PAPER 51

THEORY OF ISOTROPIC MAGNETIC TURBULENCE IN GASES

S. A. KAPLAN

Astronomical Observatory, University of L'vov, U.S.S.R.

ABSTRACT

A system of spectral equations of magnetic turbulence in gases differing from that given by Chandrasekhar is suggested. The solution of this system is examined. Correlation and structure functions of the turbulence of interstellar gases, determined according to the data on radial velocities of interstellar clouds from Adams's catalogue, are given. For motions of a scale less than the fundamental one (l less than 80 pc) the spectral function $F(k)$ is about $k^{-1.71}$, which agrees with the theoretical conclusions.

I

1. The theory of isotropic turbulence of gases in the magnetic field (gasomagnetic turbulence) can be developed at present only by means of spectral methods. The correlative method does not permit, in general, to take into account the dissipation of energy in the shock-waves, arising as a result of 'supersonic' turbulence.

The theory of the isotropic turbulence in gases can be applied only in the absence both of the mean directed flow of gases and of the mean directed magnetic field, i.e. only in the case, when all directions of the velocity vectors and of the magnetic field are equally probable. Such is the case, for instance, when an originally weak magnetic field has been increased as a result of an 'entanglement' of the magnetic lines of force caused by turbulent movements of ionized gases. The properties of motions in the interstellar space and in the nebulae can evidently be explained in the same mannner.

2. The author offered in 1953–4 the following system of spectral equations [1], [2] of magnetic turbulence in gases:

$$\epsilon_k = 2\left(\nu + \kappa_f \int_k^\infty \sqrt{\frac{F(k)}{k^3}}\, dk\right) . \int_0^k F(k)\, k^2 dk + 2\int_0^k \sqrt{[F(k)\, k^3]}\, [\zeta_f(k)\, F(k) + \mu(k)\, G(k)]\, dk, \qquad (1)$$

$$\epsilon_m = 2\left(\lambda + \kappa_g \int_k^\infty \sqrt{\frac{F(k)}{k^3}}\, dk\right) \int_0^k G(k)\, k^2\, dk$$
$$- 2\int_0^k \sqrt{[F(k)\, k^3]}\, [\zeta_g(k) + \mu(k)]\, G(k)\, dk, \qquad (2)$$

where ν is the viscosity; $\lambda = c^2/4\pi\sigma$ (σ = conductivity); $k = 2\pi/r$ (r is the characteristic scale of motion); $F(k)$ is the spectral density of kinetic energy, $G(k)$ is the spectral density of magnetic energy (referred to the unity of mass); κ_f and κ_g are some dimensionless quantities of the order of unity, $\zeta_f(k)$, $\zeta_g(k)$ and $\mu(k)$ are dimensionless, slowly varying functions;

$$0 \lesssim \zeta_g(k) \lesssim \zeta_f(k) \lesssim 1, \quad -1 \lesssim \mu(k) \lesssim +1.$$

The first two members of the right part of Eq. (1) have the same meaning as in Heisenberg's theory[3] of turbulence of incompressible fluids. The first two members of the right part of Eq. (2) are analogous. In detail, the member with κ_g describes the transmission of magnetic energy from big vortices to the lesser ones, occurring simultaneously with the decay of kinetic energy (member with κ_f) of big vortices. The member

$$2\int_0^k \mu(k) \sqrt{[F(k)\, k^3]}\, G(k)\, dk,$$

positive in (1) and negative in (2), describes an increase or a decrease of magnetic energy as a result of 'entanglements' or respectively 'dis-entanglements' of the lines of force. The function $\mu(k)$ depends, actually, upon the relation F/G, being positive at $F \gtrsim G$ and negative at $F \gtrsim G$ and $\mu = 0$ at $F \approx G$.

The member with $\zeta_f(k)$ describes the dissipation of the kinetic energy in shock-waves and the member with $\zeta_g(k)$ describes the corresponding increase of the magnetic energy. Functions ζ_f and ζ_g depend upon the relation of gas velocities to the velocity of sound and tend to acquire constant and positive values with the increase of these relations.

Finally, the members ϵ_k and ϵ_m respectively describe the total dissipations of the kinetic and magnetic energies in the vortices with wave numbers in the intervals between 0 and κ. In steady states ϵ_k = constant and ϵ_m = constant. In the case of a decay of turbulence:

$$\epsilon_k = -\frac{\partial}{\partial t}\int_0^k F(k, t)\, dk, \quad \epsilon_m = -\frac{\partial}{\partial t}\int_0^k G(k, t)\, dk. \qquad (3)$$

It is necessary to note that, though the system of spectral equations (1) and (2) is postulated arbitrarily, the choice of members with magnetic energy is substantially limited by the linear character of these equations in respect to $G(k)$ inasmuch as the corresponding equations of magnetic gas dynamics are also linear. The choice of the two members in (2), describing the changes of magnetic energy (with κ_g and μ) is due to two members from the corresponding correlative equations (11). There are several other physical considerations in favour of this choice of systems (1) and (2). The values κ_f and κ_g, as well as the functions $\zeta_f(k)$ and $\zeta_g(k)$ should be known, the function $\mu(k)$ is determined from the conditions of compatibility of these equations.

3. Systems (1) and (2) have two solutions for the spectral region of small wave numbers (analogous to the spectral region by Kolmogoroff):

(A)
$$\left.\begin{aligned} F(k) &= G(k) = F_0(k_0/k)^{[5/3+(32/27)(\zeta_f-\zeta_g)/(\kappa_f+\kappa_g)+\dots]}, \\ \mu &= -(\kappa_g\zeta_f+\kappa_f\zeta_g)/(\kappa_g+\kappa_f), \end{aligned}\right\} \tag{4}$$

F_0 and k_0 are arbitrary constants for the case of steady state. This solution corresponds to the case, when the magnetic and kinetic energies are in equilibrium.

(B)
$$\left.\begin{aligned} F(k) &= F_0(k_0/k)^{[5/3+(32/27)\zeta_f/\kappa_f+\dots]}, \\ G(k) &= \frac{2}{3}\,\frac{3\kappa_f\lambda-4\kappa_g\nu}{\kappa_g^2}\,\sqrt{(F_0k_0)}\left(\frac{k_0}{k}\right)^{[1/3+(16/27)\zeta_f/\kappa_f+\dots]}, \\ \mu &= \tfrac{3}{8}\kappa_g-\frac{5}{12}\,\frac{\kappa_g}{\kappa_f}\,\zeta_f-\zeta_g-\dots. \end{aligned}\right\} \tag{5}$$

The second solution corresponds to the case when the magnetic energy is mainly concentrated in vortices of the inner scale of turbulence (i.e. the case investigated by Batchelor in 1950 [4]). When analyzing the solution. (B) we suggested $\kappa_g \approx \kappa_f\lambda/\nu$, if $\lambda<\nu$ and $\kappa_g\approx\kappa_f$, if $\lambda \gtrsim \nu$. In this case $\epsilon_k/\epsilon_m \approx \sqrt{(F_0/k_0)}/\nu \approx Re$. Solution (B) may be conventionally called a quasi-stationary one.

If the dissipation of energy in shock waves can be disregarded (hydromagnetic turbulence) we shall have in equations (1), (2), (4) and (5) $\zeta_f=\zeta_g=0$.

The author investigated also the structure of spectra at different values of wave numbers. Unsteady magnetic turbulence in gases was also studied.

4. It was often supposed that the equilibrium of kinetic and magnetic energies (solution (A)) cannot occur in the presence of magnetic turbulence in gases, because the magnetic field of a large scale suppresses gas motions of lesser scales. This makes the movements more regular, which contradicts the statistical character of turbulence. Such regulation of the movements does not occur in the case of solution (B). We may suppose that solution (A) can also be realized, but in this case vortices of different scales

should be more isolated than in the case of absence of a magnetic field (or solution (B)). Here the space fluctuation in the density of magnetic energy and consequently of the kinetic energy must be far greater.

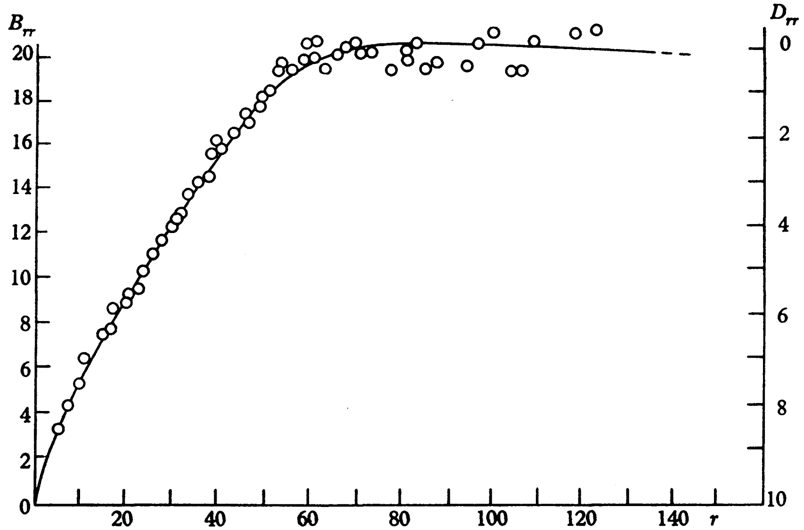

Fig. 1. The correlative and structural functions of interstellar turbulence.

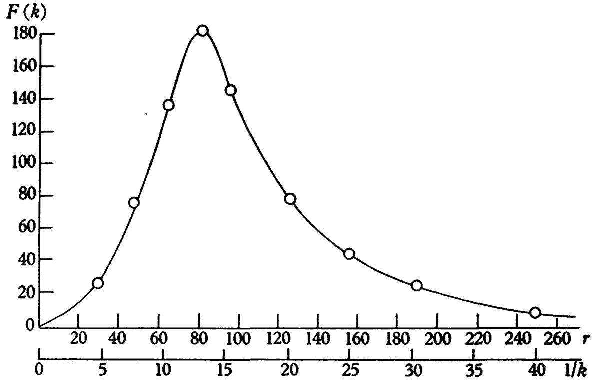

Fig. 2. The spectral function of interstellar turbulence.

5. The theory of magnetic turbulence in gases describes satisfactorily the properties of chaotic motions in interstellar gases and nebulae. Fig. 1 shows the correlation (B_{rr}) and structural (D_{rr}) functions of the turbulence of interstellar gases, found by the author [5] on the basis of Adams's catalogue of radial velocities of interstellar clouds. Fig. 2 shows spectral function

https://doi.org/10.1017/S0074180900238072 Published online by Cambridge University Press

$F(k)$ calculated according to the data of Fig. 1. In the region of motions of lesser scale than the principal one ($r < 90$ pc) we have $F(k) \sim k^{-1.71}$, which is in good agreement with (4) and (5).

2

In 1955 Chandrasekhar proposed another system of spectral equations of hydromagnetic turbulence [6]

$$\frac{1}{2}\frac{\partial F(k)}{\partial t} = \kappa\sqrt{\frac{F(k)}{k^3}}\int_0^k F(k)\,k^2 dk + \kappa\sqrt{\frac{G(k)}{k^3}}\int_0^k G(k)\,k^2 dk - \kappa F(k)\,k^2\int_k^\infty \sqrt{\frac{F(k)}{k^3}}\,dk - \kappa G(k)\,k^2\int_k^\infty\sqrt{\frac{F(k)}{k^3}}\,dk - \kappa F(k)\,k^2\int_k^\infty\sqrt{\frac{G(k)}{k^3}}\,dk - \nu F(k)\,k^2, \tag{6}$$

$$\frac{1}{2}\frac{\partial G(k)}{\partial t} = \kappa\sqrt{\frac{F(k)}{k^3}}\int_0^k G(k)\,k^2 dk + \kappa\sqrt{\frac{G(k)}{k^3}}\int_0^k F(k)\,k^2 dk - \kappa G(k)\,k^2\int_k^\infty\sqrt{\frac{G(k)}{k^3}}\,dk - \lambda G(k)\,k^2. \tag{7}$$

The symbols used in these equations are the same as defined in § 1, κ is the numerical constant equal to all members.

In our theory is also taken into account the dissipation of energy in shock-waves. But here, as we are interested in making a comparison of the system of Eqs. (1) and (2) with Chandrasekhar's we have excluded these members. Our system in this case, written in a differential form is as follows:

$$\frac{1}{2}\frac{\partial F(k)}{\partial t} = \kappa_f\sqrt{\frac{F(k)}{k^3}}\int_0^k F(k)\,k^2 dk - \kappa_f F(k)\,k^2\int_k^\infty\sqrt{\frac{F(k)}{k^3}}\,dk - \mu(k)\,G(k)\,\sqrt{[F(k)\,k^3]} - \nu F(k)\,k^2, \tag{8}$$

$$\frac{1}{2}\frac{\partial G(k)}{\partial t} = \kappa_g\sqrt{\frac{F(k)}{k^3}}\int_0^k G(k)\,k^2 dk - \kappa_g G(k)\,k^2\int_k^\infty\sqrt{\frac{F(k)}{k^3}}\,dk + \mu(k)\,G(k)\,\sqrt{[F(k)\,k^3]} - \lambda G(k)\,k^2. \tag{9}$$

It is necessary to point out that Eqs. (6) and (7) are the reduction to spectral language of the system of correlative equations of isotropic hydromagnetic turbulence, found by Chandrasekhar [6]:

$$\left.\begin{aligned} \frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial t^2} - \nu^2 D_5^2\right) Q &= -2Q\frac{\partial}{\partial r}D_5 Q - 2H\frac{\partial}{\partial r}D_5 H,\\ \left(\frac{\partial^2}{\partial t^2} - \lambda^2 D_5^2\right) H &= -2QD_5 H - 2HD_5 Q - 2\frac{\partial Q}{\partial r}\frac{\partial H}{\partial r},\end{aligned}\right\} \tag{10}$$

https://doi.org/10.1017/S0074180900238072 Published online by Cambridge University Press

where $Q(r, t)$ and $H(r, t)$ are correlative scalars, determining the correlation of second-order tensors with the components of velocity and components of strength of the magnetic field in two points of the fluid M' and M'', respectively, so that $r=|r''-r'|$ and $t=t''-t'$. D_5 is a differential operator.

The system (8) and (9) is a reduction to spectral language of another system of correlative equations, also found by Chandrasekhar [7].

$$\left.\begin{aligned} &\frac{\partial Q}{\partial t}-2\nu D_5 Q=2\left(r\frac{\partial}{\partial r}+5\right)(T-S), \\ &\frac{\partial H}{\partial t}-2\lambda D_5 H=2P. \end{aligned}\right\} \tag{11}$$

Here the correlative scalars were taken for the same moments, i.e. $t''=t'$ and T, S, P are correlative scalars, determining the correlative third-order tensors. System (11) is derived from equations of magnetic hydrodynamics, supposing the turbulence to be of homogeneous and isotropic nature, while for the derivation of system (10) Chandrasekhar used the hypothesis by Millionschchikov [8] about the relation of the fourth correlative moments to the second correlative moment. The correctness of this hypothesis in the complicated case of hydromagnetic turbulence was not clear. Moreover, it is necessary to show that this supposition (being not very correct even in the case of hydrodynamic turbulence) does not lead to wrong results. An introduction of this hypothesis may be justified, if the derivation of system (10) is the final aim, because system (10) is total and may be solved, contrary to Eq. (11). However, the use of this hypothesis was not suitable as the heuristic ground of the spectral theory, in the case when some other physical hypotheses must be made. System (11) is better for this purpose, because at its derivation no arbitrary mathematical hypothesis has been made.

System (10) did not suit as a heuristic ground for the derivation of a spectral theory also for the reason shown below. It is evident that the left part of correlative equations was directly reduced to the spectral theory. For the reduction to the spectral theory of the right part of this equation, i.e. namely of the non-linear term, the hypotheses must be introduced. Comparing the left part of Eqs. (8) and (9) with (11) we see that Eqs. (8), (9)–(11) pass directly from one to the other. In (8) and (9) and (11) the members of the left part are also the first derivative of the energy with time. But the left part of Eq. (10) is not the first derivatives of energy with time. Therefore the right part of Eq. (10) cannot be directly related to the right part of Eqs. (6) and (7).

From these considerations one may say that system (11) is much more suitable than system (10), as the heuristic ground for the postulation of the spectral theory.

But Chandrasekhar's system of spectral equations proceeds, in the main, from system (10). For instance, the term $\sqrt{[G(k)/k^3]}$ describes 'turbulent resistance', in the same way as the term $\sqrt{[F(k)/k^3]}$ describes 'turbulent viscosity'. This is explained only by the symmetry of the right part of (10) in respect to Q and H. The same explanation was given by him in respect to the equality of the values of κ in Eqs. (6) and (7). However, the appearance of the term $\sqrt{[G(k)/k^3]}$ is difficult to interpret from a physical point of view. Indeed, there is dissipation of the magnetic energy, the same as of the kinetic energy (i.e. a transfer from big scale motions to small scale motions is taking place). But it must be kept in view, that the transfer of magnetic energy between the vortices of different scales is not determined by the magnetic field (because the Maxwell equations are linear), but are connected with the motions of fluids, transferring the magnetic energy according to the principle of 'frozen' lines of magnetic forces. Thus, the 'turbulent conductivity' does not depend upon the parameters $\sqrt{[G(k)/k^3]}$ but depends upon the parameters $\sqrt{[F(k)/k^3]}$. There was an analogy with molecular viscosity and conductivity, which depend also upon the velocity and the length of the free path and does not depend upon the magnetic field (in isotropic conductor). It is also clear that the 'turbulent viscosity' $\kappa_f \sqrt{[F(k)/k^3]}$ and the 'turbulent resistance' $\kappa_g \sqrt{[F(k)/k^3]}$ could not be quite equal and we choose therefore different values for the constants κ_f and κ_g.

We want particularly to point out that, owing to the term $\sqrt{[G(k)/k^3]}$ Eqs. (6) and (7) of Chandrasekhar are not linear in respect to the magnetic energy, while the fundamental hydromagnetic equation and Maxwell's equation, as well as the correlative equations, are linear in respect to magnetic energy. The non-linearity of (10) was made artificially by introducing tensors of higher orders. The linearity of the systems of spectral equations in respect to $G(k)$ was one of the chief requirements followed in the derivation of Eqs. (8) and (9).

The member $\mu(k)\, G(k) \sqrt{[F(k)\, k^3]}$ in Eqs. (8) and (9) describes the transfer of kinetic energy into the magnetic energy, owing to 'entanglements' or 'disentanglements' of magnetic lines of force. Indeed, it is known that the density of magnetic energy increases owing to an 'entanglement' of the magnetic lines of force, proportionally to $\left(\frac{\partial v}{\partial x}\right)_H \cdot \frac{H^2}{8\pi}$, where $\left(\frac{\partial v}{\partial x}\right)_H$ is the gradient of velocity in the direction of the vectors of the magnetic field. In spectral terms this expression was written in the form given above. The presence of dimensionless functions $\mu(k)$ is explained by the necessity to describe the direction of the transfer of energy. This function may be defined by the conditions of compatibility of Eqs. (8) and (9). We may also suppose that an exchange of kinetic and magnetic energies in a definite scale of motions does not depend on the motion in other scales. Therefore, the member $\mu(k)\ G(k)\ \surd[F(k)\ k^3]$ was written in an integral form. It is possible, however, that this simple form would be insufficient in the future, we shall then easily write it in an integral form. This is not necessary at present. There are no members describing this process clearly in Chandrasekhar's system.

As a summary of all that has been said above we arrive at the conclusion that the system of spectral equations of hydromagnetic turbulence proposed by Chandrasekhar does not reflect the physical process taking place in this case. This system does not satisfy the requirements of linearity in respect to the magnetic energy. According to our opinion, the system (8) and (9) is better suited to the physical picture of phenomena of hydromagnetic turbulence.

We point out in conclusion that the solution of Eqs. (6) and (7) found by Chandrasekhar is also difficult to explain from the physical point of view. Indeed, Chandrasekhar [6] found two solutions in both of which there is the equipartition of the kinetic and magnetic energies in vortices of the biggest scales ($k \to 0$), i.e. in such scales of motions, in which systems (6) and (7) are not quite correct. The relation $G(k)/F(k)$ tends to zero in the first Chandrasekhar's solution and to 2·6 in the second solution with increasing k. In the second case the magnetic energy is always greater, than the kinetic one.

Meanwhile, we can think that on the whole the inequality $G(k) \leqslant F(k)$ must be fulfilled, because in the reverse case the magnetic field suppresses the motion of fluids. Furthermore, as the external energy is transferred into turbulence in the shape of big vortices we can think that it must be $G(k)/F(k) \to 0$ since $k \to 0$. We can suppose, at last, that among the possible solutions of spectral equations, there must be an equipartitional solution for sufficiently large intervals of wave numbers. Solutions of systems (6) and (7) do not satisfy any of these requirements.

The solutions of our system of spectral equations, given in paragraph 1, satisfy these requirements, because they are more probable as compared with Chandrasekhar's solution.

REFERENCES

[1] Kaplan, S. A. *Circ. Lvov astron. obs.* no. 25, 1953.
[2] Kaplan, S. A. *C.R. Acad. Sci. U.R.S.S.* **94**, 33, 1954; *J. Exp. Theor. Phys.* **27**, 699, 1954.
[3] Heisenberg, W. *Z. Phys.* **124**, 628, 1948.
[4] Batchelor, G. K. *Proc. Roy. Soc.* A, **201**, 405, 1950.
[5] Kaplan, S. A. *A.J. U.S.S.R.* **32**, 255, 1955.
[6] Chandrasekhar, S. *Proc. Roy. Soc.* A, **233**, 322 and 390, 1955.
[7] Chandrasekhar, S. *Proc. Roy. Soc.* A, **204**, 435, 1951.
[8] Millionschchikov, M. D. *C.R. Acad. Sci. U.R.S.S.* **32**, 611 and 615, 1941.