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UNIFORMITIES ON A PRODUCT

ANTHONY W. HAGER

All topological spaces shall be uniformizable (completely regular Hausdorff). A uniformity on X shall be viewed as a collection μ of coverings of X, via the manner of Tukey [**20**] and Isbell [**16**], and the associated uniform space denoted μX. Given the uniformizable topological space X, we shall be concerned with compatible uniformities as follows (discussed more carefully in § 1). The fine uniformity α (finest compatible with the topology); the "cardinal reflections" α_m of α (m an infinite cardinal number); α_c, the weak uniformity generated by the real-valued continuous functions.

With μ standing, generically, for one of these uniformities, we consider the question: when is $\mu(X \times Y) = \mu X \times \mu Y$? For $\mu = \alpha_{\aleph_0}$ (the finest compatible precompact uniformity), the problem is equivalent to that of when

$$\beta(X \times Y) = \beta X \times \beta Y,$$

β denoting Stone–Čech compactification; this is answered by the theorem of Glicksberg [**9**]. For $\mu = \alpha$, we have Isbell's generalization [**16**, VII.32]. For $\mu = \alpha_c$, the question was raised and partially answered in case $X = Y$ by Onuchic [**18**]. These results will be discussed in some detail in §'s 1 and 5 below.

We round out the picture by treating all cases $\mu = \alpha_m$ (§'s 3 and 4). For $m = \aleph_1$, the problem is equivalent to Onuchic's problem, whose solution we complete (§ 6). For $m = \aleph_2$, we have the following amusing special case: Let $D(k)$ be the discrete space of power k. The equation

$$\alpha_{\aleph_2}(D(\aleph_0) \times D(\aleph_2)) = \alpha_{\aleph_2} D(\aleph_0) \times \alpha_{\aleph_2} D(\aleph_2)$$

holds if and only if the Continuum Hypothesis is true (4.3). The development adds some details to the case $\mu = \alpha$ (§ 5), and the results have application to the problem of the equality $v(X \times Y) = vX \times vY$, v being Hewitt realcompactification (§ 7).

1. Background. For more detail, see [**16**, chapter VII].

To begin with, if μ is a uniformity on X, then the collection of finite μ-covers generates basically a uniformity $\mu_{\aleph_0}$ (precompact reflection [**16**]), as first shown by Tukey [**20**]; likewise, the collection of countable μ-covers generates basically a uniformity $\mu_{\aleph_1}$ ([**16**, III.ex.2], called $e\mu$ in [**16**, VII.16]), as first shown for $\mu = \alpha$ by Shirota [**19**], and in general by Ginsberg and Isbell [**8**].

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That is to say, for $m = \aleph_0$ or $\aleph_1$, each μ-cover of power $<m$ has a star-refinement in μ of power $<m$. It is not known if this statement holds for arbitrary μ and arbitrary infinite m. However, it does hold (a) for any μ if the Generalized Continuum Hypothesis holds at m (Ganter **6**), or (b) for any m if μ has a basis of σ-discrete covers, and this is true of $\mu = \alpha$ (Vidossich **21**; Shapiro's result is closely related).

While (b), for $\mu = \alpha$, is mostly what we require, some further general remarks are needed. For arbitrary μ and m (and without *GCH*), there does exist "the reflection into uniform spaces with no uniformly discrete subspace of power m" [**16**, III, Exercise 2], which we call μ_m. μ_m lives on X, is coarser than μ, and has the characteristic property: $f: \mu X \to \nu Y$ uniformly continuous, and νY without u.d. sets of power m, imply that $f: \mu_m X \to \nu Y$ is uniformly continuous. It results that the passage $\mu X \mapsto \mu_m X$ is functorial, that is, if $f: \mu X \to \nu Y$ is uniformly continuous, then so is $f: \mu_m X \to \nu_m Y$.

For some special X, there is a very simple covering description of α_m; see 1.5 below.

Now α consists of all normal covers. Isbell calls a space X pseudo-m-compact (m or infinite cardinal) if each cover from α has a subcover of power $<m$ [**16**, p. 135]. With [**16**, II.33], one shows easily that:

1.1. *These conditions on X are equivalent.*

(a) *X is pseudo-m-compact.*

(b) $\alpha = \alpha_m$.

(c) *αX contains no uniformly discrete copy of $D(m)$.*

Here, $D(m)$ is the discrete space of power m.

The pseudocompactness number pX (or covering character of αX) is $\min\{k: k > |D|$ whenever D is uniformly discrete in $\alpha X\}$. Thus X is pseudo-pX-compact but not pseudo-q-compact for $q < pX$. Note that for X discrete, $pX = |X|^+$, where m^+ denotes the cardinal successor of m.

The uniformity $\alpha_{\aleph_0}$ has for basis the finite normal covers; it is the finest compatible precompact uniformity. It is not hard to show that $\alpha_{\aleph_0}X$ carries the weak uniformity generated by $C^*(X)$, the bounded continuous real-valued functions on X. Thus the completion of $\alpha_{\aleph_0}X$ is βX, the Stone–Čech compactification.

The uniform product $\mu X \times \nu Y$ carries the weak uniformity generated by the projections π_x and π_y. Since the completion of a product is the product of completions, and since a compact space has a unique compatible uniformity, the two equations $\alpha_{\aleph_0}(X \times Y) = \alpha_{\aleph_0}X \times \alpha_{\aleph_0}Y$ and $\beta(X \times Y) = \beta X \times \beta Y$ are equivalent. Thus, the theorem of Glicksberg [**9**] can be put in the following form.

1.2 (Glicksberg). *$\alpha_{\aleph_0}(X \times Y) = \alpha_{\aleph_0}X \times \alpha_{\aleph_0}Y$ if and only if either* (a) *$X \times Y$ is pseudo-$\aleph_0$-compact, or* (b) *one space is finite.*

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Pseudo-$\aleph_0$-compact spaces usually are called pseudocompact, by definition, each continuous real-valued function is bounded, or $C(X) = C^*(X)$, $C(X)$ being the set of continuous real-valued functions on X.

A space with the property that each family of (strictly) fewer than m open sets has open intersection will be said to be pseudo-discrete in degree m, or $P(m)$. $P(\aleph_1)$ spaces usually are called P-spaces.

The following is a remarkable generalization of 1.2.

1.3 (Isbell [**16**, VII.32]). $\alpha(X \times Y) = \alpha X \times \alpha Y$ *if and only if either* (a) *there is m for which $X \times Y$ is pseudo-m-compact and $P(m)$, or* (b) *one space, say X, is discrete and Y is $P(|X|^+)$.*

In § 5, a proof of this is described, and the connection with some of the results of this paper is discussed.

Some facts about $P(m)$ spaces shall be required in the sequel. The following shows that the spaces in 1.3(a) are just short of discrete.

1.4 (Isbell [**16**, VII.31]). *If X is pseudo-m-compact and $P(m^+)$, then X is discrete (of power $<m$).*

For $P(\aleph_1)$ spaces, there is a very simple basis for each α_m.

1.5. *If X is $P(\aleph_1)$, then each uniformity α_m has a basis of partitions into $<m$ clopen sets; and $\alpha_c = \alpha_{\aleph_1}$.*

Proof. Clearly, any clopen partition is normal. Hence, a clopen partition of power $<m$ is in α_m.

On the other hand, a basis for $\alpha_m X$ consists of covers $f^{-1}(\mathscr{U})$, where $f: \alpha_m X \to \rho M$ is uniformly continuous and onto, ρM is a metric uniform space, and $\mathscr{U}$ is a uniform cover of ρM [I; I.14]. Take such a cover $f^{-1}(\mathscr{U})$. Because f is onto, ρM has no uniformly discrete set of power m (i.e., $\rho_m M = \rho M$), and so by passing to a subcover we may suppose that $\mathscr{U}$ has power $<m$ [I; II.32]. In M, points are G_δ's, so each $f^{-1}(\{p\})$ is a G_δ in X, and is open since X is $P(\aleph_1)$. This shows that $f: X \to M$ is continuous when M is given the discrete topology. Hence, $f: \alpha X \to dM$ is uniformly continuous, where d is the discrete uniformity on M. Because the passage $\mu X \mapsto \mu_m X$ is functorial, $f: \alpha_m X \to d_m M$ is uniformly continuous. Because d is fine on a discrete space, d consists of all covers, and so d_m consists of all covers of power $<m$: so $\mathscr{U} \in d_m$. Now well-order $\mathscr{U}$ as $U_0, U_1, \ldots, U_\alpha, \ldots$, set $V_\alpha = U_\alpha - \bigcup\{U_\beta: \beta < \alpha\}$, and $\mathscr{V} \equiv \{V_\alpha\}_\alpha$. Then $\mathscr{V} \in d_m$ (but $\mathscr{V}$ is probably not in ρ_m), so that $f^{-1}(\mathscr{V}) \in \alpha_m$. Clearly, $f^{-1}(\mathscr{V})$ is a clopen partition of power $<m$, and $f^{-1}(\mathscr{V})$ refines $f^{-1}(\mathscr{U})$.

Thus, the clopen partitions of power $<m$ form a basis for α_m.

Finally: In general $\alpha_c \subset \alpha_{\aleph_1}$, of course. Now $\alpha_{\aleph_1}$ has a basis of countable partitions in clopen sets. Given one, $\{U_n\}_n$, define $f: X \to R$ by $f(x) = n$ if and only if $x \in U_n$. Then $f^{-1}(\mathscr{S}(1)) = \{U_n\}_n$, so $\alpha_{\aleph_1} \subset \alpha_c$.

In [**8**], the equation $\alpha_c = \alpha$ (which implies $\alpha_c = \alpha_{\aleph_1}$), for paracompact spaces, is studied; it holds for Lindelöf spaces of finite dimension. A topological equivalent of "$\alpha_c = \alpha_{\aleph_1}$" is not known.

2. Lemmas. We require some technical preliminary results about products which shall be used throughout the paper.

A zero-set in X is a set of the form $f^{-1}(0)$, for some $f \in C(X)$.

2.1. LEMMA. *If there are compatible uniformities μ and ν such that the uniformity of $\mu X \times \nu X$ is finer than that of $\alpha_{\aleph_0}(X \times Y)$, then the projections π_x and π_y are z-closed (i.e., carry zero-sets onto closed sets).*

Proof. Suppose $Z = f^{-1}(0)$, and $x_0 \notin \pi_x(Z)$. By replacing $f(x, y)$ by $|f(x, y)/f(x_0, y)| \wedge 1$, we may assume that $0 \leqq f \leqq 1$ and that $f(x_0, y) = 1$ for each $y \in Y$.

From the hypothesis, there are open covers $\mathscr{U} \in \mu, \mathscr{V} \in \nu$ with f varying $< \frac{1}{2}$ on each $U \times V$, for $U \in \mathscr{U}$, $V \in \mathscr{V}$. Choose $U \in \mathscr{U}$ with $x_0 \in U$. Then $U \cap \pi_x(Z) = \emptyset$, so x_0 is not in the closure of $\pi_x(Z)$.

See 5.1 for more extensive comment on this lemma and [**10**; **17**; **2**] for a discussion of the condition, "π_x is z-closed".

2.2 LEMMA. *If π_x is z-closed, then for each infinite cardinal m, either X is $P(m^+)$ or Y is pseudo-m-compact.*

Proof. This is proved in [**17**, 3.1] under the assumption that neither X nor Y is discrete. In case X is discrete, X is $P(m^+)$. In case Y is discrete but not pseudo-m-compact, then $|Y| \geqq m$. Let Y_0 be a subset of power m. If X is not $P(m^+)$, then there is a family $\{V_y : y \in Y_0\}$ of closed sets with non-closed union: say $x_0 \in \text{cl} \bigcup V_y - \bigcup V_y$. Define continuous $f: X \times Y \rightarrow R$ by: for $y \in Y_0$, $f(x, y) = 0$ if $x \in V_y$, $f(x, y) = 1$ if $x = x_0$; for $y \notin Y_0$, $f(x, y) \equiv 1$. Then $\bigcup V_y \subset \pi_x(f^{-1}(0))$, and $x_0 \notin \pi_x(f^{-1}(0))$, so that $x_0 \in \text{cl}\, \pi_x(f^{-1}(0)) - \pi_x(f^{-1}(0))$.

2.3 LEMMA. *Let X be $P(m)$ and not discrete, and let X and Y be pseudo-m-compact. Then, if π_x is z-closed, $X \times Y$ is pseudo-m-compact.*

This is proved in [**17**, 3.4], though stated in weaker form. The converse holds [**10**, 4.2]; compare [**17**, 3.4]. See § 5 on this.

Several times in the sequel, we shall reduce a problem $\alpha_m(X \times Y) = \alpha_m X \times \alpha_m Y$ to the corresponding problem for discrete spaces, and the following two lemmas deal with this.

2.4 LEMMA. *If E is uniformly discrete in αZ, then the relativization of $\alpha_m Z$ to E is $\alpha_m E$.*

Proof. Let μ be the relativization of $\alpha_m Z$ to E. μ consists of covers $\mathscr{U} \cap E$ where $\mathscr{U} \in \alpha_m Z$. Each of these is, clearly, normal in E and of power $< m$, i.e., in $\alpha_m E$.

On the other hand: The hypothesis says there is a cover $\mathscr{D} \in \alpha Z$ such that $\mathscr{D} \cap E = \{\{e\}: e \in E\}$. Let $\mathscr{V} \in \alpha_m E$. For $V \in \mathscr{V}$, let

$$U_V = \bigcup\{D \in \mathscr{D}: D \cap E \subset V\}.$$

Let $\mathscr{U} = \{U_V: V \in \mathscr{V}\}$. Evidently, $\mathscr{U} \cap E = \mathscr{V}$, $\mathscr{U}$ has power $|\mathscr{V}| + 1 < m$, and $\mathscr{U} \in \alpha Z$ because $\mathscr{D}$ refines $\mathscr{U}$. Thus, $\mathscr{V} \in \mu$.

2.5 LEMMA. *Let D_1 and D_2 be uniformly discrete in αX and αY, respectively. If $\alpha_m(X \times Y) = \alpha_m X \times \alpha_m Y$, then $\alpha_m(D_1 \times D_2) = \alpha_m D_1 \times \alpha_m D_2$.*

Proof. By 2.4, the relativization of $\alpha_m X$ and $\alpha_m Y$ to D_1 and D_2 are $\alpha_m D_1$ and $\alpha_m D_2$. Now the relativization of a product is certainly the product of relativizations, so $\alpha_m X \times \alpha_m Y$, relativized, is $\alpha_m D_1 \times \alpha_m D_2$.

Now, $D_1 \times D_2$ is, clearly, uniformly discrete in $\alpha X \times \alpha Y$, and hence uniformly discrete in the finer uniformity of $\alpha(X \times Y)$. By 2.4, relativizing $\alpha_m(X \times Y)$ yields $\alpha_m(D_1 \times D_2)$.

The result follows.

3. The uniformities α_m, non-discrete case.

Here the situation is rather simple.

3.1 THEOREM. *If neither X nor Y is discrete, then $\alpha_m(X \times Y) = \alpha_m X \times \alpha_m Y$ if and only if there is $k \leqq m$ such $X \times Y$ is pesudo-k-compact and $P(k)$.*

Proof. Suppose that $X \times Y$ is pseudo-k-compact; then so are X and Y. If $k \leqq m$, then for all three spaces, $\alpha = \alpha_k = \alpha_m$. The result now follows by Isbell's Theorem 1.3 (the case $k = \aleph_0$ being Glicksberg's Theorem 1.2). (In § 5 we show how to prove directly the appropriate part of 1.3.)

Conversely, let $a_m(X \times Y) = \alpha_m X \times \alpha_m Y$. Since $\alpha_{\aleph_0} \subset \alpha_m$, the hypothesis of 2.1 holds choosing α_m for μ and ν. Thus π_x and π_y are z-closed.

Now, since neither X nor Y is discrete, the family $\{q: X$ and Y are $P(q)\}$ has supremum k. And X and Y are both $P(k)$, while one, say X, is not $P(k^+)$. 2.2 asserts that Y is pseudo-k-compact. Thus Y too cannot be $P(k^+)$ (for, if Y were, Y would be discrete (1.4)). Again by 2.2, X is pseudo-k-compact. By 2.3, $X \times Y$ is pseudo-k-compact.

We show $k \leqq m$. If $k > m$, then $k \geqq m^+$ and X and Y are $P(m^+)$. Thus neither is pseudo-m-compact (for, if one were, it would be discrete (1.4)). Thus, each of αX and αY contain a uniformity discrete copy of $D(m)$, the discrete space of power m. By 2.5, we must have $\alpha_m(D(m) \times D(m)) = \alpha_m D(m) \times \alpha_m D(m)$. This is false, by the general result 4.4, or by the following direct argument.

Let $f: D(m) \times D(m) \to \{0, 1\}$ be the characteristic function of the diagonal. Evidently, f is uniformly continuous relative to $\alpha_{\aleph_0}$, hence relative to α_m. But any *product* cover on each member of which f varies $\leqq \frac{1}{2}$ must contain every $\{(x, x)\}$ $(x \in D(m))$, and thus have power $\geqq m$. So f is not uniformly continuous relative to $\alpha_m D(m) \times \alpha_m D(m)$.

4. The uniformities α_m, discrete case. When one space is discrete the situation is more complicated and the complete answer depends on the disposition of equations like $2^{\aleph_0} = \aleph_1$ (the Continuum Hypothesis) in 4.1(b) and more explicitly in 4.3 below. The following is the main result; we postpone its proof.

4.1 THEOREM. *Let X be discrete with $|X| < m$. Then, $\alpha_m(X \times Y) = \alpha_m X \times \alpha_m Y$ if and only if Y is $P(|X|^+)$ and either* (a) *$pY \leqq m$, or* (b) *$pY > m$, and $k^{|X|} < m$ whenever $k < m$.*

Using (a) we have:

4.2 COROLLARY. *$\alpha_{n^+}(D(n) \times Y) = \alpha_{n^+}D(n) \times \alpha_{n^+}Y$ if and only if Y is pseudo-n^+-compact and $P(n^+)$.*

Using (b), and specializing:

4.3 COROLLARY. *The Continuum Hypothesis is equivalent to the equation $\alpha_{\aleph_2}(D(\aleph_0) \times D(\aleph_2)) = \alpha_{\aleph_2}D(\aleph_0) \times \alpha_{\aleph_2}D(\aleph_2)$.*

In the same way, one derives an equivalent to the Generalized Continuum Hypothesis ($2^{\aleph_\gamma} = \aleph_{\gamma+1}$) by replacing in 4.3, $\aleph_0$ by $\aleph_\gamma$ and $\aleph_2$ by $\aleph_{\gamma+2}$.

4.4 THEOREM. *Let X be discrete with $|X| \geqq m$. Then*

$$\alpha_m(X \times Y) = \alpha_m X \times \alpha_m Y$$

if and only if Y is discrete with $k^{|Y|} < m$ when $k < m$.

Proof. Suppose that $\alpha_m(X \times Y) = \alpha_m X \times \alpha_m Y$. As at the start of the proof of 3.1, π_y is z-closed. X is not pseudo-$|X|$-compact, so Y is $P(|X|^+)$, by 2.2. Now $pY > m$ would imply both αX and αY containing a uniformly discrete copy of $D(m)$, and by 2.5 this is a contradiction. So $pY \leqq m$. Thus Y is pseudo-$|X|$-compact and hence discrete, by 1.4. Since $|Y| < m$, we can use 4.1, reversing the roles of X and Y. Since $pX > m$, we are in case (b), which asserts the desired condition.

The converse follows by 4.1(b) (reversing the roles of X and Y).

4.5 COROLLARY. *For $m = \aleph_0$ or $\aleph_1$, $\alpha_m(D(m) \times Y) = \alpha_m D(m) \times \alpha_m Y$ if and only if Y is finite.*

Proof. Use $k = 2$ in 4.4, and the fact that $2^{\aleph_0} \geqq \aleph_1$.

Proof of 4.1. Let Y be $P(|X|^+)$, and $\mathscr{G} \in \alpha_m(X \times Y)$. We are to produce $\mathscr{U} \in \alpha_m X, \mathscr{V} \in \alpha_m Y$ with $\mathscr{U} \times \mathscr{V} < \mathscr{G}$.

Since $|X| <$ m, we shall take $\mathscr{U} = \{\{x\} : x \in X\}$ and proceed as follows. For $x \in X$, let $\mathscr{G}_x$ be the projection on Y of the trace of $\mathscr{G}$ on $\{x\} \times Y$. In case X is finite, set $\mathscr{V} = \bigwedge\{\mathscr{G}_x : x \in X\}$. Otherwise, Y is at least $P(\aleph_1)$; so also are X, and $X \times Y$, and we may assume $\mathscr{G}$ a partition into clopen sets by 1.5. Then $\mathscr{V}' = \bigwedge\{\mathscr{G}_x : x \in X\}$ is a partition, consists of open sets because Y is $P(|X|^+)$, and hence is normal. The power of $\mathscr{V}'$ is $\leqq |\mathscr{G}|^{|X|}$, which in case (b) is $<m$. Then set $\mathscr{V} = \mathscr{V}'$. In case (a), ignore the power of $\mathscr{V}'$ and take $\mathscr{V}$ to be a normal refinement of $\mathscr{V}'$ of power $<pY \leqq m$.

Conversely, suppose $\alpha_m(X \times Y) = \alpha_m X \times \alpha_m Y$. Since $\alpha_{\aleph_0} \subset \alpha_m$, the hypothesis of 2.1 holds choosing α_m for μ and ν. Hence π_y is z-closed. Since X is not pseudo-$|X|$-compact, Y is $P(|X|^+)$ by 2.2.

Suppose that (a) and (b) both fail. Then, $pY > m$ and there is $k < m$ with $k^{|X|} \geqq m$. Now αY contains a uniformly discrete copy of $D(m)$. Using 2.5, we infer that $\alpha_m(X \times D(m)) = \alpha_m X \times \alpha_m D(m)$. We now show this to be false.

Fix a set K with $|K| = k$. Since $k^{|X|} \geqq m$, we may view $D(m)$ as a set of functions $f: X \to K$. Given $x \in X$ and $\gamma \in K$, let $E_\gamma^x \equiv \{f: \gamma = f(x)\}$. Then $\{E_\gamma^x: \gamma \in K\}$ is a partition of $D(m)$ into $\leqq k$ sets. Thus

$$\mathscr{G} \equiv \{\{x\} \times E_\gamma^x: x \in X, \gamma \in K\}$$

is a partition (hence a normal cover) of $X \times D(m)$ into $\leqq k \cdot |X| < m$ sets; that is, $\mathscr{G} \in \alpha_m(X \times D(m))$. Suppose $\mathscr{V} \in \alpha_m D(m)$ and

$$\{\{x\} \times V: x \in X, V \in \mathscr{V}\} < \mathscr{G}.$$

Then, given $V \in \mathscr{V}$, for each x, there is one $\gamma = \gamma(V, x)$ with $V \subset E_\gamma^x$; so $V \subset \bigcap \{E^x_{\gamma(V,x)}: x \in X\}$. Thus, if $f \in V$, $f(x) = \gamma(V, x)$ for each x, and $V = \{f\}$ follows. So each member of $\mathscr{V}$ is a singleton and $|\mathscr{V}| = m$, which is a contradiction.

5. Remarks on the fine uniformity. The following theorem clarifies the connection between Isbell's Theorem 1.3, Lemma 2.1, and the results of §'s 3 and 4.

5.1 PROPOSITION. *These conditions are equivalent.*

(a) *$\alpha X \times \alpha Y$ is fine.*

(b) *There is m such that the uniformity of $\alpha_m X \times \alpha_m Y$ is finer than that of $\alpha_{\aleph_0}(X \times Y)$.*

(c) *There is m such that $\alpha_m(X \times Y) = \alpha_m X \times \alpha_m Y$.*

Let us give a proof of this. The implications (a) $\Rightarrow$ (c) $\Rightarrow$ (b) are obvious. To show (b) $\Rightarrow$ (a), observe that by 2.1, both projections are z-closed. According to the following, this suffices.

5.2 (Noble [**17**, 3.5]). *If π_x and π_y are z-closed, then $\alpha X \times \alpha Y$ is fine.*

Proof. (The following proof is a little simpler than Noble's.) According to [**10**, 1.1], π_x is z-closed if and only if the semi-uniform product [**16**] $\alpha X * \alpha Y$ is fine. In general, the uniform product $\mu X \times \nu Y$ carries the coarsest uniformity finer than both semi-uniform products. Hence $\alpha X \times \alpha Y$ is fine.

Now, note that in the proofs of 3.1, 4.1, and 4.4 the only application of Isbell's Theorem 1.3 is in 3.1, and there is used only the implication (1.3(a) $\Rightarrow \alpha X \times \alpha Y$ is fine). To prove this, we need show only that π_x and π_y are z-closed, by 5.2. This follows by twice applying [**10**, 4.2], which asserts:

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if $X \times Y$ is pseudo-k-compact and X is $P(k)$, then π_x is z-closed. See also [**17**, 3.5].

Isbell's theorem now follows quite readily from 3.1, 4.1, and 5.1. The proof so given is, of course, somewhat similar to Noble's [**17**], but rather more direct.

For the moment, introduce the pseudo-discreteness number of a space: $dX = \max\{q: X \text{ is } P(q)\}$. Then X is $P(dX)$ but not $P(dX^+)$ (and this defines dX).

3.1 admits the following rephrasing.

5.2 PROPOSITION. *Suppose neither X nor Y is discrete. Then $\alpha X \times \alpha Y$ is fine if and only if the six numbers p and d for the spaces $X \times Y$, X, and Y are all the same.*

Now, when $\alpha X \times \alpha Y$ is fine, let m_1 be the least cardinal for which 5.1(c) holds. Evidently, $p(X \times Y) \geqq m_1$. From 3.1, we have

5.3 PROPOSITION. *If $\alpha X \times \alpha Y$ is fine, and neither X nor Y is discrete, then $p(X \times Y) = m_1$.*

But 4.1(b) yields the following.

5.4 *Example.* With $X = N$, $Y = D(2^{\aleph_0})$, and $m = 2^{\aleph_0}$, we have

$$\alpha_m(X \times Y) = \alpha_m X \times \alpha_m Y,$$

while $p(X \times Y) = m^+ > m = m_1$.

6. The uniformity α_c. In [**18**], Onuchic has shown that for the equality $\alpha_c(X \times X) = \alpha_c X \times \alpha_c X$, it is sufficient that $X \times X$ be pseudo-$\aleph_0$-compact, or Lindelöf and $P(\aleph_1)$, and necessary that X be either pseudo-$\aleph_0$-compact or pseudo-$\aleph_1$-compact and $P(\aleph_1)$. We shall sketch out the proof of the following.

6.1 THEOREM. *The following conditions are equivalent.*

(a) $\alpha_c(X \times Y) = \alpha_c X \times \alpha_c Y$.

(b) $\alpha_{\aleph_1}(X \times Y) = \alpha_{\aleph_1} X \times \alpha_{\aleph_1} Y$.

(c) *Either* (i) *one of X, Y is finite or $X \times Y$ is pseudo-$\aleph_0$-compact, or*
(ii) *$X \times Y$ is pseudo-$\aleph_1$-compact and $P(\aleph_1)$.*

(The conditions (c)(i) are the hypotheses of Glicksberg's Theorem 1.2, and since $\alpha_{c^*} = \alpha_{\aleph_0}$ always, are equivalent to the equality $\alpha_{c^*}(X \times Y) = \alpha_{c^*} X \times \alpha_{c^*} Y$, which is the "bounded version" of the equation under discussion.)

Proof. The equivalence of (b) of (c) follows quickly from 3.1, 4.1, and 4.2.

Assume (c). Except when one space is finite, the hypothesis guarantees by 1.5 that $\alpha_c = \alpha_{\aleph_1}$, for X, Y, and $X \times Y$. Thus (b) and (a) are the same statement. When one space is finite, say $X = \{x_1, \ldots, x_n\}$, and $\mathscr{G} \in \alpha_c(X \times Y)$, let $\mathscr{G}_i$ be the projection on Y of the trace of $\mathscr{G}$ on $\{x_i\} \times Y$. Evidently $\mathscr{G}_i \in \alpha_c Y$, so

$$\mathscr{V} = \bigwedge_{i=1}^{n} \mathscr{G}_i \in \alpha_c Y$$

as well. With $\mathscr{U} = \{\{x\}: x \in X\}$, $\mathscr{U} \in \alpha_c X$ and $\mathscr{U} \times \mathscr{V} < \mathscr{G}$, as desired.

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To prove (a) implies (c), it seems to be necessary to repeat much of the proof that (b) implies (c). If (a) holds, then the hypotheses of 2.1 hold (since $\alpha_{\aleph_0} \subset \alpha_c$) and both projections are z-closed. Using 2.3 with $m = \aleph_0$, either X is $P(\aleph_1)$ or Y is pseudo-$\aleph_0$-compact and either X is pseudo-$\aleph_0$-compact or Y is $P(\aleph_1)$. If one of X, Y is *both* pseudo-$\aleph_0$-compact and $P(\aleph_1)$, it is finite (1.4). If neither is finite and both are pseudo-$\aleph_0$-compact, then $X \times Y$ is finite, by 2.4. So assume that both spaces are infinite and $P(\aleph_1)$. So $X \times Y$ is $P(\aleph_1)$, and by 1.5, the hypothesis (a) is equivalent to (b). Since (b) implies (c), the result follows.

7. The equation: (*) $v(K \times Y) = vK \times vY$. Here, v denotes Hewitt's realcompactification [**12**; **1**]. The question of when (*) holds has lately attracted attention [**1**; **3**; **10**; **11**; **14**; **15**]. There would appear to be no simple answer.

In [**7**, chapter 15], vX is shown to be naturally homeomorphic to the topological space of the completion of $\alpha_c X$; α_c is often called the Nachbin uniformity, and the completion of $\alpha_c X$, the Nachbin completion; see [**7**, p. 271]. Now, the equation $\alpha_c(X \times Y) = \alpha_c X \times \alpha_c Y$ holds if and only if the completions of $\alpha_c(X \times Y)$ and $\alpha_c X \times \alpha_c Y$ are uniformly isomorphic by an extension of the identity map on $X \times Y$; and this implies, but is not implied by, (*). Thus the problem treated in § 6 is the uniform version of the problem of (*). But much more can be said.

In [**19**], Shirota proves that the completions of $\alpha_c X$ and $\alpha_{\aleph_1} X$ are homeomorphic, and in case X has no closed discrete set of measurable power, homeomorphic to the completion of αX. In [**16**, VII.18], Isbell generalizes, proving a theorem which implies: if m is nonmeasurable and $> \aleph_1$, then the completion of $\alpha_m X$ is homeomorphic to vX.

Thus, as above, sufficient topological conditions for (*) follow from any of 1.2, 1.3, 3.1, 4.1, 4.4.

This, too, can be generalized, as follows. It is easy to see that the completion of either of the semi-uniform products [**16**] of μX and νY is homeomorphic to the completion of $\mu X \times \nu Y$; that is, to the product of the completions. Thus, whenever $\alpha X * \alpha Y$ (or $\alpha Y * \alpha X$) is fine, and X and Y have no closed discrete sets of measurable power, the equation (*) holds. Sufficient conditions for $\alpha X * \alpha Y$ to be fine are given in [**10**]; e.g., if for some k, X is $P(k)$ and either $X \times Y$ is pseudo-k-compact, or Y is "weakly k-compact". (Recall from § 5 or [**10**, 1.1] that $\alpha X * \alpha Y$ is fine if and only if π_x is z-closed. Compare this derivation of (*) with that in [**3**].)

8. Infinite products. We include these comments at the referee's request. Let Λ be an infinite index set, and for each $\lambda \in \Lambda$, let X_λ be a topological space with at least two points. (The case of finite Λ is treated easily using the results of §'s 3 and 4.) Glicksberg [**9**] and Isbell [**16**], have considered the equations (1) $\alpha_{\aleph_0} \Pi X_\lambda = \Pi \alpha_{\aleph_0} X_\lambda$, (2) $\alpha \Pi X_\lambda = \Pi \alpha X_\lambda$, respectively, showing the equivalence of (1), (2), and (3) ΠX_λ is pseudo-$\aleph_0$-compact. (These extend 1.2 and 1.3.

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See [**16**, Chapter VII].) To these we add the infinite analogues of the conditions of 5.1: (4) There is m such that the uniformity of $\Pi\ \alpha_m X_\lambda$ is finer (= not coarser) than that of $\alpha_{\aleph_0}\ \Pi\ X_\lambda$. (5) There is m such that $\alpha_m\ \Pi\ X_\lambda = \Pi\ \alpha_m X_\lambda$.

We shall not give an independent proof of the equivalence of these five conditions, but assume the equivalence of (1), (2), (3).

It is clear that (1) $\Rightarrow$ (5) $\Rightarrow$ (4). We show (4) $\Rightarrow$ (1). Assume (4), let $\lambda_0 \in \Lambda$, and apply 2.1 to

$$X_{\lambda_0} \times \prod_{\lambda \neq \lambda_0} X_\lambda;$$

we find that the projection Π_{λ_0} onto X_{λ_0} is z-closed. Use 2.2 with

$$m = \aleph_0, \qquad X = \prod_{\lambda \neq \lambda_0} X_\lambda, \qquad Y = X_{\lambda_0};$$

X, like any nontrivial infinite product, is not $P(\aleph_1)$ (exercise), so X_{λ_0} is pseudo-$\aleph_0$-compact. This is true for each λ_0, so every X_λ is pseudo-$\aleph_0$-compact, and for each λ,

$$\alpha X_\lambda = \alpha_m X_\lambda = \alpha_{\aleph_0} X_\lambda.$$

So (4) reduces to:

$$\Pi\ \alpha_{\aleph_0} X_\lambda = \alpha_{\aleph_0}\ \Pi\ X_\lambda.$$

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Wesleyan University,
Middleton, Connecticut