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Mean Curvature Comparison with L^1-norms of Ricci Curvature

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Abstract. We prove an analogue of mean curvature comparison theorem in the case where the Ricci curvature below a positive constant is small in L^1-norm.

1 Introduction

The comparison geometry is a branch of Riemannian geometry that is related with investigating the structure of spaces satisfying some curvature conditions. Starting with the success of the Rauch comparison theorem, much developments have been made up to now and produced many applications such as sphere theorem, compactness theorem, finiteness theorem and so on. In particular, the mean curvature comparison theorem has played an important role in such theorems and it is deeply related to the volume comparison theorem, which is also an important result in the comparison geometry.

In 1998, P. Petersen and C. Sprouse [PS] generalized the classical Heinze-Karcher volume comparison result for hypersurfaces to a situation where one has an integral bound for the part of the Ricci curvature which lies below a given positive number. To obtain this result, they first generalized the classical mean curvature comparison theorem. In order to state these specifically, we need some notation as follows.

(M,g) is a complete Riemannian manifold with metric g. At each point x in this manifold, we denote by $\mathrm{Ric}_-(x)$ the lowest eigenvalue for the Ricci tensor at x. Let $S_x \subset T_xM$ denote the space of unit tangent vectors at x and $d(\theta)$ be the distance from x to the cut point in the direction $\theta \in S_x = S^{n-1} \subset T_xM$.

Then we define $\omega(r,\theta)$ by pulling back the volume form $d\,\mathrm{vol}$ of M to $U_x = \{(r,\theta) \in T_xM : 0 < r < d(\theta), \theta \in S_x\}$, *i.e.*,

$$d\,\mathrm{vol} = \omega(r,\theta)\,dt\,d\theta,$$

where $d\theta$ is the standard volume form on $S_x = S^{n-1}$.

For convenience, we define $\omega(r,\theta)$ to be zero for $r > d(\theta)$.

Let $\omega_k(r,\theta)$ be the $\omega(r,\theta)$ of the space form $\mathbb{S}^n_k$ of dimension n with constant curvature $k > 0$. We then know that $\omega' = h\omega$ (resp. $\omega'_k = h_k\omega_k$), where h (resp. h_k) is the mean curvature of the level sets of distance function on (M,g) (resp. $\mathbb{S}^n_k$).

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Recall that the classical mean curvature comparison theorem says that if $\mathrm{Ric}_- \geq (n-1)k$, then $h(r,\theta) \leq h_k(r)$.

In order to generalize this, the following are defined.

$$\psi_k(r,\theta) = \big(h(r,\theta) - h_k(r)\big)_+,$$
$$\rho_k(r,\theta) = \big((n-1)k - \mathrm{Ric}_-(r,\theta)\big)_+,$$

where $u_+ = \max(0,u)$ is the positive part of the function u.

With these notations, P. Petersen and C. Sprouse proved the desired mean curvature estimates as follows.

Theorem 1.1 (PS) *For all $n \geq 2$, $p > \frac{n}{2}$, $k > 0$, $r + r_0 < \frac{\pi}{\sqrt{k}}$, we have an estimate of the form*

$$\int_0^r \psi_k^{2p}(t,\theta)\omega\, dt \leq C(n,p,k,r) \int_0^r \rho_k^p(t,\theta)\omega\, dt,$$

where $C(n,p,k,r)$ is an explicit constant depending only on the variables indicated and θ is fixed.

The above theorem shows that the classical mean curvature comparison can be generalized to a situation where the amount of Ricci curvature which lies below $(n-1)k$ is small in L^p-sense for $p > \frac{n}{2}$.

For some analytic reason, the condition $p > \frac{n}{2}(\geq 1)$ in the above theorem is essential and the proof of the above theorem strongly relies on the condition that $p > \frac{n}{2}$, where the case $p = 1$ is excluded.

Generally, the geometry of manifolds with bounded Ricci curvature in L^1-sense is known to be not so interesting.

Indeed, S. Gallot [G] showed several examples that the geometry of manifolds which has small $\bar{k}(p,k,R) = \sup_{x\in M} \frac{1}{\mathrm{vol}\, B(x,R)} \int_{B(x,R)} \rho_k^p \, d\,\mathrm{vol}$, for any $p \leq \frac{n}{2}$, $R > 0$ does not give any interesting results.

Recently, C. Sprouse however managed to show that if one assumes that the manifold has $\mathrm{Ric}_- \geq -(n-1)k$ $(k > 0)$, then it suffices to assume that the amount of Ricci curvature which lies below $(n-1)$ in L^1-norm is small in order to get a diameter bound close to π. The precise statement of this theorem is as follows.

Theorem 1.2 (S) *Assume (M,g) is a complete Riemannian n-manifold with* $\mathrm{Ric}_- \geq -(n-1)k$ $(k > 0)$. *Then for given $\epsilon > 0$, $R > 0$ there exists an $\delta = \delta(\epsilon,R,k,n)$ such that if*

$$\sup_{x\in M} \frac{1}{\mathrm{vol}\, B(x,R)} \int_{B(x,R)} \big((n-1) - \mathrm{Ric}_-\big)_+ \, d\,\mathrm{vol} < \delta(\epsilon,R,k,n),$$

then (M,g) is compact with $\mathrm{diam}(M) < \pi + \epsilon$.

Motivated by this result, we investigate in this paper that if one assumes the manifold has $\mathrm{Ric}_- \geq -(n-1)k$ $(k > 0)$, then we can generalize the classical mean curvature comparison to a situation where we have Ricci curvature bounded in L^1-norm. Let's mention our main result.

Theorem 1.3 *Assume (M, g) is a complete Riemannian n-manifold with* $\mathrm{Ric}_{-} \geq -(n-1)k\,(k > 0)$. *Then for given $\epsilon > 0$ and $R \in (0, \pi)$, there exists a $\delta = \delta(\epsilon, R, k, n)$ such that if $\int_{B(x,R)} \rho_1 \, d\,\mathrm{vol} < \delta$, then $\int_{B(x,R)} \psi_1 \, d\,\mathrm{vol} < \epsilon$.*

Note that Theorem 1.2 gives a diameter structure for manifolds in the case where Ricci curvature below a positive constant is small in L^1-sense averaged over the metric balls. Here, averaging the 'bad' part of Ric_{-} over metric balls is reasonable, since one can have small $\int_{B(x,R)}((n-1)-\mathrm{Ric}_{-})_{+} \, d\,\mathrm{vol}$ by simply having small volume of $B(x, R)$.

In fact, it is shown in [CK] that for any complete Riemannian manifold with non-negative Ricci curvature, $\inf_x \mathrm{vol}\, B(x, 1)$ can be zero.

As a corollary of Theorem 1.3, we can however provide a corresponding volume structure of the space in the Theorem 1.2, where the requirement of averaging the 'bad' part of Ric_{-} over metric balls is not necessary.

Corollary 1.4 *For given $R > \pi$, $\epsilon > 0$, $k > 0$, and an integer n, there exists a $\delta = \delta(\epsilon, R, k, n)$ such that if M is a complete n-manifold with $\int_{B(x,R)} \rho_1 \, d\,\mathrm{vol} < \delta$,* $\mathrm{Ric}_{-} \geq -(n-1)k\,(k > 0)$, *then* $\mathrm{vol}\big(B(x, R) - B(x, \pi)\big) < \epsilon$ *for all $x \in M$.*

2 Proof of Theorem 1.3

For the proof of Theorem 1.3, we will use the Paeng's method in [P]. Consider a sequence (M_i, g_i, x_i) of Riemannian n-manifolds with metrics g_i and $x_i \in M_i$ such that $\mathrm{Ric}_{M_i} \geq -(n-1)k\,(k > 0)$.

Let $\psi_i(r, \theta)$ and $\rho_i(r, \theta)$ be the $\psi_1(r, \theta)$ and $\rho_1(r, \theta)$ of (M_i, g_i) respectively. Then it suffices to show that if $\int_{B(x_i,R)} \rho_i \, d\,\mathrm{vol}$ converges to zero, then $\int_{B(x_i,R)} \psi_i \, d\,\mathrm{vol}$ also converges to zero.

Note that for any $\delta > 0$, $\mathrm{vol}\,(E^i_\delta) := \mathrm{vol}\,\{x \in B(x_i, R) : \rho_i(x) > \delta\}$ converges to zero, since

$$\int_{B(x_i,R)} \rho_i \, d\,\mathrm{vol} > \int_{E^i_\delta} \rho_i \, d\,\mathrm{vol} > \int_{E^i_\delta} \delta \, d\,\mathrm{vol} = \delta\,\mathrm{vol}\,(E^i_\delta).$$

Consider now a sequence $\{\delta_i(> 0)\}$ such that $\omega_{-k}(\delta_i) = \sqrt[8]{\epsilon_i}$, where $\epsilon_i := \mathrm{vol}\,(E^i_\delta)$. Let μ be the measure on $\gamma^i_\theta(t) = \exp_{x_i} t\theta$ and $d^i(\theta)$ be the distance from x_i to the cut point in the direction $\theta \in S^{n-1} \subset T_{x_i}M_i$.

We also write $\{\gamma^i_\theta(t) : a \leq t \leq b\}$ as $\gamma^i_\theta([a, b])$. Then we define

$$S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta) = \inf\Big\{ s : s > \delta_i, (\theta \in \Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c, \mu\big(\gamma^i_\theta([\delta_i, s]) \cap E^i_\delta\big) \geq \sqrt[4]{\epsilon_i} \Big\},$$

where

$$\Phi_{\sqrt[4]{\epsilon_i},\delta_i} = \Big\{ \theta \in S^{n-1} \subset T_{x_i}M_i : \mu\big(\gamma^i_\theta([\delta_i, \min(R, d^i(\theta))]) \cap E^i_\delta\big) < \sqrt[4]{\epsilon_i} \Big\}.$$

We first show the following lemma.

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Lemma 2.1

$$\lim_{i\to\infty} \operatorname{vol}\left\{ \exp_{x_i} t\theta : \theta \in (\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c, S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta) \le t \le \min\big(R, d^i(\theta)\big) \right\} = 0.$$

If $\operatorname{vol}(\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c$ converges to zero, then there is nothing to prove.

So we may assume that $\lim_{i\to\infty} \operatorname{vol}(\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c > 0$. Let

$$\Psi_{\sqrt[4]{\epsilon_i},\delta_i} = \left\{ \theta \in (\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c : \int_{\gamma_\theta^i([\delta_i, d^i(\theta)])\cap E_\delta^i} \omega(r,\theta)\, dr \ge \sqrt{\epsilon_i} \right\}$$

Then we have $\epsilon_i = \operatorname{vol}(E_\delta^i) > \sqrt{\epsilon_i}\operatorname{vol}(\Psi_{\sqrt[4]{\epsilon_i},\delta_i})$, which implies that $\operatorname{vol}(\Psi_{\sqrt[4]{\epsilon_i},\delta_i})$ converges to zero.

Thus we may assume that for every direction $\theta \in (\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c$,

$$\int_{\gamma_\theta^i([\delta_i, d^i(\theta)])\cap E_\delta^i} \omega(r,\theta)\, dr < \sqrt{\epsilon_i}.$$

We then know that there exists a $c_i > \delta_i$ such that

$$\omega(c_i,\theta) < \sqrt{\epsilon_i}/\sqrt[4]{\epsilon_i} = \sqrt[4]{\epsilon_i}$$

and

$$\mu\big(\gamma_\theta^i([\delta_i, c_i]) \cap E_\delta^i\big) \le \sqrt[4]{\epsilon_i}.$$

From this fact, we also know that $c_i \le S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta)$.

Now since $\operatorname{Ric}_{M_i} \ge -(n-1)k$, we have for any r with $S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta) \le r \le \pi$ and $\theta \in (\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c$,

$$\omega(r,\theta) \le \frac{\omega_{-k}(\pi)}{\omega_{-k}(\delta_i)}\omega(c_i,\theta) \le \frac{\omega_{-k}(\pi)}{\omega_{-k}(\delta_i)}\sqrt[4]{\epsilon_i} = \omega_{-k}(\pi)\sqrt[8]{\epsilon_i},$$

which converges to zero.

Thus we obtain the desired result. ∎

Now we consider

$$\begin{aligned} \int_{B(x_i,R)} \psi_i \, d\operatorname{vol} = \int_{B(x_i,\delta_i)} & \psi_i\omega_i \, dr\, d\theta \\ & + \int_{\Phi_{\sqrt[4]{\epsilon_i},\delta_i}} \int_{\delta_i}^{\min(R,d^i(\theta))} \psi_i\omega_i \, dr\, d\theta \\ & + \int_{(\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\delta_i}^{\min(R,d^i(\theta))} \psi_i\omega_i \, dr\, d\theta. \end{aligned} \tag{2.1}$$

Note that we can without loss of generality assume that $\delta_i \le d^i(\theta)$ and $\lim_{i\to\infty} d^i(\theta) > 0$ for all $\theta \in S^{n-1}$ in (2.1).

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We will estimate each term of the above sum in (2.1) as below.

First, we know that the first term in the above sum tends to zero as $i \to \infty$, since $\mathrm{Ric}_{M_i} \geq -(n-1)k$ and $\delta_i \to 0$.

Now let's estimate the second term in the above sum as follows.

Note first that in $(E^i_\delta)^c$, we have $(n-1) - \mathrm{Ric}_{M_i}(\partial_r, \partial_r) \leq \delta$, where $\partial_r = \partial/\partial_r$ is the radial unit vector, since $\rho_i(x) < \delta$.

Thus $\mathrm{Ric}_{M_i}(\partial_r, \partial_r) \geq (n-1)\big(1-\tau(\delta)\big)$ for some $\tau(\delta)$ which tends to zero as $\delta \to 0$.

So we have the following inequality for any $\theta \in \Phi_{\sqrt[4]{\epsilon_i}, \delta_i}$,

$$\int_{\gamma^i_\theta([\delta_i,R]) \cap (E^i_\delta)^c} \frac{(\frac{h_i}{n-1})'}{\big(1-\tau(\delta)\big) + (\frac{h_i}{n-1})^2}\, dr \leq \int_{\gamma^i_\theta([\delta_i,R]) \cap (E^i_\delta)^c} -1\, dr.$$

On the other hand, from the inequality

$$h_i' + \frac{h_i^2}{n-1} \leq -\mathrm{Ric}_{M_i}(\partial_r, \partial_r) \leq (n-1)k,$$

we know that

$$\frac{(\frac{h_i}{n-1})'}{\big(1-\tau(\delta)\big) + (\frac{h_i}{n-1})^2} \leq \frac{k - \frac{h_i^2}{(n-1)^2}}{\big(1-\tau(\delta)\big) + (\frac{h_i}{n-1})^2} < \frac{k}{1-\tau(\delta)}.$$

Consequently, we have

$$\begin{aligned}\int_0^r \frac{(\frac{h_i}{n-1})'}{\big(1-\tau(\delta)\big) + (\frac{h_i}{n-1})^2}\, dr &\leq -r + (\delta_i + \sqrt[4]{\epsilon_i}) + \left(\frac{k}{1-\tau(\delta)}\right)(\delta_i + \sqrt[4]{\epsilon_i}) \\ &= -r + C(\delta_i + \sqrt[4]{\epsilon_i}),\end{aligned}$$

where $C = 1 + \frac{k}{1-\tau(\delta)} > 1$ for small δ and $r < R$.

The above inequality shows that

$$\begin{aligned} h_i(r,\theta) &\leq (n-1)\sqrt{1-\tau(\delta)}\cot\sqrt{1-\tau(\delta)}\big(r - C(\delta_i + \sqrt[4]{\epsilon_i})\big) \\ &= h_{1-\tau(\delta)}\big(r - C(\delta_i + \sqrt[4]{\epsilon_i})\big),\end{aligned}$$

for $C(\delta_i + \sqrt[4]{\epsilon_i}) < r \leq \min\big(d^i(\theta), R\big)$.

Now if we put $\tau_i := C(\delta_i + \sqrt[4]{\epsilon_i})$, then we can write as follows.

$$h_i(t,\theta) - h_1(t) = h_i(t,\theta) - h_{1-\tau(\delta)}(t-\tau_i) + h_{1-\tau(\delta)}(t-\tau_i) - h_1(t).$$

Noting that $h_\lambda(t,\theta) \sim \frac{n-1}{t}$ as $t \to 0$ for any $\lambda \in \mathbb{R}$, it is easy to check that

$$h_{1-\tau(\delta)}(t-\tau_i) - h_1(t) < \eta\big(\tau(\delta)\big),$$

which converges to 0 as $\tau(\delta) \to 0$.

Since $h_i(t,\theta) - h_{1-\tau(\delta)}(t-\tau_i)$ is negative, we can conclude that $\big(h_i(t,\theta) - h_1(t)\big)_+$ can arbitrarily be small on $[\sqrt[3]{\tau_i}, \min(d^i(\theta), R)]$ for sufficiently large i.

We now write

$$\int_{\Phi_{\sqrt[4]{\epsilon_i},\delta_i}} \int_{\delta_i}^{\min(R,d^i(\theta))} \psi_i\omega_i \, dr\, d\theta = \int_{\Phi_{\sqrt[4]{\epsilon_i},\delta_i}} \int_{\delta_i}^{\sqrt[3]{\tau_i}} \psi_i\omega_i \, dr\, d\theta + \int_{\Phi_{\sqrt[4]{\epsilon_i},\delta_i}} \int_{\sqrt[3]{\tau_i}}^{\min(R,d^i(\theta))} \psi_i\omega_i \, dr\, d\theta.$$

Then we can see that the first and second terms of the above sum converge to zero as $i \to \infty$.

Now let's estimate the third term of (2.1) similarly as above.

$$\int_{(\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\delta_i}^{\min(R,d^i(\theta))} \psi_i\omega_i \, dr\, d\theta = \int_{(\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\delta_i}^{S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta)} \psi_i\omega_i \, dr\, d\theta + \int_{(\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta)}^{\min(R,d^i(\theta))} \psi_i\omega_i \, dr\, d\theta.$$

We know that the second term of the above sum converges to zero by Lemma 2.1 which was shown previously.

For the first term of the above sum, we assume that $S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta) > 0$ for all i and $\theta \in S^{n-1}$. We can then split the first term of the above sum as

$$\int_{(\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\delta_i}^{\sqrt[3]{\tau_i}} \psi_i\omega_i \, dr\, d\theta + \int_{(\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\sqrt[3]{\tau_i}}^{S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta)} \psi_i\omega_i \, dr\, d\theta.$$

Clearly the first term of the above sum converges to zero.

Since $\mu(\gamma_\theta^i\big([\delta_i, S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta)]\big) \cap E_\delta^i) = \sqrt[4]{\epsilon_i}$, we can apply the same mean curvature estimates as for $\theta \in \Phi_{\sqrt[4]{\epsilon_i},\delta_i}$ to the second term of the above sum. So we see that the second term of the above sum also converges to zero. Now we have arrived at the desired result.

3 Proof of Corollary 1.4

Consider a sequence (M_i, g_i, x_i) of Riemannian n-manifolds with metrics g_i such that $\int_{B(x_i,R)} \rho_i \, d\,\mathrm{vol}$ converges to zero and $\mathrm{Ric}_{M_i} \geq -(n-1)k\ (k > 0)$.

We use the same notation as in the proof of Theorem 1.3.

Let $\pi \leq r \leq R$ and $\lim_{i\to\infty} d^i(\theta) > \pi$.

We first estimate $\omega_i(r,\theta)$ for $\theta \in \Phi_{\sqrt[4]{\epsilon_i},\delta_i}$.

Note that $\omega_i(r,\theta) \leq \frac{\omega_{-k}(R,\theta)}{\omega_{-k}(\pi-\alpha,\theta)}\omega_i(\pi-\alpha,\theta)$ for any small $\alpha > 0$, since $\mathrm{Ric}_{M_i} \geq -(n-1)k\ (k > 0)$.

From the proof of Theorem 1.3, we know that $\big(h_i(t,\theta)-h_1(t)\big)_+$ can be arbitrarily small on $[\sqrt[3]{\tau_i}, \pi-\alpha]$ for sufficiently large i.

So we may let $h_i(t,\theta) \le h_1(t) + \eta_i$ on $[\sqrt[3]{\tau_i}, \pi-\alpha]$ for sufficiently small $\eta_i > 0$, which implies that $(\log \frac{\omega_i}{\omega_1})' \le \eta_i$.

Integrating both sides from $\sqrt[3]{\tau_i}$ to $\pi-\alpha$, we obtain

$$\log \frac{\omega_i(\pi-\alpha,\theta)}{\omega_1(\pi-\alpha)} - \log \frac{\omega_i(\sqrt[3]{\tau_i},\theta)}{\omega_1(\sqrt[3]{\tau_i})} \le \eta_i(\pi-\alpha-\sqrt[3]{\tau_i}).$$

So we have

$$\omega_i(\pi-\alpha,\theta) \le \exp\{\eta_i(\pi-\alpha-\sqrt[3]{\tau_i})\}\frac{\omega_i(\sqrt[3]{\tau_i},\theta)}{\omega_1(\sqrt[3]{\tau_i})}\omega_1(\pi-\alpha),$$

which can be arbitrarily small if we choose $\alpha > 0$ suitably. This again means that $\omega_i(r,\theta)$ can be arbitrarily small for $\theta \in \Phi_{\sqrt[4]{\epsilon_i},\delta_i}$ by the above inequality for $\omega_i(r,\theta)$.

Next, estimate $\omega_i(r,\theta)$ for $\theta \in (\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c$.

If $\lim_{i\to\infty} S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta) < \pi$, the proof of Lemma 2.1 says that $\omega_i(\pi-\alpha,\theta)$ can be arbitrarily small for $\alpha > 0$ with $S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta) \le \pi-\alpha$. So the above argument also holds for this case.

On the other hand, if $\lim_{i\to\infty} S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta) \ge \pi$, then the same mean curvature estimates as for the case $\theta \in \Phi_{\sqrt[4]{\epsilon_i},\delta_i}$ in the above holds for $\sqrt[3]{\tau_i} \le r \le \pi-\alpha$, which leads the same estimates for $\omega_i(r,\theta)$ as above.

Consequently, we arrive at the conclusion that $\omega_i(r,\theta)$ can be arbitrarily small for all r with $\pi \le r \le R$ and $\theta \in S^{n-1}$.

Corollary 1.4 now follows immediately.

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