TRANSFORMATION GROUPS WITH $(n-1)$-DIMENSIONAL ORBITS ON NON-COMPACT MANIFOLDS

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Introduction

When a Lie group G operates on a differentiable manifold M as a Lie transformation group, the *orbit of a point p in M under G*, or the *G-orbit* of p, is by definition the submanifold $G(p) = \{G(p)\,;\ g \in G\}$. The purpose of this paper is to characterize the structure of a non-compact manifold M such that there exists a compact orbit of dimension $(n-1)$, $n = \dim M$, under a connected Lie transformation group G, which is assumed to be compact or an isometry group of a Riemannian metric on M. When G is compact there exists on M a G-invariant Riemannian metric, and so we shall always consider G as an isometry group. In order to state our main theorem we need another definition: a Riemann manifold M is said *isotropic* (or *H-isotropic*) at a point p in M when there exists an isometry group H of M such that, for any two unit vectors X and Y at p, H contains an isometry carrying X to Y (Some authors use this terminology in a different sense). Now the main theorem (Theorem 3) reads: If there exists a compact $(n-1)$-dimensional G-orbit then M admits a fibre bundle structure over a compact orbit $B = G(b)$, $b \in B$, associated with the principal bundle $(G, G/H, H)$ where H is the isotropy subgroup at b, the fibre being diffeomorphic to the euclidean space on which the structure group H operates as a linear group. The fibre is a submanifold of M containing b and H-isotropic at b, if $\dim B > n-1$. The hypothesis of the theorem can be replaced by a more geometric one: G leaves invariant and operates transitively on a connected component of any submanifold consisting of the points at a constant distance from a fixed compact submanifold.

P. S. Mostert proved a similiar theorem [5] in a different formulation (see

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Corollary 5.8) in case G is compact and M had not necessarily a differentiable structure. J. L. Koszul [2] showed the existence of the bundle structure in a neighborhood of an orbit of an arbitrary dimension when G is compact.

Our formulation allows us to derive, for instance, a theorem of Montgomery and Zippin (see Corollary 5.4) and will be convenient for our purpose: in forthcoming papers [6], [7] we shall determine 1) M as a differentiable mnaifold (compact or not) under an additional condition that there exists a 0-dimensional G-orbit and 2) G as a transformation group under another additional condition that M is homeomorphic to a sphere.

Contents of various sections. In 1 we shall explain conventions and definitions together with known properties on geodesics. Existence of an $(n-1)$-dimensional orbit will turn out to be equivalent to existence of a G-orbit B such that G, operating naturally on the tangent bundle of M, is transitive on a connected component of the set of unit vectors normal to B (Corollary 4.9). In 2 we shall establish that for any point p in M there exists a geodesic of the minimum length joining p to B (Theorem 1). Section 3 is devoted to demonstrate that such a geodesic is unique unless B is $(n-1)$-dimensional and two-sided (Theorem 2). (The G-orbits are $(n-1)$-dimensional and two-sided except at most one orbit). From these two theorems follows the main theorem (Section 4). In the last section one will find several corollaries to the main theorem.

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1. Preliminaries

The letter M will be reserved for a connected differentiable manifold of differentiability class C^3, whose dimension will be denoted by n. G will denote a connected Lie group which operates on M as a C^3-transformation group. Without special mention, G is assumed to be an isometry group of a Riemannian metric on M of differentiability class C^2 (The condition that G is a Lie group is superfluous because of a theorem of Myers and Steenrod and that of Kuranishi and Yamabe). A submanifold is said compact when it is compact in its inner topology. Given a connected submanifold B of M, we denote by

$N'(B)$ the set of all unit normal vectors of B; i.e. every vector X in $N'(B)$ has its origin on B and is orthogonal to the tangent space of B. $N'(B)$ is given the topology induced from the tangent bundle $T(M)$ of M. $N'(B)$ is a compact submanifold of $T(M)$ when B is compact. We call $N'(B)$ the *normal bundle* of B. $N'(B)$ has at most two connected components, any one of which is called a *connected normal bundle* of B, denoted by $N(B)$. G operates naturally on $T(M)$. If B is a G-orbit then $N(B)$ is invariant under G. If $N(B)$ is a G-orbit then so is the other component of $N'(B)$.

Given a (piecewise differentiable) curve α we write $|\alpha|$ for the length of α. For two points p and q we put $d(p, q)$ = the distance between them $= \inf_{\alpha \text{ joining } p \text{ to } q} |\alpha|$, and $d(p, B)$ = the distance from p to $B = \inf_{b \in B} d(p, b)$, which is denoted by $d(p)$ when no ambiguity is to fear. By definition a *minimum geodesic* γ (from p to B) is a geodesic joining p to some point b in B with $\gamma = d(p)$. Then b is called the initial point of γ, and p is said to admit a minimum geodesic. More generally, by a perpendicular γ (to B) we mean a geodesic issuing from a point b in B and orthogonal to B at b. γ has not always the end point but b, while a minimum geodesic is compact. For a curve γ having two points p and q on it, $\gamma(p, q)$ will denote the subarc of γ lying between them, which will be well defined always when considered in the sequel. For a non-negative number c, N_c shall denote the set $\{q \in M;\ d(q) = c\}$, which depends on B. Given a point p in M, we write $N_{(p)}$ for the connected component containing p of $N_{d(p)}$.

Denoting by T' the set of all unit vectors on $M = \bigcup_{x \in M} N(\{x\})$, and by R_+ the set of all non-negative numbers, we here recall three well known properties of geodesics, for which we need C^2-differentiability of the metric tensor on M:

(G. 1) Given $X \in T'$ and $s \in R_+$, the point p is uniquely determined (if p exists) by the condition that p lies on a geodesic with X as the initial tangent and at the arc length s from the origin of X.

(G. 2) The above defined map ϕ: $(X, s) \to p$ is continuous, the definition domain of ϕ being a neighborhood W of $T' \times \{0\}$ in $T' \times R_+$.

(G. 2 a) ϕ is differentiable on $W - T' \times \{0\}$.

(G. 3) There exists an open covering $\{U_\lambda\}$ of M such that every two points p and q in a U_λ are joined by a geodesic γ with $|\gamma| = d(p, q)$. γ is the unique curve with this property.

Further we need two known properties:

(G. 4) If a curve α joining two points p and q in M satisfies $|\alpha| = d(p, q)$, then α is a geodesic.

This follows immediately from (G. 3).

(G. 5) A minimum geodesic γ to a submanifold B is a perpendicular.

We give an outline of the proof. Let b be the initial point of γ. Consider a normal coordinate system ν with center b, a diffeomorphism of a neighborhood U of b onto an open subset V of the euclidean space, and note that the given Riemannian metric tensor μ is asymptotic to the metric μ' induced by ν from the euclidean metric on V if U is sufficiently small. We have $\mu(0) = \mu'(0)$. μ and μ' have in common the geodesics starting at b and the angle β between γ and B. There exist point sequences $\{p_k\}$ on γ and $\{b_k\}$ on B both converging to b such that $\lim[d'(b, b_k)/d'(b, p_k)] = \sin \beta$ where d' is the distance function corresponding to μ'. One concludes $\sin \beta = 1$ from this together with three relations: $d'(b, p_k) = d(b, p_k)$, $\lim[d'(b, b_k)/d(b, b_k)] = 1$, and $d(p_k, b) \leqq d(p_k, b)$ (due to the assumption on γ).

2. Existence of minimum geodesics.

THEOREM 1. *Let G be a connected isometry group of a Riemann manifold M, and B a compact submanifold of M. If a connected normal bundle $N(B)$ is a G-orbit then any point of M admits a minimum geodesic to B.*

We shall prove this by establishing several lemmas.

(2.1) Let N be the subset of M consisting of the points p such that any point q with $d(q) \leqq d(p)$ admits a minimum geodesic. Since N contains B which is of course assumed to be nonvacuous, *N is nonvacuous.*

(2.2) *An arbitrary point p in the closure $\overline{N}$ of N admits a minimum geodesic to B.*

Let $\{p_k\}$ be a point sequence in N which converges to p. If we have $d(p) \leqq d(p_k)$ for some k then p admits a minimum geodesic by the definition of N. We thus assume $d(p_k) < d(p)$ for all k. Belonging to N, p_k admits a minimum geodesic, γ_k. The initial unit tangent vector X_k of γ_k belongs to $N'(B)$ by (G. 5). Since $N'(B)$ is compact, one may assume that all X_k belong to

$N(B)$ and the sequence $\{X_k\}$ converges to some X in $N(B)$. From the fact that the connected Lie group G is transitive on the manifold $N(B)$, it follows that there exists a sequence $\{g_k\}$ in G converging to the identity such that each g_k carries X_k to X. The isometry g_k carries γ_k into the maximal geodesic, γ, with the initial tangent X by (G. 1), γ being maximal in the sense that any geodesic with the initial tangent X is contained in γ. The point $g_k(p_k)$ is the end point of $g_k\gamma_k \subset \gamma$ other than the origin b of X. The point sequence $\{g_k(p_k)\}$ on γ converges to p. Further the arc length $|g_k(\gamma_k)|$ which equals $|\gamma_k| = d(p_k)$ converges to $d(p)$. Hence p belongs to γ and we have $|\gamma(p, b)| = d(p)$; in other words $\gamma(p, b)$ is a minimum geodesic from p.

(2.3) *N is closed.*

Let p be an arbitrary point in $\bar{N}-B$ and q a point of $M-B$ with $d(q) \leqq d(p)$. There exists a point sequence $\{q_k\} \subset M$ converging to q and satisfying $d(q_k) < d(q)$ for all k, [because otherwise there would exist a positive number $e < d(q)$ such that any point x with $d(x, q) < e$ satisfies $d(q) \leqq d(x)$. Then any curve joining q to B must be longer than $d(q) + e$, contrary to the fact that $d(q)$ equals the greatest lower bound of the length of such curves]. Hence we have $d(q_k) < d(p)$. Since p belongs to $\bar{N}$, this shows that q_k belongs to N. Thus q adheres to N, and so q admits a minimum geodesic owing to (2.2). This gives that N contains p.

(2.4) *N_c is compact, $c \in R_+$, if $N_c \cap N$ is nonvacuous.*

N_c is then contained by N. The map ϕ in (G. 2) commutes with any element g of G; i.e. one has $\phi(gX, s) = g\phi(X, s)$. Since G is transitive on $N(B)$, it follows that ϕ is defined on $N'(B) \times \{c\}$ or on its connected component and has N_c as its image or a connected component of its image. Hence N_c, a continuous image of a compact set, is compact.

(2.5) *N is open.*

Let p be an arbitrary point of N. Set $c = d(p)$. N_c is then contained in N. To establish (2.5) it is sufficient to show that a neighborhood of N_c is contained in N. An arbitrary point q of N_c belongs to some open set $U = U_r$ mentioned in (G. 3). Denote by $V(q)$ a neighborhood $\subset U$ of q such that every point x in $V(q)$ satisfies $2d(x, q) < d(x, r)$ for any boundary point r of U.

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There exists then a point y in N_c with $d(x, y) = d(x, N_c)$ by (2.4). y belongs to U, for otherwise any curve α joining x to y intersects the boundary of U and so we have $2d(x, y) \leqq 2d(x, q) < |\alpha|$, which is a contradiction. Hence, by the definition of U, there exists a geodesic γ, joining x to y with $|\gamma_1| = d(x, y) = d(x, N_c)$. Belonging to $N_c \subset N$, y admits a minimum geodesic γ_2 to B. We consider the curve $\gamma = \gamma_1 \cup \gamma_2$. When $d(x) > c$, γ will turn out to be a minimum geodesic to B. In fact any curve α joining x to a point p of B must then intersect N_c at a point, z, and we have $|\gamma_1| = d(x, N_c) \leqq |\alpha(x, z)|$ and $|\gamma_2| = c \leqq |\alpha(z, p)|$. Hence one has $|\gamma| = d(x)$; thus γ is a minimum geodesic by (G. 4). We have proved that any point in $V(q)$ admits a minimum geodesic. The compact set N_c is covered by $\cup_{q \in N_c} V(q)$. Therefore N_c is contained in the interior of N.

Now Theorem 1 is clear; by (2.1), (2.3) and (2.5), N coincides with M.

Corollary 2.1. *Let G be a connected isometry group of a Riemann manifold M, and B be a compact submanifold of M. Then the following these conditions are equivalent:*

1) *A connected normal bundle $N(B)$ is an orbit under G;*

2) *For any point p of M the set $N_{(p)}$ is a compact G-orbit;*

3) *For any number $e > 0$ there exists a point p with $0 < d(p) < e$ such that $N_{(p)}$ is a G-orbit.*

Put $d(p) = c$. Assuming 1) we will prove 2). By Theorem 1, $N_{(p)}$ is the ϕ-image of $N'(B) \times c$ or its connected component $N(B) \times c$. Hence $N_{(p)}$ is compact. Denoting by X the initial unit tangent of a minimum geodesic from p, we get $G(p) = G\phi(X, c) = \phi(G(X), c) = \phi(N(B), c)$. In the latter case above we thus obtain $G(p) = N_{(p)}$. In the former case we have $N_{(p)} = \phi(N(B), c) \cup \phi(N'(B) - N(B), c)$, and $\phi(N'(B) - N(B), c)$ is compact. Hence $G(p) = \phi(N(B), c)$ contains an open subset of $N_{(p)}$. Since G is an isometry group of $N_{(p)}$ it follows that $G(p) = N_{(p)}$, and 2) is proved. 2) implies 3) obviously. Finally we shall derive 1) from 3). Since B is compact, there exists by (G. 3) a neighborhood V such that any point in V admits a minimum geodesic to B. Let γ be a minimum geodesic from a point q in V. By 3) we may assume that there exists a point p on γ, $0 < d(p) = c < d(q)$, such that $N_{(p)}$ is a G-orbit contained by V. For an arbitrary point in $N_{(p)}$ the minimum geodesic is unique. We may prove this for p only. Suppose that γ_1 and γ_2 are different minimum geodesics from p to B. Then $\gamma(q, p) \cup \gamma_1$ and $\gamma(p, q) \cup \gamma_2$ are minimum geodesics. Hence they are geodesics, contrary to (G. 1). Denote by $N(B)$ the connected normal bundle of B containing the initial tangent of γ. ϕ is defined on $N(B) \times c$ and a one-to-one map onto $N_{(p)}$. Since ϕ commutes with each g in G, this gives that $N(B)$ is a G-orbit.

3. Uniqueness of minimum geodesics

THEOREM 2. *Under the hypothesis of Theorem 1, assume that the normal bundle $N'(B)$ is connected and M is not compact. Then every point of M admits only one perpendicular to B.*

The fact $N'(B) = N(B)$ and (G. 1) give:

(3.1) *For two perpendiculars γ_1 and γ_2 to B there exists an isometry g in G such that $g\gamma_1$ contains or is contained by γ_2.*

(3.2) *Every perpendicular γ to B is a minimum geodesic to B.*

Proof. We may assume that γ is maximal. Let γ' be the maximal subarc of γ such that, for any point p of γ', the subarc $\gamma(p, b)$, b being the initial point of γ, is a minimum geodesic from p to B. Patently γ' contains b, and γ' is nonvacuous. It is easy to see that γ' is closed in γ (in its inner topology). Assume that $\gamma' \neq \gamma$. Then γ' is compact. Since $N'(B)$ is also compact, $\cup_{g \in G} g\gamma'$ is compact. M being non-compact, it follows from Theorem 1 that there exists a minimum geodesic γ'' to B with $|\gamma'| < |\gamma''|$. By (3.1), some isometry g in G carries γ'' into γ, and $g\gamma''$ contains γ' as a proper subset, contrary to the definition of γ'.

From this proof one deduce:

(3.3) Any minimum geodesic is a proper subset of another minimum geodesic.

(3.4) Every point p of M admits only one minimum geodesic.

Proof. Assume that p admits two minimum geodesics γ_1 and γ_2. By (3.3) γ_1 is a proper subset of another minimum geodesic γ. Denote by γ_0 the subarc of γ such that $\gamma = \gamma_0 \cup \gamma_1$ and $\gamma_0 \cap \gamma_1 = \{p\}$. Since $|\gamma_0 \cup \gamma_2| = |\gamma_0| + |\gamma_2| = |\gamma_0| + |\gamma_1| = |\gamma_0 \cup \gamma_1|$, we find that $\gamma_0 \cup \gamma_2$ is a minimum geodesic. But $\gamma_0 \cup \gamma_2$ is not differentiable at p, contrary to (G. 4).

Now Theorem 2 follows from (3.2) and (3.4).

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Remark 3.1. $N'(B)$ is connected if dim $B<n-1$.

COROLLARY 3.2. *If a non-compact Riemann manifold M is isotropic at a point o, then M is homeomorphic to the euclidean space.*

Put $B=O$. $N(B)$ is an orbit of some connected isometry group G of M. If $1<n$, $N'(B)$ is connected and Theorems 1 and 2 imply that the normal coordinate system ν with center o extends to a homeomorphism of the whole space M. When $n=1$, M is obviously C^1-diffeomorphic to the euclidean space.

Remark 3.3. If further M is homogeneous, then the isotropy subgroup of the isometry group of M is irreducible. Since M is not compact, one can apply Matsushima's theorem (unpublished) which states that M is then symmetric. The corresponding compact symmetric space M_c is isotropic at each point. Hence M_c is one of the spaces determined by Wang [9]. Modifying his method and using Yamabe's theorem one can easily obtain the same conclusion for a locally and finitely compact metric space. But more general results have already been obtained by J. Tits [8] and H. Freudenthal [1].

4. The main theorem

When H is a closed subgroup of a Lie group G, $(G, G/H, H)$ shall denote the principal bundle over G/H relative to the projection of G onto G/H.

THEOREM 3. *Let G be a connected isometry group G of a non-compact n-dimensional Riemann manifold M with an* $(n-1)$*-dimensional compact orbit, then M has a fibre bundle structure such that 1) the base space is a compact G-orbit, 2) the associated principal bundle is* $(G, G/H, H)$ *where H is the isotropy subgroup of G at a point b of B, 3) the fibre E (containing b) is a submanifold of M which is* C^1*-diffeomorphic to the euclidean space of dimension n-dim B, 4) the structure group H acts on E as a linear group in terms of some coordinate system of E, and finally 5), if dim* $B<n-1$*, then H acts transitively on the (unit sphere with center* $b=E\cap B$ *in the tangent space to E at b*; E is thus isotropic at b.

(4.1) *Every G-orbit is compact* by Corollary 2.1.

We distinguish two cases;

Case I: *All G-orbits are* $(n-1)$*-dimensional.*

Case II: *There exists a G-orbit B of dimension* $<n-1$.

In case I, we fix an arbitrary point p and denote by E the union of the two perpendiculars to $P = G(p)$ issuing from p. E is a geodesic without end points. If E contains a point q of P other than p we denote by b the middle point of the subarc $E(p, q)$; otherwise we put $b = p$. We set $B = G(b)$. In case II, we fix an arbitrary point b in B, and denote by E the union of all perpendicular to B issuing from b.

(4.2) *G is transitive on $N(B)$ and, in case I, $N(P)$.*

In case I, the orbits B and P being $(n-1)$-dimensional, G is transitive on $N(B)$ and $N(P)$. In case II, B adheres to the union of $(n-1)$-dimensional orbits as is easily seen. If $G(p)$ is $(n-1)$-dimensional, $G(p)$ is open in the subset $N_{(p)}$ which is left invariant by the isometry group G. Hence $N_{(p)}$ coincides with $G(p)$. Corollary 2.1 shows that G is transitive on $N(B)$.

(4.3) *The set $A = B \cap E$ contains b only.*

Case I. If A contains two points, the set $E \cap P$ contains at least three points, $p_i (i = 1, 2, 3)$. Assume that p_2 lies on minimum geodesic γ to P by Theorem 1 which applies due to (4.1) and (4.2). An isometry transforming the initial point of γ to p_2 transforms γ into $E(p_1, p_3)$. As in the proof of (3.2), we can infer that M is compact, contrary to the assumption.

Case II. If A contains a point x other than b, then E contains a geodesic γ joining x to b by the definition of E Hence there exist two perpendiculars γ and x from x to B, contrary to Theorem 2 which applies owing to (4.1), (4.2) and Remark 3.1.

(4.4) *Let h be an isometry in G. If $h(E)$ intersects E, then $h(E)$ coincides with E.*

In case *I* they are maximal geodesics. (4.4) follows from the fact that a geodesic γ, which is orthogonal to the orbit under a connected isometry group G at a point, is orthogonal to the G-orbit at any point of γ ([10]; p. 48). In case II, $h(E)$ is the union of all perpendiculars to B issuing from $h(b)$. Hence if $h(b) = b$ then $h(E) = E$. If a point x belongs to $h(E) \cap E$, then there exist perpendiculars from x to b and to $h(b)$. By Theorem 2, we have $h(b) = b$. (4.4) is proved.

Let ρ be a map of $G \times E$ into M defined by $\rho(g, x) = g(x)$.

(4.5) *ρ is onto.*

Given a point y in M there exists a minimum geodesic γ to B. Let g be the isometry in G which carries b to the initial point of γ. Then $x = g^{-1}(y)$ belongs to E and we have $\rho(g, x) = y$.

(4.6) *We have $\rho(g, x) = \rho(g', x')$ if and only if $h = g^{-1}g'$ belongs to H and $x = h(x')$.*

Assume $\rho(g, x) = \rho(g', x')$. Then $x = h(x')$ and x belongs to $E \cap h(E)$. Hence we find $h(E) = E$ by (4.4). This gives that $h(b)$ belongs to $E \cap h(B) = E \cap B$. By (4.3) we get $h(b) = b$; i.e. h belongs to H. The converse is evident.

Now from (4.5) and (4.6) we conclude that M is a fiber bundle with fibre E and associated with the principal bundle $(G, G/H, H)$; we have proved 1) and 2).

(4.7) *The assertion* 3) *in Theorem 3 is true.*

In case I, 3) is obvious. In case II we consider a normal coordinate system ν with center b, a C^1-diffeomorphism of a neighborhood V of b in M onto an open subset of the n-dimensional euclidean space. $W = E \cap V$ is a closed submanifold of V. The restriction ν' of ν to W extends to E in such a way that every perpendicular to $\{b\}$ is isometrically mapped to a geodesic in the eudlidean space of dimension $(n - \dim B)$. This extension ν'' is well defined and one-to-one due to Theorem 2. ν'' is diffeomorphic owing to the facts that ν''^{-1} is differentiable by (G.2a) and that a neighborhood of any $N_{(p)}$, $p \notin B$, is the direct product (as a differentiable manifold) of $N_{(p)}$ and a geodesic orthogonal to $N_{(p)}$ by (G.2a).

(4.8) *The assertion* 4) *in Theorem 3 is true.*

ν'' can be regarded naturally as a map into the tangent space T to E at b. The operations of H on E correspond to the operations of the linear isotropy group on T (which we confound with H).

(4.9) The assertion 5) in Theorem 3 is true.

The unit sphere in T coincides with $T \cap N(B)$. $N(B)$ is a G-orbit. An h of H is characterized in G by the property that h carries a vector in T to another in T. It follows that H is transitive on $T \cap N(B)$.

5. Corollaries to the main theorem

COROLLARY 5.1. *Let G' be a compact C^3-transformation group of an n-dimensional connected non-compact paracompact differentiable manifold M of class C^3. If there exists a G'-orbit of dimension $n-1$, then the conclusion of Theorem 3 holds good.*

In fact there exists on M a G'-invariant Riemannian metric of class C^2 by Whitney's theorem and compactness of G. Further the identity component G of G' admits an $(n-1)$-dimensional compact orbit because G' is compact. Thus Theorem 3 applies.

COROLLARY 5.2. *In Theorem 3, any point x of M can be joined to B by exactly one perpendicular γ_x.*

In case dim $B < n-1$, this is nothing but Theorem 2. In case dim $B = n-1$, our corollary follows from Theorem 3; γ_x is contained in (hence coincides locally with) the fibre containing x.

COROLLARY 5.3. *In Theorem 3 the subspace B is a strong deformation retract of M. In particular the singular homology groups of B and M are isomorphic*; $H(B) = H(M)$. (So are their homotopy groups).

Let f be the retraction, which is the projection of the bundle space M onto the base space B. Given t in the closed interval $[0, 1]$ and x in M, $D(t, x)$ shall denote the point on the perpendicular γ_x (from x) at the distance $td(x)$ from B. D is well defined due to Corollary 5.2. The composition of f and the inclusion map $B \to M$ is homotopic to the identity by D. In fact for an arbitrary point $p \in M$ there exists a neighborhood U of $f(p)$ in B, a local cross-section $c: U \to G$, and a continuous map $e: f^{-1}(U) \to E$ such that we have $x = \rho(cf(x), e(x))$, ρ being defined above (4.5). We get a similarity transformation with multiplicity $t (\neq 0)$ if we restrict the transformation $x \in M \to D(t, x) \in M$ to E (more precisely, if one further induces this restricted map to the tangent space T to E at b). It follows D is continuous on $[0, 1] \times E$. On $[0, 1] \times f^{-1}(U)$ we have $D(t, x) = \rho(cf(x), D(t, e(x))$ and find that D is continuous, whence D is a homotopy. Therefore B is a strong deformation retract of M.

COROLLARY 5.4. *If, in Theorem 3, M is homeomorphic to the euclidean space E^n, then G admits a fixed point and G is a linear group.* (Montgomery and Zippin [4] proved an analogous theorem assuming compactness of G and no differentiability of G and M. (See [3] also).

By Corollary 4.3, we have
$H(B) = H(M) = H(E^n) = H_0(E^n)$, i.e. $H(B) = H_0(B) =$ the integers, where H_0 denotes the 0-dimensional homology group. B is a compact manifold and again by Corollary 4.3, B is simply connected; in particular B is orientable. It follows that B contains just one point.

Corollary 5.5. (The converse of the preceding corollary). *If, in Theorem 3, G admits a fixed point then M is diffeomorphic to the euclidean space.*

Then B contains just one point and we get $M = E$ (the fibre).

Corollary 5.6. *In Theorem 3 the normal bundle $N'(B)$ is connected if and only if $M - B$ is connected. When this is the case, $N_{(p)} = G(p)$, $p \in M - B$, is a k-sphere bundle over B, $k = n - \dim B - 1$, the structure group being transitive on the fibre. In the other case M is a trivial bundle $= B \times$ (a straight line).* (A zero-dimensional sphere is understood to consist of two points).

Corollary 5.7. *In Theorem 3 the isotropy subgroup H_b at a point b in B is characterized by the property to be maximal in the sense that if the isotropy subgroup H_x at a point x contains H_b then it coincides with H_b. The other isotropy subgroups are all conjugate to each other.*

Corollary 5.8. *In Theorem 3, the orbit space M/G is homeomorphic to (i) an open interval or (ii) the half open interval $J = (0, 1)$. The case (i) occurs if and only if $M - B$ is not connected. In case (ii) there exist subgroups H and K with $K \subset H$ such that M is homeomorphic to $(G/K) \times J$ with $(G/K) \times \{0\}$ identified to $G/H \times \{0\}$ by the relation $(gK, 0) \equiv (hK, 0)$ if $h \in gH$.* (Mostert [5] proved an analogous theorem in assuming compactness of G and no differentiability of M).

In fact M/G can be identified with E if $M - B$ is not connected, and with a maximal perpendicular γ to B if $M - B$ is connected. Let H be the subgroup in Theorem 3, and K the isotropy subgroup at a point on $\gamma - B$. Then our corollary will be evident.

COROLLARY 5.9. *Let G be a connected isometry group of a Riemann manifold M of dimension n. Then the following three conditions are equivalent.*

a) *All orbits C are compact, and G is transitive on $N(C)$.*

b) *There exists a compact G-orbit B such that G is transitive on a connected normal bundle $N(B)$.*

c) *There exists an $(n-1)$-dimensional compact orbit.*

Clearly a) implies b). Assume b). Consider a non-compact G-invariant neighborhood V of B. Applying Corollary 5.6 to V, we find c) deduced. Finally assume c). Then Theorem 3 applies to M if M is not compact and to $M-A$ if M is compact where A is a G-orbit such that the isotropy subgroup at a point of A is maximal, $M-A$ being then connected. By Corollary 5.6 all orbits but B and A are compact and $(n-1)$-dimensional. Hence G is transitive on their connected normal bundles. By Corollary 2.1 G is transitive on $N(B)$. Considering $M-B$ instead of M we also find that G is transitive on $N(A)$.

Question 1. I do not know whether the assumption of compactness in Theorem 3 is indispensable or not.

Question 2. It would be desirable to generalize the whole theory to the case where M is a topological manifold with a metric such that there exist geodesics satisfying local prolongeability, uniqueness of prolongation, etc. and M is locally convex, etc.

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