X')/X') \subseteq K/X$, satisfying the hypothesis of 2.8. We have $X' \supseteq X$, $X' = X$, $\bar{h}$ is an automorphism of P/X.

Assuming that $H \neq X$, let $u \notin X$ be an element of H so that

$$\alpha(u + X) = 0 + H.$$

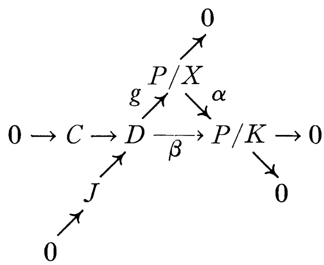
Now $u + X = h(t)$ with $t \notin X = (\ker h) = (\ker f)$. We have

$$f(t) \neq 0 + H = \alpha h(t)$$

in contradiction of the diagram's commutativity. $H = X$ has been proved.

2.10. THEOREM. *The extension P/X over P/K satisfies* (5B).

Proof. In the following commutative diagram of exact sequences:



the map α is the natural one and the co-rationality of D over P/K is expressed by β. (5B) requires for this situation that $J = (\ker g) = 0$. This we shall prove.

By the commutativity of the diagram and exactness of the sequences, we have $\beta(J) = \alpha g(J) \subseteq K = \beta(C)$, whence $J \subseteq C$.

By Theorem 2.7, (5A) holds, so that we have a chain of submodules of P

(*) $$X \subseteq G \subseteq J' \subseteq K$$

and an isomorphism h of D onto P/G which induces $D/J \cong P/J'$ and

(**) $$J \cong J'/G.$$

$J' \subseteq K$ was obtained from $J \subseteq C$ and the correspondence under h between C and K stipulated by (5A). We have $P/X \cong D/J \cong P/J'$ so that, considering (*) and 2.9, $X = J'$. Thus $J' = G$, and by (**), $J = 0$.

2.11. *Remark.* The following weaker form of the property (5B) established in 2.10 should be noted: No canonically co-rational extension P/H of P/K exists with H properly contained in X.

2.12. THEOREM (Summary). *Let the exact sequence*

$$0 \to K \to P \to M \to 0$$

indicate that P is a projective cover of the S-module M. Let

$$W = \mathrm{Hom}_S(P, K), \qquad X = \textstyle\sum_{f \in W} f(P).$$

Then P/X is the maximal co-rational extension over M and is unique in the following strong sense: $P/X \cong N$ if N satisfies either (5A) *or* (5B). *Moreover:*

(1) *If $X \subset H \subseteq K$, P/H is not isomorphic with P/X.*

(2) *If $P/H \underset{\sigma}{\cong} P/X$ for a submodule H that does not contain X, then $\sigma((X + H)/H$ does not lie in K/X.* (Such additional occurrences of the maximal co-rational extension can exist and this is the subject of 2.13.)

The references for 2.12 are 1.1, 2.4, 2.7, 2.8, 2.9, and 2.10.

2.13. *Example.* Let S be a commutative ring with a unit element having a unique maximal ideal J. Thus J is small in S and it is easily proved, using 2.12, that the S-module S/J is its own maximal co-rational extension. In order to illustrate 2.12(2), let the S-modules M and M' be isomorphic with S and let X and X' be the submodules that are isomorphic with J under the respective isomorphisms. Then X is small in $P = M \oplus M'$, P is a projective S-module, and P/X is its own maximal co-rational extension. P/X' and P/X are S-isomorphic and we have the promised additional occurrence of the maximal co-rational extension with X not contained in X'. The condition "K/X does not contain $\sigma((X + H)/H)$" of 2.12(2) reads "X/X does not contain $\sigma((X + X')/X')$" in this example and is right for every isomorphism σ of P/X' onto P/X.

3. A module that has no maximal co-rational extension. Let Z and Q denote the ring of integers and the field of rational numbers, respectively. We shall show that the Z-module 2_∞ (the smallest subgroup of Q/Z that contains 2^n for each negative integer n) has no maximal co-rational extension. We observe that 2_∞ is isomorphic to each of its non-zero homomorphic images; that for $n = 0, 1, 2, \ldots$, there is a unique subgroup F_n of 2_∞ isomorphic with the cyclic group of order 2^n; and that the finiteness of these subgroups implies $\mathrm{Hom}_Z(2_\infty, F_n/*) = 0$, whence $2_\infty \to 2_\infty/F_n \to 0$ expresses a co-rational extension over $2_\infty \cong 2_\infty/F_n$.

3.1. We assume now that there is a maximal co-rational extension C over 2_∞. The following statements will be derived from this assumption and will be shown in 3.2 to imply a contradiction:

(a) C is a maximal co-rational extension over $C/K \cong 2_\infty$.

(b) For $n = 0, 1, \ldots$, C has subgroups G_n such that $C/G_n \cong 2_\infty$ and K/G_n is cyclic with order 2^n.

(c) $K = G_0 \supset G_1 \supset \ldots \supset G_n \supset G_{n+1} \supset \ldots.$

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(d) There exists $\sigma \in K$ such that for $n = 1, 2, \ldots, K$ is generated by $G_n \cup \{\sigma\}$.

(e) C has subgroups $K_0 = K \subset K_1 \subset K_2 \subset \ldots$, such that K_n/K is cyclic with order 2^n; K_{n+1} is generated by K_n and one element; the union of the K_n is C.

(f) Every proper subgroup F of C is necessarily contained in some K_n and, for the least such n, $K_n = K + F$.

(g) $C = K + W$ where $W = \{3x | x \in C\}$.

(h) $\sigma = 3\tau$ for some element τ of C, where σ is the element named in (d).

(a) implies the remaining statements, each of which is to be obtained from the ones preceding it, but the lengthy proof of (c) is made last. (b) follows from (a) if we apply (5A) to the Z-module C and the co-rational extensions over 2_∞ mentioned in the opening statement. It is easy to obtain (d) from (b) and (c). (e) follows from the isomorphism between C/K and 2_∞.

To prove (f), let F be a subgroup of C and let $x \in F$ belong to K_n and not to K_{n-1}. If $0 < m \leqslant n$, $2^{n-m}x$ belongs to $K_m \cap F$ and not to K_{m-1} so that, using (e), $K_m \subseteq K_{m-1} + F$. $K_n \subseteq K + F$ follows and we must have $K_n = K + F$ if F is contained in K_n and not in K_{n-1} for some non-negative integer n. If the subgroup F lies in no K_n, these results imply that $K_n \subseteq K + F$ $(n = 1, 2, \ldots)$, so that $C = \cup K_n = K + F$. The co-rationality of C over C/K implies that $C/F \cong K/(K \cap F)$ is zero; F is not a proper subgroup.

To prove (g), define $3K_n = \{3x | x \in K_n\}$ and let $x \notin K_{n-1}$ belong to K_n, $n > 0$. Then $(x - 3x) \in K_{n-1}$ since K_n/K_{n-1} has order two. Thus for each positive integer n,

$$K_n = K_{n-1} + 3K_n = K + 3K_1 + \ldots + 3K_n \subseteq K + W,$$

where $W = \{3x | x \in C\}$. This yields (g).

If (h) is false, $C/W \cong K/(K \cap W) \neq 0$, which contradicts (a).

If (c) is false, there is a least positive integer n such that G_{n+1} is not contained in G_n. There exist unique subgroups $G'_0, \ldots, G'_n$ with $G_{n+1} \subseteq G'_i \subseteq K$ and $K/G_i \cong K/G'_i$, $1 \leqslant i \leqslant n$. Let $j < n$ be the non-negative integer such that $G_{j+1} \neq G'_{j+1}$ but $G_j = G'_j$, which we shall call Q. Put $N = G_{j+1} \cap G'_{j+1}$. Let $r \in G_{j+1}$, $r \notin G'_{j+1}$; $s \in G'_{j+1}$, $s \notin G_{j+1}$. The isomorphism of C/G'_{j+1} with 2_∞ such that Q/G'_{j+1} is the two-element group implies that

$$(*) \qquad 2t \equiv r \pmod{G'_{j+1}}$$

has solutions and that they must have order two (mod Q). Thus

$$T = \{t | t \notin Q, 2t \in Q\}$$

is the solution set for (*). By a similar argument, it also is the solution set for

$$(**) \qquad 2t \equiv s \pmod{G_{j+1}}.$$

Clearly T contains any solution of

$$(\#) \qquad 2t \equiv r \pmod{N},$$

and we suppose that there is a solution t_1 of

$$2t_1 \equiv r \pmod{N}, \qquad N \subseteq G_{j+1}.$$

Then $2t_1$ belongs to G_{j+1} (since r does) and, since t_1 is a solution of (**), we have $s \in G_{j+1}$, which is a contradiction. We have proved that (#) has no solutions; we cannot have $r = 2x$, $x \in C$.

Let $V = \{2x | x \in C\}$. The isomorphism of C/Q with 2_∞ implies that $C = V + Q$. Then

$$C/V = (Q + V)/V \cong Q/(Q \cap V) \neq 0,$$

since $r \in Q$, $\notin V$. Thus we have a non-zero element of

$$\mathrm{Hom}_Z(C, Q/*) \subseteq \mathrm{Hom}_Z(C, K/*)$$

in contradiction of (a). This completes the proof of (c).

3.2. We shall now obtain a contradiction from statements (a) through (h) of 3.1, thereby disproving the existence of a maximal co-rational extension over 2_∞. From (b), (c), (d), and (h) we have

(1) $\sigma(=3\tau$ for some element τ of $C)$ and G_n generate K where K/G_n is cyclic of order 2^n and $G_n \supset G_{n+1}$, $n = 1, 2, \ldots$.

Let $G' = \bigcap G_n$. Then $z\sigma \in G'$ implies that the integer z is zero. Applying Zorn's lemma, we obtain:

(2) There is a subgroup H of C maximal with respect to the property: $z\sigma \in H$ implies that the integer z is zero. Since $\sigma \notin H$, 3.1(f) implies that there is a non-negative integer j such that $H \subseteq K_j$ and $K_j = H + K$. Since $\sigma \in K$, we have

(3) $H \subseteq K_j, \qquad K_j/H \cong K/(K \cap H) \neq 0.$

In the following observations $\alpha, \beta, \ldots$ belong to C and $m, n, p, q, z, \ldots$ are in the ring Z of integers:

(4) If $\alpha \notin H$, we must have an equivalence: $q\alpha \equiv p\sigma \pmod{H}$, $p \neq 0$. Otherwise $Z\alpha + H$ contains $z\sigma$ only if $z = 0$, contradicting the maximality of H.

(5) $m\alpha \in H$, $m \neq 0$, imply $\alpha \in H$.

Otherwise, we would have from (4):

$$n\alpha - z\sigma \in H, \quad z \neq 0; \qquad mz\sigma - mn\alpha \in H, \quad mz\sigma \in H \quad (mz \neq 0).$$

(6) If, modulo H, $q\alpha \equiv p\sigma$ and $q'\alpha \equiv p'\sigma$, then, modulo H, $p'q\alpha \equiv pq'\alpha$. Thus, if also $\alpha \notin H$, we have $pq' = p'q$; cf. (5).

(7) If, modulo H, $q\alpha \equiv p\sigma$ and $q'\alpha' \equiv p'\sigma$, then

$$q'q(\alpha - \alpha') \equiv (q'p - qp')\sigma \pmod{H}.$$

Let Y_1 be the set of elements β of C such that $\beta \in H$ or, for some integers q and m with $q \notin 3Z$, we have $q\beta - m\sigma \in H$. Let Y_2 be the set of elements β of C such that $\beta \in H$ or, for some integers q and m with $q \notin 2Z$, we have $q\beta - m\sigma \in H$. By (7) the difference of two elements of Y_i belongs to Y_i, $i = 1, 2$. Thus Y_1 and Y_2 are subgroups of C.

We claim that $Y_2 \subseteq K_j$ where K_j contains H; cf. (3). Otherwise,

$$(2n + 1)\alpha - m\sigma \in H$$

for some integers m and n and some $\alpha \notin K_j$. Let k be the positive integer such that $\alpha' = 2^k\alpha \notin K_j$ but $2\alpha' \in K_j$. Then $(2n + 1)\alpha' - 2^k m\sigma \in H$. Since σ, $2\alpha'$, and $(2n + 1)\alpha' - 2^k m\sigma$ belong to K_j, so must α'; this is a contradiction.

We next prove that $C = Y_1 + Y_2$. Clearly $\beta \in Y_1 + Y_2$ if $q\beta - m\sigma \in H$ where 6 does not divide q. Alternatively, let $\beta \in C$ and integers m, q, $a > 0$, $b > 0$, satisfy $2^a 3^b q\beta - m\sigma \in H$, where q and 6 are relatively prime. Let integers c and d satisfy $2^a c + 3^b d = 1$. Since H contains both $3^b q(2^a c\beta) - cm\sigma$ and $2^a q(3^b d\beta) - dm\sigma$ and since $3^b q \notin 2Z$ and $2^a q \notin 3Z$, it follows that $\beta (= 2^a c\beta + 3^b d\beta)$ belongs to $Y_1 + Y_2$.

We claim that $Y_1 \neq C$. Suppose, for the element τ mentioned in (1) that $q\tau - m\sigma \in H$. Then from $3\tau - \sigma \in H$ and (6) follows $q = 3m$, whence $\tau \notin Y_1$.

Evidently $Y_1 \neq C = Y_1 + Y_2$ provides a non-zero isomorphism

$$C/Y_1 \cong Y_2/(Y_1 \cap Y_2).$$

Since $Y_2 \subseteq K_j$ and $H \subseteq (Y_1 \cap Y_2)$, the displayed isomorphism combines with $K_j/H \cong K/(K \cap H) \neq 0$ to produce a non-zero element of $\mathrm{Hom}_Z(C, K/^*)$; cf. (3). This contradicts the co-rationality of C over C/K implied by 3.1(a). Thus the Z-module 2_∞ does not have a maximal co-rational extension.

References

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