THE HIGHER HOMOTOPY GROUPS OF THE p-SPUN TREFOIL KNOT

by W. A. McCALLUM

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1. Introduction. In this paper we show that the $(p+1)$st homotopy group of the p-spun trefoil knot is nontrivial. This result was obtained for $p = 1$ in [**1**] using duality arguments. Here we take a totally different approach via the algorithm given in [**3**] and a module representation giving a simpler and more natural argument.

The first homotopy group of the complement of the trefoil knot k contained in the standard 3-Ball is given by

$$\pi_1(B^3 - k) = (x, t : txt = xtx)$$

DEFINITION 1.1. One obtains the $(p+1)$-dimensional knot K^{p+1} by p-spinning a knot k as follows:

$$S^{p+3} = (S^p \times B^3) \cup (D^{p+1} \times \partial B^3)$$

identified along:

$$S^p \times \partial B^3 = \partial D^{p+1} \times \partial B^3$$

and:

$$K^{p+1} = (S^p \times k) \cup (D^{p+1} \times \partial k)$$

identified along:

$$S^p \times \partial k = \partial D^{p+1} \times \partial k.$$

LEMMA 1.2. $\pi_1(S^{p+3} - K^{p+1}) = \pi_1(B^3 - k)$.

Proof. See [**4**].

LEMMA 1.3. $\pi_{p+1}(S^{p+3} - K^{p+1}) = (X : (1 - t + xt)X)$.

Proof. Via [**3**] we have

$$\pi_{p+1}(S^{p+3} - K^{p+1}) = \left(X, \frac{\partial r}{\partial x} X\right),$$

where $r = xtxt^{-1}x^{-1}t^{-1}$, which will yield the lemma.

2. Module representations.

DEFINITION 2.1. If M and M' are left modules over rings R and R' respectively, then a pair of homomorphisms (ϕ, σ) is a module representation of M in M' if and only if

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(1) ϕ: $R \to R'$ is a ring homomorphism.
(2) σ: $M \to M'$ is a group homomorphism, and
(3) $\sigma(rm) = \phi(r)\sigma(m)$ for all $m \in M$ and $r \in R$.

Given a representation (ϕ, σ), we say that σ is induced by ϕ, as condition (3) ensures that σ is a left-module map, where the action of R on M' is via ϕ.

THEOREM 2.2. *The $(p+1)$st homotopy group of the p-spun trefoil knot is nontrivial.*

Proof. Consider the group of the trefoil $H = (t, x : txt = xtx)$ and form the group ring over the integers $ZH = R$. If we p-spin the trefoil, we obtain the $(p+1)$st homotopy module

$$M = (X : (1-t+xt)X).$$

Consider the group S_4 and form the group ring $ZS_4 = R'$. The group $M' = Z_8 + Z_8 + Z_8 + Z_8$ (Z_8 = Integers mod 8) can be considered as a left R'-module by letting S_4 act on M' by permuting the natural basis for M'. A single nontrivial representation will prove the theorem. However, we find all module representations of M in M' such that the homomorphism ϕ: $R \to R'$ is induced by a group homomorphism of H into S_4 and such that $\phi(t) = (1234)$. If $\phi(t) = (1234)$, then $\phi(x)$ must be a four-cycle because of the relation $xtx = txt$. Five of the six choices give rise to module representations of M in M'. These choices are

	$\phi(x)$	$\sigma(X)$
(1)	(1234)	0
(2)	(1342)	$(-4, -2, -3, 1)X$
(3)	(1423)	$(-2, -3, 1, -4)X$
(4)	(1243)	$(-3, 1, -4, -2)X$
(5)	(1324)	$(1, -4, -2, -3)X$

Here X is an arbitrary element of Z_8. For similar calculations of $\sigma(X)$ see [2]. For each $\phi(X)$ we may calculate $\sigma(X)$ by observing that $\sigma((1-t+xt)X) = 0$.

In case (1) we have

$$(1-(1234)+(13)(24))\sigma(X) = 0,$$

which leads to the following system of equations

$$y_1 - y_4 + y_3 = 0$$

$$y_2 - y_1 + y_4 = 0$$

$$y_3 - y_2 + y_1 = 0$$

$$y_4 - y_3 + y_2 = 0,$$

where $\sigma(X) = (y_1, y_2, y_3, y_4)$.

In case (2) we have

$$(1-(1234)+(143))\sigma(X) = 0,$$

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which leads to the following system of equations

$$y_1 - y_4 + y_3 = 0$$

$$y_2 - y_1 + y_2 = 0$$

$$y_3 - y_2 + y_4 = 0$$

$$y_4 - y_3 + y_1 = 0.$$

Calculating these two maps we obtain (1) and (2) above. The calculations of the remaining maps may be simplified by observing that each of the maps ϕ defined in (3)–(5) is a conjugate by (1234) of the preceding map and then showing that the choice of $Y = \sigma(X)$ is $\phi(t)Y'$, where Y' is the choice for the preceding map.

Thus we see that the $(p+1)$st homotopy group of the p-spun trefoil knot has several nontrivial representations in the module M', which provides another more generalized proof and result of the major result of [**1**], the nontrivialness of higher homotopy groups of higher dimensional knots.

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KANSAS STATE UNIVERSITY

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