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THE STRONG ϕ TOPOLOGY ON SYMMETRIC SEQUENCE SPACES

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1. Introduction. The strong ϕ topology. Let S be a linear space of real sequences written in functional notation

$$s = (s(j)) = (s(1), s(2), \dots).$$

There is a natural duality between S and the space ϕ of sequences which are eventually 0 given by the equation

$$\langle s, t\rangle = \sum_j s(j)t(j) \quad s \in S, t \in \phi.$$

The series has only a finite number of nonzero terms since t is in ϕ.

A subset B of ϕ is called *S-bounded* if

$$p_B(s) = \sup\left\{\left|\sum_j s(j)t(j)\right| : t \in B\right\} < \infty$$

for each s in S.

The *strong* ϕ ($\beta\phi-$) topology on S is the locally convex topology determined by all seminorms of the form p_B as B ranges over all S-bounded subsets of ϕ. Most familiar sequence spaces bear the $\beta\phi$ topology, e.g., ϕ with the strongest locally convex topology, the l^p-spaces $1 \leqq p \leqq \infty$ with the BK topology, and ω (all sequences) with the topology of coordinate-wise convergence, but not l^p $(0 < p < 1)$ with the FK topology. The concept of $\beta\phi$ topology is related to the concept of norming biorthogonal sequence; see, e.g., **[11]**.

A space S of sequences is called symmetric if the sequence s_π is in S for every s in S and every permutation π on the set of indices. Here s_π is the sequence given by

$$s_\pi = (s(\pi(1)), s(\pi(2)), \dots).$$

Symmetric sequence spaces are considered in the 1934 paper of Köthe and Toeplitz **[3]**, in the three papers of Garling **[4-6]** and two papers of the author **[8, 9]**. Besides ϕ, ω and the l^p-spaces, two additional types of symmetric sequence spaces, Lorentz sequence spaces and Orlicz sequence spaces, have been the object of investigation. See, for example, **[7]**.

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The purpose of this paper is to determine whether symmetric sequence spaces are barrelled in the $\beta\phi$ topology. We shall prove that the answer is "yes" for three of the four categories of symmetric spaces, but the answer in general is "no." The main positive result in this paper is Theorem 4.3 which asserts that every symmetric space of bounded sequences which contains a nonconvergent sequence is barrelled in its $\beta\phi$ topology, and the $\beta\phi$ topology coincides with the relative topology of the BK-space l^∞ of bounded sequences. This is a generalization of a result of Seever [**13**] for the particular space of finitely valued sequences. The main negative result is an example of a nonseparable symmetric BK-space which is not barrelled in its $\beta\phi$ topology.

For any sequence space S is the *α-dual* or *Köthe dual* of S is the space S^α determined by the equation

$$S^\alpha = \left\{t: \sum_j |s(j)t(j)| < \infty, \forall\, s \in S\right\}.$$

A sequence space S is called *perfect* if $S^{\alpha\alpha} = S$. Köthe and Toeplitz [**3**] showed in 1934 that if S is perfect and symmetric then $S = \phi$, $S = \omega$, $S = l^\infty$ or $l^1 \subseteq S \subsetneq c_0$. This permits us to classify symmetric sequence spaces S which may not be perfect in terms of S^α (which is perfect).

S is *very large* if $S^\alpha = \phi$

S is *large* if $S^\alpha = l^1$

S is *medium* if $l^1 \subsetneq S^\alpha \subsetneq c_0$

S is *small* if $S^\alpha = l^\infty$.

If S is a symmetric sequence space and $S^\alpha = \omega$ then $S \subset \phi$ so either $S = \phi$ or

$$S = \left\{s \in \phi: \sum_j s(j) = 0\right\}.$$

Henceforth we assume that all sequence spaces mentioned contain ϕ. Thus if $S^\alpha = \omega$ and $S \supset \phi$, $S = \phi$; the $\beta\phi$ topology on ϕ is the strongest locally convex topology whose various properties are well known.

We shall see that for small, large and very large symmetric sequence spaces the $\beta\phi$ topology is the relative FK topology of l^1, l^∞ and ω respectively, and all of these spaces are barrelled. On the other hand, the collection of medium spaces admits a variety of topologies, some of which are not barrelled.

2. Very large symmetric sequence spaces ($S^\alpha = \phi$). It is easy to see that a symmetric sequence space is very large if and only if it contains an unbounded sequence. The space ω is very large, and seems to be the only very large space mentioned in the literature. Here is an example which shows that many very large symmetric sequence spaces are possible.

2.1 Example of a very large sequence space distinct from ω. Let D consist of all finite linear combinations of rational sequences. Then D is a very large symmetric sequence space. If u is a sequence of real numbers which is linearly independent over the rationals, then u is not in D. To see this suppose

$$u = \sum_{n=1}^{k} a_n v_n$$

where each v_n is a sequence of rationals and each a_n is a real number $n = 1, 2, \dots, k$. But this means that each $u(j)$ is a finite rational combination of $\{a_1, \dots, a_k\}$ contradicting the assumption that u is linearly independent over the rationals.

2.2 THEOREM. *If S is a symmetric sequence space which contains an unbounded sequence, then*

(a) *S is very large*;

(b) *the $\beta\phi$ topology on S is the relative topology of ω (the product topology)*;

(c) *S is barrelled in the $\beta\phi$ topology.*

Proof. We omit the straightforward proof of conclusion (a).

Conclusion (b) follows from the proof of Proposition 2 of **[5]** which does not use the fact that (e_n) forms a basis for the space.

(c) First we shall prove that if B is an S-bounded subset of ϕ then

(i) For each n

$$\sup\{\, |x(n)| : x \in B\} = M_n < \infty.$$

(ii) There is N such that for each x in B and $j > N$, $x(j) = 0$.

Assertion (i) is true since by our standing assumption e_n is in S for each n. To establish (ii), we assume the contrary, for the sake of obtaining a contradiction. This means we assume there is a sequence (x_n) in B and a sequence of indices (i_n) such that $i_n > i_{n-1} + 1$ and $x_n(i_n)$ is the last nonzero term in x_n for each n. Let v be an unbounded sequence in S. We define a permutation θ on the set of indices by induction. For $n < i_1$ let $\theta(n) = n$; let $\theta(i_1)$ be the smallest index h_1 such that

$$|v(h_1)x_1(i_1)| > 1 + \sum_{j<i_1} |v(\theta(j))|M_j.$$

If $\theta(n)$ has been defined for $n < i_k$ let $\theta(i_k)$ be the smallest index h_k such that

$$|v(h_k)x_k(i_k)| > k + \sum_{j<i_k} |v(\theta(j))|M_j.$$

Finally, let $\theta(i_k + 1)$ be the smallest index in the complement of $\{\theta(j) : j \leqq$

$i_k\}$ to ensure that θ is onto. Then we have

$$\left|\sum_j x_n(j)v_\theta(j)\right| \geqq |x_n(i_n)v_\theta(i_n)| - \left|\sum_{j<i_n} x_n(j)v_\theta(j)\right|$$
$$\geqq |x_n(i_n)v_\theta(i_n)| - \sum_{j<i_n} |v_\theta(j)|M_j$$
$$> n.$$

This shows v_θ is not bounded on B, contradicting the fact that B is S-bounded.

Since the $\beta\phi$ topology on S is the relative ω topology, it follows that the dual space of S is the space ϕ with the natural duality. If B is an S-bounded subset of ϕ, then it satisfies (i) and (ii) so it is ω-bounded; see [**3**]. But the $\beta\phi$ topology on ω is an FK topology so that it is barrelled. Therefore, B is equicontinuous. Since S-bounded implies S-equicontinuous it follows that S is barrelled.

3. Small symmetric sequence spaces ($S^\alpha = l^\infty$). Examples of small symmetric sequence spaces are $l^p (0 < p \leqq 1)$. In [**10**] it is shown that the intersection of all small symmetric sequence spaces is ϕ. In other words, for each sequence u not in ϕ there is a small symmetric sequence space which does not contain u.

3.1 LEMMA. *Suppose S is a symmetric sequence space which properly contains ϕ but is contained in l^1. If A is an unbounded subset of l^∞ i.e., if*

$$\sup\{\sup_j|x(j)|:x \in A\} = \infty,$$

then there is s in S such that

$$(3.1)\quad \sup\left\{\left|\sum_j s(j)x(j)\right|:x \in A\right\} = \infty.$$

Proof. For each $k = 1, 2, \dots$, let

$$M(k) = \sup\{|x(k)|:x \in A\}.$$

If for any k, $M(k) = \infty$ then we may take s to be e_k and conclude the proof. Thus for the remainder of the proof we may assume $M(k) < \infty$ for each k. Of course, since A is unbounded in l^∞ it follows that $\sup_k M(k) = \infty$.

Let t be any sequence in S but not in ϕ. Let $h_1 < h_2 < \dots$ be a sequence of positive integers such that for each j,

$$h_j - h_{j-1} > 1 \text{ and } |t(h_j)| < |t(h_{j-1})|.$$

Let π be the permutation on the integers which interchanges h_{2n-1} and h_{2n}

for all $n = 1, 2, \ldots$ and leaves other integers the same. If $v = t - t_\pi$ then v is in S,

$$v(j) = 0 \text{ for } j \notin \{h_1, h_2, \ldots\} \quad \text{and}$$

$$v(h_{2n-1}) = -v(h_{2n}) \neq 0 \text{ for } n = 1, 2, \ldots.$$

Let $\{n_1, n_2, \ldots\}$ be a sequence of integers for which

$$\text{(3.2)} \quad \sum_{j>m} |v(h_{2n_j-1})| + |v(h_{2n_j})| < 2^{-(m+1)}|v(h_{n_m})|.$$

This is possible because $S \subset l^1$. Denote by θ the permutation which interchanges h_{2n_j-1} and h_{2n_j} and leaves the remaining integers unchanged. If

$$u = \frac{1}{2}(v - v_\theta),$$

then u is in S,

$$u(j) = 0 \text{ for } j \notin \{h_{2n_j-1}, h_{2n_j}, j = 1, 2, \ldots\} \quad \text{and}$$

$$u(h_{2n_j-1}) = -u(h_{2n_j}) \neq 0.$$

Denote $|u(h_{2n_k-1})|$ by a_k. Then because of (3.2)

$$\sum_{k>m} a_k < 2^{-m}a_m.$$

We shall now define a sequence s which is a permutation of u and satisfies (3.1). Let x_1 be any sequence in A such that

$$\|x_1\| > \max\{1/a_1, M(1), M(2)\} + 1.$$

Here $\| \ \|$ denotes the norm in l^∞, the sup-norm. Let m_1 be the smallest positive integer such that

$$|x_1(m_1)| > \|x_1\| - 1/2.$$

Since $|x_1(m_1)|$ is larger than $M(1)$ and $M(2)$ it follows that $m_1 > 2$. Let

$$s(1) = -(\operatorname{sgn} x_1(m_1))a_1$$

$$s(j) = 0 \quad 1 < j < m_1$$

$$s(m_1) = (\operatorname{sgn} x_1(m_1))a_1.$$

Suppose we have defined x_h, m_h for $h < n$ and $s(j)$ for $j \leqq m_{n-1}$. Let x_n in A be such that

$$\|x_n\| > 1 + \max\left\{a_n^{-1}(1 - 2^{-n})^{-1}\left(2^n + \sum_{j<m_{n-1}} |s(j)|M(j)\right.\right.$$

$$+ a_nM(m_{n-1} + 1)\Big), M(1), \ldots, M(m_{n-1} + 2)\Big\}.$$

Let m_n be the smallest positive integer such that

$$|x_n(m_n)| > \|x_n\| - 2^{-n}.$$

Note that $m_n > m_{n-1} + 2$. Let

$$s(m_{n-1} + 1) = (-\text{sgn } x_n(m_n))a_n$$

$$s(j) = 0 \quad m_{n-1} + 1 < j < m_n$$

$$s(m_n) = (\text{sgn } x_n(m_n))a_n.$$

Then s is a permutation of u since it exhausts the nonzero elements $\pm a_n$ and contains infinitely many 0's as well. This implies $s \in S$. For each n we have

$$\begin{aligned}\left|\sum_j s(j)x_n(j)\right| &\geqq |s(m_n)x_n(m_n)| \\ &\quad - \sum_{j<m_n} |s(j)x_n(j)| - \sum_{j>m_n} |s(j)x_n(j)| \\ &\geqq a_n(\|x_n\| - 2^{-n}) - \sum_{j<m_{n-1}} |s(j)|M(j) \\ &\quad - a_nM(m_{n-1} + 1) - \|x_n\| \sum_{j>m_n} |s(j)| \\ &\geqq a_n(\|x_n\| - 2^{-n}) - \sum_{j<m_{n-1}} |s(j)|M(j) \\ &\quad - a_nM(m_{n-1} + 1) - \|x_n\|2^{-n}a_n \\ &\geqq a_n((1 - 2^{-n})\|x_n\| - 2^{-n}) \\ &\quad - \sum_{j<m_{n-1}} |s(j)|M(j) - a_nM(m_{n-1} + 1) \\ &\geqq 2^n - 2^{-n}a_n.\end{aligned}$$

Consequently we conclude that (3.1) holds for s.

3.2 THEOREM. *If S is a small symmetric sequence space then*
(a) *The $\beta\phi$ topology on S is the relative topology of the BK space l^1.*
(b) *S is barrelled in the $\beta\phi$ topology.*

Proof. If B is an S-bounded subset of ϕ then by Lemma 3.1 B is absorbed by the set

$$U_\infty = \{x \in \phi : \sup_n |x(n)| \leqq 1\}.$$

Since S is contained in l^1, U_∞ is S-bounded. Therefore, the $\beta\phi$ topology on S is determined by the norm

$$p_{U_\infty}(u) = \sup\left\{ \left| \sum_j u(j)x(j) \right| : x \in U_\infty \right\}$$

$$= \sum_j |u(j)|.$$

This confirms conclusion (a).

Conclusion (b) now follows from Lemma 3.1 just as (c) of Theorem 2.2 follows from (i) and (ii) in the proof of that Theorem. Since the $\beta\phi$ topology on S is the relative topology of l^1, the dual space of S is l^∞. If B is an S-bounded subset of m then by Lemma 3.1 it is uniformly bounded hence equicontinuous. Therefore, S is barrelled.

4. Large symmetric sequence spaces ($S^\alpha = l^1$). There are three classes of large symmetric sequence spaces

I $S \subset c_0$

II $S \subset c$ convergent sequences, but $S \not\subset c_0$

III $S \subset l^\infty$, but S contains a nonconvergent sequence.

We first consider large symmetric sequence spaces of the first class. An example of such a space not c_0 is $c_0 \cap D$ where D is given by 2.1.

4.1 LEMMA. *Suppose S is a large symmetric sequence space. A subset B of l^1 is S-bounded if and only if*

$$\text{(4.1)} \quad \sup\left\{ \sum_j |y(j)| : y \in B \right\} < \infty.$$

Proof. Condition (4.1) implies B is bounded in the BK topology of l^1 hence l^∞-bounded. Since $S^\alpha = l^1$, $s \subset S^{\alpha\alpha} = l^\infty$ so B is S-bounded. This shows (4.1) is sufficient.

If x is in S then the set $\langle x \rangle$ consisting of all sequences x_π where π ranges over all permutations is l^1-bounded because $\langle x \rangle$ is uniformly bounded. Since $\langle x \rangle$ is bounded, by Satz 1, Section 5 of **[3]** it is completely bounded. Hence, if B is an S-bounded subset of l^1, so is

$$\langle B \rangle = \{y_\pi : y \in B, \pi \text{ is a permutation}\}.$$

This is because for x in S

$$\sup\left\{\left|\sum_j x(j)y_\pi(j)\right| : y \in B\right\}$$
$$= \sup\left\{\left|\sum_j u(j)y(j)\right| : u \in \langle x\rangle, y \in B\right\} < \infty.$$

Now assume, for the sake of obtaining a contradiction, that B is an S-bounded subset of l^1 which does not satisfy (4.1). For each n let x_n in B satisfy

$$\sum_j |x_n(j)| > 4^n + 1.$$

Since each x_n is in l^1 we can find a sequence (π_n) of permutations and a sequence (M_n) of disjoint subsets of indices such that

$$\sum_{j \notin M_n} |(x_n)_{\pi_n}(j)| < 1.$$

It follows that

$$\sum_{j \in M_n} |(x_n)_{\pi_n}(j)| > 4^n$$

for each n. Each $(x_n)_{\pi_n}$ is a member of the set $\langle B\rangle$ which is S-bounded. Therefore, the partial sums of

$$\sum_n 2^{-n}(x_n)_{\pi_n}$$

form a Cauchy sequence in the $\sigma(l^1, S)$ topology on l^1. By Satz 2, Section 4 of [**3**] there is x in l^1 such that

$$\sum_n 2^{-n}(x_n)_{\pi_n} = x$$

in the $\sigma(l^1, S)$ topology. Since $\phi \subset S$,

$$\sum_n 2^{-n}(x_n)_{\pi_n}(j) = x(j) \quad \text{for each } j,$$

we have

$$\sum_{j \in M_n} |x(j)| \geqq 2^{-n} \sum_{j \in M_n} |(x_n)_{\pi_n}(j)| - \sum_{m \neq n} 2^{-m} \sum_{j \in M_n} |(x_m)_{\pi_m}(j)|$$
$$\geqq 2^n - 1.$$

This contradicts the fact that x is in l^1.

The following theorem follows from Lemma 4.1 very much as Theorem 3.2 follows from 3.1. Therefore, we omit the proof.

4.2 THEOREM. *If S is a large symmetric sequence space which is contained in c_0 then*

(a) *The $\beta\phi$ topology on S is the relative topology of the BK space c_0;*

(b) *S is barrelled in the $\beta\phi$ topology.*

If S is of class II then $S = [e]$ the span of $e = (1, 1, \dots)$ or $S = T \oplus [e]$ where T is a symmetric sequence space which is small, medium or large of class I. This S is barrelled if and only if T is. This essentially reduces the study of large symmetric sequence spaces of class II to those which are small, medium, or large, class I.

For symmetric sequence spaces of class III we have the following result.

4.3 THEOREM. *If S is a large symmetric sequence space which contains a divergent sequence then*

(a) *The $\beta\phi$ topology on S is the relative topology of l^∞;*

(b) *S is dense in l^∞;*

(c) *S is barrelled in the $\beta\phi$ topology.*

The remainder of this section is devoted to the proof of 4.3. First we establish several lemmas.

4.4 LEMMA. *Suppose S is a symmetric space of sequences which contains ϕ and also contains a divergent sequence. For each subset M of indices and $\epsilon > 0$ there is a sequence $e_{M,\epsilon}$ in S such that*

(a) $|e_{M,\epsilon}(j)| < \epsilon$ *for $j \notin M$;*

(b) $|e_{M,\epsilon}(j) - 1| < \epsilon$ *for $j \in M$;*

(c) $e_M - e_{M,\epsilon} \in c_0$ *where $e_M(j) = 1$ for $j \in M$ and $e_M(j) = 0$ for $j \notin M$.*

Proof. We first prove the lemma under the assumption that M is infinite and has an infinite complement. Let s be a bounded nonconvergent sequence in S. Let $h_1 < h_2 < \dots$ and $k_1 < k_2 < \dots$ be two sequences of indices such that $h_n < k_n < h_{n+1}$ for each n while $\lim_n s(h_n) = a$ and $\lim_n s(k_n) = b$ exist and are distinct. Let π be the permutation which interchanges $h(n)$ and $k(n)$ for $n = 1, 2, \dots$ and leaves the other integers the same. Let $t = s - s_\pi$; then

$$t(h_n) = -t(k_n),\ \lim_n t(h_n) = a - b,\ \lim_n t(k_n) = b - a.$$

If $u = (a - b)^{-1}t$ we have

$$\lim_n u(h_n) = 1 \text{ and } \lim_n u(k_n) = -1.$$

Given $\epsilon > 0$, let N be such that

$$|u(h_n) - 1| < \epsilon/2 \text{ and } |u(k_n) + 1| < \epsilon/2$$

if h_n or $k_n > N$. Let v be the sequence for which $v(j) = 0$ for $j \leqq N$ and $v(j) = u(j)$ for $j > N$. Since S contains ϕ, v is in S. For simplicity we shall assume that h_1 and k_1 are greater than N. Let θ be a permutation on the indices which (1) leaves each index in $\sim(\{h_n\} \cup \{k_n\})$, the complement of $\{h_n\} \cup \{k_n\}$, the same; (2) maps h_{2n} onto k_n for each $n = 1, 2, \ldots$; (3) maps k_n onto h_{2n} for each $n = 1, 2, \ldots$; (4) leaves each h_{2n-1} unchanged. If $w = (v + v_\theta)/2$ then $w(j) = 0$ for j in $\sim(\{h_n\} \cup \{k_n\})$;

$$|w(j)| = |v(k_n) + v(h_{2n})|/2 \leqq |v(k_n) + 1|/2 + |v(h_{2n}) - 1|/2$$
$$< \epsilon/2 \text{ for } j = k_n;$$
$$|w(j)| = |u(h_{2n}) + u(k_n)|/2 < \epsilon/2 \text{ for } j = h_{2n};$$

while for $j = h_{2n-1}$,

$$|v(j) - 1| = |v(h_{2n-1}) - 1| < \epsilon/2.$$

Since $\{h_{2n-1}\}$ is an infinite subset of indices with an infinite complement, there is a permutation ρ which takes $\{h_{2n-1}\}$ onto M and the complement of $\{h_{2n-1}\}$ onto the complement of M. If $e_{M,\epsilon} = w_\rho$, $e_{M,\epsilon}$ satisfies (a), (b) and (c).

If M is a finite set of indices then $e_M \in \phi \subset S$. If M has a finite complement let σ be the permutation of indices which maps $\{h_{2n-1}\}$ onto $\sim\{h_{2n-1}\}$ and let

$$e_{M,\epsilon} = w + w_\sigma - e_{\sim K}.$$

The following lemma is found on p. 108 of [**12**] as well as in the book of Köthe [**2**] and the works of Bourbaki [**1**].

4.5 LEMMA. *Let (f_n) be a sequence of continuous linear functions on l^∞ and let (M_j) be a sequence of finite subsets of indices. There is a set M of indices which is a union of a subsequence of (M_j) such that whenever s is a member of m with support on M we have*

$$f_n(s) = \sum_{j \in M} s(j) f_n(e_j).$$

Conclusion of the proof of Theorem 4.3. Let l_0^∞ be the space of finitely valued sequences. It is well known (see, e.g., [**15**]) that l_0^∞ is dense in l^∞. It is clear that l_0^∞ is the linear span of all sequences e_M as M ranges over all over sets of indices. By Lemma 4.4 if S is a large symmetric sequence space which contains a divergent sequence, e_M is in the closure of S in l^∞ for each M. Therefore, S must be dense in l^∞. This establishes conclusion (b).

In order to verify conclusions (a) and (c) we shall prove that if B is a subset of $(l^\infty)^*$, the dual space of l^∞, which is S-bounded, then B is l^∞-bounded. Then (a) and (c) will follow by the same argument that works for small symmetric sequence spaces.

Suppose, for the sake of obtaining a contradiction, that B is a subset of $(l^\infty)^*$ which is S-bounded but not l^∞-bounded. Then there is a subset M of indices such that

$$\sup\{\,|f(e_M)|:f \in B\} = \infty.$$

See Lemma 7.2 of [**14**]. By Lemma 4.4 there is $e_{M,1/2}$ in S such that

$$e_M - e_{M,1/2} = v \in c_0.$$

Since

$$\sup\{\,|f(e_M)|:f \in B\} = \infty$$

and

$$\sup\{\,|f(e_{M,1/2})|:f \in B\} < \infty$$

it follows that

$$\sup\{\,|f(v)|:f \in B\} = \infty.$$

But since $v \in c_0$,

$$f(v) = \sum_j v(j)f(e_j)$$

for each f in B. This implies that

$$\sup\left\{\sum_j |f(e_j)|:f \in \mathrm{B}\right\} = \infty.$$

However, since e_M is in $\phi \subset S$ for each finite subset M of indices we conclude that

$$\sup\left\{\sum_{j\in M} |f(e_j)|:f \in B\right\} = b(M) < \infty$$

for each finite M.

Let f_1 be any member of B such that

$$\sum_j |f_1(e_j)| > 2$$

and let N_1 be any integer such that

$$\sum_{j\leqq N_1} |f_1(e_j)| > 2, \quad \sum_{j>N_1} |f_1(e_j)| < 1/2.$$

If $f_1, \ldots, f_n$ in B and integers $N_n, \ldots, N_n$ have been constructed, let f_{n+1} be any member of B such that

$$\sum_j |f_{n+1}(e_j)| > 2^{n+1} + 1 + 4b(\{1, 2, \ldots, N_n\}).$$

Let N_{n+1} be any integer such that

$$\sum_j |f_{n+1}(e_j)| < 2^{-n}.$$

Then it follows that

$$(4.2) \qquad \sum_{N_n<j\leqq N_{n-1}} |f_{n+1}(e_j)| > 2^{n+1} + 3b(\{1, 2, \dots, N_n\}).$$

By induction we construct functionals (f_n) and indices (N_n) which satisfy (4.2) for all n.

For each k let $M_k = \{j:N_{k-1} < j \leqq N_k\}$ and suppose

$$M = \bigcup_{h=1}^{\infty} M_{k_h}$$

satisfies the conclusion of Lemma 4.3. Let w be a sequence in S such that

$$w(j) = 0, j \notin \bigcup_{h=1}^{\infty} M_{k_h}, |w(j)| < 1/2, j \in M_{k_h}, h \text{ even},$$

$$|\text{sgn} f_{k_h}(e_j) - w(j)| < 1/2, j \in M_{k_h}, h \text{ odd}.$$

Here we use Lemma 4.4 and the fact we can find in S a sequence with finitely many zeros and transfer them to the complement of M. Then if h is odd,

$$\begin{aligned} |f_{k_h}(w)| &= \left|\sum_{j\in M} f_{k_h}(e_j)w(j)\right| \\ &\geqq (1/2)\sum_{j\in M_{k_h}} |f_{k_h}(e_j)| - (3/2)\sum_{j\notin M_{k_h}} |f_{k_h}(e_j)| \\ &\geqq (1/2)(2^{n-1} + 3b(\{1, 2, \dots, N_{k_h-1}\})) \\ &- (3/2)\sum_{j\leqq N_{k_h-1}} |f_{k_h}(e_j)| - (3/2)\sum_{j>N_k} |f(e_j)| \\ &\geqq 2^n - 3\cdot 2^{-n-1}. \end{aligned}$$

Therefore,

$$\sup_h |f_{k_h}(w)| = \infty$$

contradicting the fact that B is S-bounded.

5. Medium symmetric sequence spaces ($l^1 \subsetneq S^\alpha \subsetneq c_0$). Most interesting symmetric sequence spaces are medium; e.g., l^p ($1 \leqq p < \infty$), Lorentz spaces, Orlicz spaces.

If x is a sequence in c_0 then by $\hat{x}$ we denote the sequence consisting of nonzero members of $\{|x(1)|, |x(2)|, \ldots\}$ in decreasing order with repetitions allowed. If x and y are in c_0 then for any permutations π and θ we have

$$\sum_j |x_\pi(j)y_\theta(j)| \leqq \sum_j \hat{x}(j)\hat{y}(j);$$

here we understand that if the left hand side of the inequality is infinite the right hand side is also.

5.1 LEMMA. *Let S be a medium symmetric sequence space. A subset B of S^α is S-bounded if and only if for each x in X*

$$\text{(5.1)} \quad \sup\left\{\sum_j |x(j)y(j)| : y \in B\right\} < \infty.$$

Proof. If x is in S then the set $\langle x \rangle$ of all permutations of x is S^α-bounded. To demonstrate this, let y be any member of S^α. Let x_θ be any permutation of α such that

$$|x_\theta(1)| \geqq |x_\theta(3)| \geqq |x_\theta(5)| \geqq \ldots$$

and

$$\sum_j |x_\theta(2j)| < 1/(\sup_j |y(j)| + 1).$$

Let z be the sequence for which $(z(1), z(3), \ldots)$ are the nonzero members of $(|y(1)|, |y(3)|, \ldots)$ in descending order and $y(2j) = 0$ for each j. It is not hard to verify that z is in S^α since S^α is symmetric and normal. For each $j = 1, 2, \ldots$ let

$$u(j) = \operatorname{sgn} x_\theta(j);$$

then $uz = (u(j)z(j))$ is also in S^α. For any permutation π of indices

$$\left|\sum_j x_\pi(j)y(j)\right| \leqq \sum_j |y_\pi(j)y(j)|$$

$$\leqq \sum_j x_\theta(j)z(j)u(j) + 1 < \infty.$$

Therefore,

$$\sup_\pi \left|\sum_j x_\pi(j)y(j)\right| < \infty.$$

If B is an S-bounded subset of s^α so is $\langle B \rangle$ consisting of all y_π as y ranges over B since

$$\sup\left\{\left|\sum_j x(j)y_\pi(j)\right|:y \in B\right\}$$

$$= \sup\left\{\left|\sum_j u(j)y(j)\right|:u \in \langle x\rangle, y \in B\right\} < \infty.$$

The last inequality follows from Satz 1, Section 5 of [**3**] since $\langle x\rangle$ must be completely bounded.

Suppose now for the sake of obtaining a contradiction that B is an S-bounded subset of S^α for which (5.1) does not hold. Then there is x in S for which

$$(5.2) \quad \sup\left\{\sum_j |x(j)y(j)|:y \in B\right\} = \infty.$$

Since $\phi \subset S$,

$$\sup\{|y(j)|:y \in B\} < \infty \text{ for all } j$$

so that if M is any finite set of indices

$$\sup\left\{\sum_{j\in M} |x(j)y(j)|:y \in B\right\} = \infty.$$

We define by induction a sequence y_n in B and a sequence $\{M_n\}$ of disjoint finite subsets of indices such that

$$(5.3) \quad \sum_{j\in M_n} |x(j)y_n(j)| > 4^n$$

(5.4) $\bigcup_n M_n$ has an infinite complement.

Since each y_n is in c_0 we can find an infinite subset K_n of indices such that

$$\sum_{j\in K_n} |y_n(j)| < 4^{-n}/\sup_j|x(j)|.$$

Using the fact that the complement of $\cup_n M_n$ is infinite we can determine sequences z_n and a partition (H_n) of the set of indices such that (a) each z_n is a permutation of y_n; (b) $z_n(j) = y_n(j)$ for $j \in M_n$; (c) $M_n \subset H_n$ for each n; (d) each H_n is infinite; (e) for $j \notin H_n$, $z_n(j) = y_n(i)$ for some $i \in K_n$. The series

$$\sum_n 2^{-n}z_n$$

converges in the $\sigma(S^\alpha, S)$ topology since $\{z_n\}$ is bounded, being a part of $\langle B\rangle$ and S^α is $\sigma(S^\alpha, S)$ complete by Satz 2 of Section 4 of [**3**]. If

$$z = \sum_n 2^{-n} z_n$$

we have

$$\sum_{j \in M_n} |x(j)z(j)| \geqq \sum_{j \in M_n} 2^{-n}|x(j)z_n(j)|$$
$$- \sum_{m \neq n} 2^{-m} \sum_{j \in M_n} |x(j)z_m(j)|$$
$$\geqq 2^n - \sum_{m \neq n} 2^{-m} \cdot 4^{-n} > 2^n - 1.$$

This contradicts the assumption that z is in S^α. Therefore (5.1) must be valid.

5.2 THEOREM. *If S is a medium symmetric sequence space then the $\beta\phi$ topology on S coincides with the topology $\beta(S, S^\alpha)$ on S determined by the polars in S of S-bounded subsets of S^α.*

Proof. Since $\phi \subset S$, $\beta(S, S^\alpha)$ is a stronger topology than $\beta(S, \phi)$.

Suppose B is an S-bounded subset of S^α. The *normal cover* C of B defined by

$$C = \{uy:y \in B, |u(j)| \leqq 1 \text{ for each } j\}$$

is also S-bounded by Lemma 5.1. Therefore, $D = C \cap \phi$ *is an* S-bounded subset of ϕ. For x in S

$$p_D(x) = \sup\left\{\left|\sum_j x(j)y(j)\right|:y \in D\right\}$$
$$= \sup\left\{\sum_j |x(j)y(j)|:y \in D\right\}$$
$$= \sup\left\{\sum_j |x(j)y(j)|:y \in C\right\}$$
$$= p_C(x) \geqq p_B(x).$$

Therefore p_B is $\beta(s, \phi)$ continuous.

A topological sequence space S containing ϕ is said to have AD if ϕ is dense in S; S is said to have AK if for each x in S

$$\lim_n[x \leqq n] = \lim_n \sum_{j=1}^n x(j)e_j = x;$$

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S is said to have UAK if $\sum_j x(j)e_j$ converges unconditionally to x.

5.3 THEOREM. *If S is a medium symmetric sequence space which has* AD *in the $\beta\phi$ topology then*

(a) *S has* UAK in the $\beta\phi$ topology,

(b) *the dual space S' of S is represented by S^α with the usual duality $f \leftrightarrow y$*

$$f(x) = \sum_j x(j)y(j), \quad f \in S', y \in S^\alpha, x \in S;$$

(c) *S is barrelled in the $\beta\phi$ topology.*

Proof. (a) Suppose x is in S and p is a continuous seminorm on S of the form

$$p(x) = \sup\left\{\left|\sum_j x(j)y(j)\right| : y \in B\right\}$$

where B is an S-bounded subset of ϕ. By Lemma 5.1 the seminorm q given by

$$q(x) = \sup\left\{\sum_j |x(j)y(j)| : y \in B\right\}$$

$$= \sup\left\{\left|\sum_y x(j)y(j)\right| : y \in C\right\}$$

where C is the normal cover of B is also continuous in the $\beta\phi$ topology. Since S has AD there is $u \in \phi$ such that $q(x - u) \leqq 1$. If $M = \{j:u(j) \neq 0^*\}$ then for all y in B

$$\sum_{j \notin M} |x(j)y(j)| \leqq \sum_j |(x(j) - u(j))y(j)| \leqq 1.$$

Therefore, if $K \cap M = \emptyset$

$$p(x[K]) \leqq q(x[K]) \leqq 1,$$

which implies $\sum_j x(j)e_j$ converges unconditionally to x.

(b) For each f in S' and x in S

$$f(x) = \sum_j x(j)f(e_j)$$

and $(f(e_j))$ is in S^α since the series converges absolutely. On the other hand if y is in S^α the linear functional defined by

$$f(x) \sum_j x(j)y(j)$$

is continuous with respect to the seminorm

$$p(x) = \sup_n \left| \sum_{j=1}^{n} x(j)y(j) \right|$$

which is continuous in the $\beta\phi$ topology.

(c) If B is an S-bounded subset of S' then B corresponds to an S-bounded subset of S^α. Therefore B is equicontinuous by Theorem 5.2.

6. An example. In this section we describe an example of a symmetric sequence space which is not barrelled in the $\beta\phi$ topology. Such a space must be a medium symmetric sequence space which does not have AD.

6.1 LEMMA. *There exists a sequence* (u_n) *in* c_0 *such that* (a) *each* u_n *is positive and decreasing with* $u_n(1) = 1$ *for each* n; (b) $\sum_j u_n(j) = \infty$ *for each* n; (c) *for each* n *there is an increasing sequence* m_n *of indices such that*

$$\lim_p \frac{\sum_{j=1}^{m_n(p)} u_k(j)}{\sum_{j=1}^{m_n(p)} u_n(j)} = 0 \quad \textit{if } k \neq n$$

and

$$\frac{\sum_{j=1}^{m_n(p)} u_k(j)}{\sum_{j=1}^{m_n(p)} u_n(j)} \leqq 1 \quad \textit{for all } k.$$

Proof. Note that we are using functional notation $m_n(p)$ to describe sequences of indices.

We first establish the existence of a sequence (u_n) in c_0 which satisfies (a), (b) and (c′): for each n there are increasing sequences $m_{n,n}, m_{n,n+1}, \ldots$ of indices such that

(i) $m_{n,r+1}$ is a subsequence of $m_{n,r}$ for each r;

$$\text{(6.1) (ii)} \quad \lim_p \frac{\sum_{j=1}^{m_{n,n}(p)} u_k(j)}{\sum_{j=1}^{m_{n,n}(p)} u_n(j)} = 0 \quad k < n$$

$$(6.2)\quad \lim_p \frac{\sum_{j=1}^{m_{n,n+h}(p)} u_{n+h}(j)}{\sum_{j=1}^{m_{n,n+h}(p)} u_n(j)} = 0 \quad h = 1, 2, \dots .$$

Each quotient in (6.1) and (6.2) is no greater than 1.

We proceed by induction. Let $u_1(j) = 1/j$ for $j = 1, 2, \dots$. Suppose that $u_1, \dots, u_{n-1}$ have been defined which satisfy (a), (b) and (c′). We must now define a sequence u_n in c_0 that satisfies (a) and (b), an increasing sequence $m_{n,n}$ of indices and subsequences $m_{k,n}$ of $m_{k,n-1}$ for $k = 1, 2, \dots, n - 1$, such that (6.1) and (6.2) are satisfied and each quotient is $\leqq 1$. For each r and $k \leqq n - 1$ let

$$U_k(r) = \sum_{j=1}^{r} u_k(j)$$

and let

$$V(r) = \max_k U_k(r).$$

We define u_n, $m_{n,n}$ and $m_{k,n}$ inductively. Let $u_n(1) = 1$. Since each u_k, $k < n$ is in c_0 there is an index $m_{n,n}(1)$ such that

$$m_{n,n}(1) > 2V(m_{n,n}(1)).$$

Let $u_n(j) = 1$ for $j \leqq m_{n,n}(1)$. Since

$$\sum_j u_1(j) = \infty$$

there is an index $m_{1,n-1}(h)$ such that

$$\sum_{j=1}^{m_{1,n-1}(h)} u_1(j) > 4m_{n,n}(1) = 4 \sum_{j=1}^{m_{n,n}(1)} u_n(j).$$

Let

$$c = 3 \sum_{j=1}^{m_{n,n}(1)} u_n(j)/(m_{1,n-1}(h) - m_{n,n}(1)).$$

Define $m_{1,n}(1)$ to be $m_{1,n-1}(h)$ and $u_n(j) = c$ for $m_{n,n}(1) < j \leqq m_{1,n}(1)$. Suppose we have defined

$$m_{1,n}(1) < m_{2,n}(1) < \dots < m_{k-1,n}(1)$$

and $u_{n,n}(j)$ for $j \leqq m_{k-1,n}(1)$. Let $m_{n-1,k}(h)$ be an index $> m_{n,k-1}(1)$ such that

$$\sum_{j=1}^{m_{n-1,k}(h)} u_k(j) > 4 \sum_{j=1}^{m_{n,k-1}(1)} u_n(j).$$

Let $m_{n,k}(1) = m_{n-1,k}(h)$; let

$$c = 3 \sum_{j=1}^{m_{n,k-1}(1)} u_n(j)/(m_{n,k}(1) - m_{n,k-1}(1))$$

and let $u_n(j) = c$ for $m_{n,k-1}(1) < j \leqq m_{n,k}(1)$.

Now suppose we have defined $m_{n,n}(q) < m_{1,n}(q) < \ldots < m_{n-1,n}(q)$ and $u_n(j)$ for $j \leqq m_{n-1,n}(q)$ such that

$$\sum_{j=1}^{m_{n,n}(q)} u_k(j) \Big/ \sum_{j=1}^{m_{n,n}(q)} u_n(j) < 1/q \tag{6.3}$$

and

$$\sum_{j=1}^{m_{k,n}(q)} u_n(j) \Big/ \sum_{j=1}^{m_{k,n}(q)} u_k(j) < 1/q \tag{6.4}$$

$$k = 1, 2, \ldots, n - 1.$$

We accomplished this for $q = 1$ in the preceeding paragraph. Let $m_{n,n}(q + 1)$ be an index which is greater than

$$(q + 1)V(m_{n,n-1}(q + 1))/u_n(m_{n-1,n}(q))$$

and let

$$u(j) = u_n(m_{n-1,n}(q)) \text{ for } m_{n-1,n}(q) < j \leqq m_{n,n}(q + 1).$$

If we have defined $m_{n,n}(q + 1) < m_{1,n}(q + 1) \ldots < m_{k-1,n}(q + 1)$ so that (6.3) and (6.4) are satisfied for $q + 1$ and $k \leqq h - 1$, let

$$m_{h,n-1}(r) > m_{h-1,n}(q + 1)$$

be such that

$$2(q + 1) \sum_{j=1}^{m_{h-1,n}(q+1)} u_n(j) < \sum_{j=1}^{m_{h,n-1}(r)} u_h(j).$$

Let

$$m_{h,n}(q + 1) = m_{h,n-1}(r);$$

let

$$c = 2 \sum_{j=1}^{m_{h-1,n}(q+1)} u_n(j)/(m_{h,n}(q + 1) - m_{h-1,n}(q + 1)),$$

and let

$$u_n(j) = \min(c, u_n(m_{h-1,n}(q+1)))$$

for $m_{h-1,n}(q+1) < j \leqq m_{h,n}(q+1)$. Then we have

$$\sum_{j=1}^{m_{h,n}(q+1)} u_n(j) \leqq 2 \sum_{j=1}^{m_{h-1,n}(q+1)} u_n(j)$$

$$< (1/q) \sum_{j=1}^{m_{h,n}(q+1)} u_h(j).$$

This completes the proof of a sequence which satisfies (a), (b) and (c′).

To complete the proof of the lemma we define sequences m_n for each n. Let

$$m_n(j) = m_{n,n+j-1}(j) \text{ for each } j.$$

Then $m_n[j \geqq h]$ is a subsequence of $m_{n,n+j-1}$ for all h so that (c) follows from(c′).

6.2 *Example*. A symmetric sequence space which is not barrelled in the $\beta\phi$ topology. Let (u_n) be a sequence in c_0 which satisfies the conclusion of Lemma 6.1. Let (v_n) be the collection of all finite sections of the u_n. That is, each v_n is equal to $u_p[\leqq k]$ for some p and some k. Let $w_n = u_n/n$ for each n. Let S consist of all sequences

$$s = \sum_n s_n + \sum_n t_n$$

such that

$$\sum_n p_n(s_n) + \sum_n q_n(t_n) < \infty$$

where

$$p_n(t) = \sup_k \sum_{j=1}^{k} \hat{s}(j) \Big/ \sum_{j=1}^{k} w_n(j)$$

$$q_n(t) = \sup_k \sum_{j=1}^{k} \hat{s}(j) \Big/ \sum_{j=1}^{k} v_n(j).$$

Then S is a BK-space with the norm

$$\|s\| = \inf\left\{ \sum_n p_n(s_n) + \sum_n q_n(t_n) : \sum_n s_n + \sum_n t_n = s \right\}.$$

We omit the proof that S is a BK space as well as the proof that S is symmetric and normal.

For each n and each k, $u_n[\leqq k]$ is some v_m so it is in the unit ball U of S. On the other hand if

$$\sum_n s_n + \sum_n t_n = u_k$$

then for all m

$$\sum_{j=1}^{m} \sum_n s_n(j) + \sum_{j=1}^{m} \sum_n t_n(j) = \sum_{j=1}^{m} u_k(j)$$

$$\sum_n \sum_{j=1}^{m} s_n(j) + \sum_n \sum_{j=1}^{m} t_n(j) = \sum_{j=1}^{m} u_k(j).$$

Thus for each $m_k(r)$

$$(6.5) \quad \sum_n \left(\sum_{j=1}^{m_k(r)} s_n(j) \Big/ \sum_{j=1}^{m_k(r)} w_n(j) \right) \left(\sum_{j=1}^{m_k(r)} w_n(j) \Big/ \sum_{j=1}^{m_k(r)} u_k(j) \right)$$

$$+ \sum_n \left(\sum_{j=1}^{m_k(r)} t_n(j) \Big/ \sum_{j=1}^{m_k(r)} v_n(j) \right) \left(\sum_{j=1}^{m_k(r)} v_n(j) \Big/ \sum_{j=1}^{m_k(r)} u_k(j) \right) = 1.$$

If $\sum_n p_n(s_n)$ and $\sum_n q_n(t_n)$ are finite the limit on the left hand side of (6.5) as $k \to \infty$ is

$$\sum_n p_n(s_n) \lim_k \left(\sum_{j=1}^{m_k(r)} w_n(j) \Big/ \sum_{j=1}^{m_k(r)} u_k(j) \right)$$

$$+ \sum_n q_n(t_n) \lim_k \left(\sum_{j=1}^{m_k(r)} v_n(j) \Big/ \sum_{j=1}^{m_k(r)} u_k(j) \right).$$

Because of Lemma 6.1

$$\lim_k \left(\sum_{j=1}^{m_k(r)} w_n(j) \Big/ \sum_{j=1}^{m_k(r)} u_k(j) \right) = \begin{cases} 1/k & \text{if } n = k \\ 0 & \text{if } n \neq k. \end{cases}$$

Since $v_n(j)$ is eventually 0 and $\sum_j u_k(j) = \infty$ for each k

$$\lim_k \left(\sum_{j=1}^{m_k(r)} v_n(j) \Big/ \sum_{j=1}^{m_k(r)} u_k(j) \right) = 0$$

for all n. Therefore, we conclude that if

$$\sum_n s_n + \sum_n t_n = u_k,$$

$p_k(s_k)$ is at least equal to k. This implies that

$$||u_k|| \geqq k \text{ for each } k.$$

If S were barrelled in the $\beta\phi$ topology there would be an S-bounded subset B of ϕ and positive numbers m and M such that

$$mp(s) \leqq ||s|| \leqq Mp(s)$$

for s in S where

$$p(s) = \sup\left\{ \left| \sum_j s(j)t(j) \right| : t \in B \right\}.$$

Since $||u_k[\leqq n]|| \leqq 1$ for each n and each k it follows that $p(u_k[\leqq n])$ $\leqq M$ for each n and each k. But because of the form of p it results that $p(u_k) \leqq M$ for all k contradicting the fact that (u_k) is unbounded in $(S, || ||)$.

References

1. N. Bourbaki, *Eléments de mathématique*, Livre V, *Espaces vectouriels topologiques*, 2 Vols. Act. Sci. et Ind. V 1189, 1229 (1953, 55).
2. G. Köthe, *Topological vector spaces I*, Grundleheren d. Math. Wissen. *159* (Springer, New York, Berlin, Heidelberg, 1969).
3. G. Köthe and O. Toeplitz, *Lineare Räume mit unendlich vielen Koordinaten und Ringe unendlicher Matrizen*, J. f. Math. *171* (1934), 193-226.
4. D. J. H. Garling, *On symmetric sequence spaces*, Proceedings of the London Math. Soc., (3) *16* (1966), 85-106.
5. ——— *Symmetric bases of locally convex spaces*, Studia Mathematica *30* (1968), 164-181.
6. ——— *A class of reflexive symmetric* BK-*spaces*, Can. J. Math. *21* (1969), 602-608.
7. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I, sequence spaces* (Springer, Berlin, Heidelberg, New York, 1977).
8. W. H. Ruckle, *Symmetric coordinate spaces and symmetric bases*, Can. J. Math. *19* (1967), 828-838.
9. ——— *On perfect symmetric sequence spaces*, Math. Annalen *175* (1968), 121-126.
10. ——— FK-*spaces in which the sequence of coordinate vectors is bounded*, Can. J. Math. *25* (1973), 973-978.
11. ——— *Representation and series summability of complete biorthogonal sequences*, Pacific J. Math. *34* (1970), 511-528.
12. ——— *Sequence spaces* (London, Pitman, 1981).
13. G. L. Seever, *Measures on F-spaces*, Trans. A.M.S. *133* (1968), 267-280.
14. I. Singer, *Bases in Banach spaces II* (Springer, Berlin, Heidelberg, New York, 1981).
15. A. Wilansky, *Functional analysis* (Blaisdell, New York, 1964).

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