OPERATORS OF THE FORM $PAQ - QAP$

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1. In this note the Hilbert spaces under consideration are complex, and the operators referred to are bounded, linear operators. If $\mathfrak{H}$ is a Hilbert space, then the algebra of all operators on $\mathfrak{H}$ is denoted by $\mathfrak{L}(\mathfrak{H})$.

It is known (**1**) that if $\mathfrak{H}$ is any Hilbert space, then the class of *commutators* on $\mathfrak{H}$, i.e., the class of all operators that can be written in the form $PQ - QP$ for some $P, Q \in \mathfrak{L}(\mathfrak{H})$, can be exactly described. A similar problem is that of characterizing all operators on $\mathfrak{H}$ that can be written in the form $PAQ - QAP$ for some $P, A, Q \in \mathfrak{L}(\mathfrak{H})$. If no restrictions are placed on the operators P, A, and Q, it is relatively easy to see that for $\dim \mathfrak{H} > 1$, every operator in $\mathfrak{L}(\mathfrak{H})$ can be written in this form. (A very brief and pretty proof of this fact for infinite-dimensional $\mathfrak{H}$ was shown to us by Paul Federbush; it is reproduced in Remark 3.3.)

Since every commutator $PQ - QP$ is automatically a commutator of invertible operators by virtue of the identity

$$PQ - QP = (P + \lambda)(Q + \mu) - (Q + \mu)(P + \lambda),$$

valid for every pair of scalars λ and μ, it is natural to ask which operators can be written in the form $PAQ - QAP$ with *invertible* P, A, and Q. This problem is somewhat more difficult, and it is the purpose of this note to furnish the solution by proving the following theorem.

THEOREM. *If $\mathfrak{H}$ is a Hilbert space of dimension greater than one, and T is any operator on $\mathfrak{H}$, then there exist invertible operators P, A, Q on $\mathfrak{H}$ satisfying $T = PAQ - QAP$.*

This theorem settles a question posed to us by Olga Taussky-Todd, to whom we are indebted for several interesting conversations.

The proof of the theorem splits naturally into cases depending on the dimension of $\mathfrak{H}$. In the finite-dimensional case, the proof depends on the following lemmas and (**4**, Theorem III).

LEMMA 1.1. *If $T \neq 0$ is an operator on an n-dimensional complex Hilbert space $\mathfrak{H}$ $(1 < n < \infty)$, then there exists an orthonormal basis for $\mathfrak{H}$ relative to which the matrix $(\alpha_{ij})_{i,j=1}^n$ of T satisfies $\alpha_{11} \neq 0 \neq \alpha_{22}$.*

Proof. Consider the numerical range (or field of values) $W(T)$ of T. If $W(T)$ consists of a single point $\{\lambda\}$, then T is the scalar operator $T = \lambda 1$ and

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the result is obvious. Otherwise, $W(T)$ contains at least two points, and thus the line segment joining them. Hence, $W(T)$ contains a number $\alpha_{11} \neq 0$ such that $\alpha_{11} \neq$ trace T. Let x_1 be a unit vector such that $(Tx_1, x_1) = \alpha_{11}$, and extend $\{x_1\}$ to an orthonormal basis $\{x_1, \ldots, x_n\}$ for $\mathfrak{H}$. Since $\alpha_{11} \neq$ trace T, there must be some k $(2 \leqq k \leqq n)$ such that $(Tx_k, x_k) = \beta \neq 0$. If we now interchange x_2 and x_k, then $(Tx_2, x_2) = \beta = \alpha_{22} \neq 0$, and the proof is complete.

LEMMA 1.2. *If T is an operator on an n-dimensional Hilbert space $\mathfrak{H}$ $(1 < n < \infty)$, then there exist invertible operators X and Y on $\mathfrak{H}$ such that* determinant X = determinant Y *and such that $T = X - Y$.*

Proof. If $T = 0$, the result is clear. We suppose that $T \neq 0$, and use the preceding lemma to pick a basis for $\mathfrak{H}$ relative to which the matrix (α_{ij}) of T satisfies $\alpha_{11} \neq 0 \neq \alpha_{22}$. We write

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & & & \vdots \\ \vdots & & & \vdots \\ \alpha_{n1} & \cdots & \cdots & \alpha_{nn} \end{pmatrix} = \begin{pmatrix} \alpha_{11}+d_1 & & & 0 \\ \alpha_{21} & \alpha_{22}+d_2 & & \\ \vdots & & \ddots & \\ \alpha_{n1} & & & \alpha_{nn}+d_n \end{pmatrix} - \begin{pmatrix} d_1 & -\alpha_{12} & \cdots & -\alpha_{1n} \\ & d_2 & & \vdots \\ & & \ddots & \vdots \\ 0 & & & d_n \end{pmatrix}$$

where the numbers $d_1, \ldots, d_n$ are to be determined. Let X and Y be the operators having these matrices (relative to the given basis), and note that to complete the proof, it suffices to show that the numbers d_i can be chosen so that

(1) $d_1 d_2 \ldots d_n \neq 0$ and

(2) $(\alpha_{11} + d_1)(\alpha_{22}+d_2) \ldots (\alpha_{nn} + d_n) = d_1 d_2 \ldots d_n$.

This amounts to choosing each $d_i \neq 0$ so that (2) is satisfied. If $n = 2$, this is equivalent to choosing non-zero numbers d_1 and d_2 such that

$$\alpha_{11} d_2 + \alpha_{22} d_1 + \alpha_{11}\alpha_{22} = 0,$$

and this is a task that is easily accomplished since $\alpha_{11} \neq 0 \neq \alpha_{22}$. If $n > 2$ we first choose d_i, $3 \leqq i \leqq n$, subject only to the conditions $d_i \neq 0 \neq d_i + a_{ii}$. Next, we arrange things so that $\beta = d_2 \ldots d_n$ is unequal to

$$\gamma = (\alpha_{22} + d_2) \ldots (\alpha_{nn} + d_n).$$

To this end let $\xi = d_3 \ldots d_n$ and $\eta = (\alpha_{33} + d_3) \ldots (\alpha_{nn} + d_n)$. Then we need $d_2 \xi \neq (\alpha_{22} + d_2)\eta$, and since ξ, η, and α_{22} are all non-zero, it is easy to see that we can choose d_2 so as to satisfy this requirement and also to satisfy $d_2 \neq 0 \neq d_2 + \alpha_{22}$. Assume this done; to complete the proof it then suffices to choose $d_1 \neq 0$ so that

$$d_1 \beta = (\alpha_{11} + d_1)\gamma,$$

and since $\beta \neq \gamma \neq 0$ and $\alpha_{11} \neq 0$, this is possible.

COROLLARY 1.3. *The theorem is true if $\mathfrak{H}$ is finite-dimensional.*

Proof. Let T be an operator on $\mathfrak{H}$. By Lemma 1.2 there exist invertible operators X and Y on $\mathfrak{H}$ with equal determinants such that $T = X - Y$. According to (**4**, Theorem III), there exist operators P, A, and Q such that $X = PAQ$ and $Y = QAP$; the invertibility of X and Y guarantees that P, A, and Q are invertible. Thus $T = PAQ - QAP$, as desired.

2. The separable case. We turn now to the case in which $\mathfrak{H}$ is a separable, infinite-dimensional, space. According to (**1**, Theorem 3), an operator T on $\mathfrak{H}$ is a commutator if it is not of the form $\lambda + K$ for some non-zero scalar λ and compact operator K. For such a commutator T there exist operators P_1 and Q_1 such that $T = P_1Q_1 - Q_1P_1$. Since, as noted before, for any scalar μ we also have that

$$T = (P_1 + \mu)(Q_1 + \mu) - (Q_1 + \mu)(P_1 + \mu),$$

μ_0 can be chosen so that the operators $P = P_1 + \mu_0$ and $Q = Q_1 + \mu_0$ are invertible. If we then define $A = 1$, we have that

$$T = PAQ - QAP$$

with invertible P, A, and Q. Thus, it suffices to prove the theorem for operators T of the form $T = \lambda + K$, where $\lambda \neq 0$ and K is compact.

We shall have occasion to write T as a matrix with operator entries, and in so doing, we observe the usual conventions. If $\mathfrak{H}$ is written as the direct sum

$$\mathfrak{H} = \mathfrak{H}_1 \oplus \ldots \oplus \mathfrak{H}_m,$$

and if E_i denotes the projection of $\mathfrak{H}$ onto $\mathfrak{H}_i$, then we write $T = (T_{ij})_{i,j=1}^{m}$, where T_{ij} denotes the linear operator

$$T_{ij} = E_iTE_j|\mathfrak{H}_j.$$

The following lemma begins our program.

LEMMA 2.1. *Let $T \in \mathfrak{L}(\mathfrak{H})$ be of the form $T = \lambda + K$ for $\lambda \neq 0$ and K compact, and let $\epsilon > 0$. Then there exists a finite-dimensional subspace $\mathfrak{K}$ of $\mathfrak{H}$ such that if $\mathfrak{L}$ and $\mathfrak{M}$ are subspaces satisfying $\mathfrak{L} \subset \mathfrak{K}^\perp$ and $\mathfrak{M} \subset \mathfrak{L}^\perp$, and if E and F denote the projections of $\mathfrak{H}$ onto $\mathfrak{L}$ and $\mathfrak{M}$, respectively, then ETF, FTE, and $ETE - \lambda E$ all have norm less than ϵ.*

Proof. It is well known that there exist finite-dimensional projections P with the property that

$$\|K - PKP\| < \epsilon.$$

(Indeed, if $\{P_n\}$ is any sequence of projections converging strongly to the identity operator, then $\|K - P_nKP_n\| \to 0$.) Fix any one such projection P_0, choose its range for the subspace $\mathfrak{K}$, and denote by L the operator $K - P_0KP_0$. Then with $\mathfrak{L}$, $\mathfrak{M}$, E, and F as in the statement of the lemma, we have that $EK = EL$ and $KE = LE$. Hence, EKF, FKE, and EKE all have norm less than ϵ, and the result follows.

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PROPOSITION 2.2. *Let $T \in \mathfrak{L}(\mathfrak{H})$ be of the form $T = \lambda + K$ for $\lambda \neq 0$ and K compact. Then there exists a decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ of $\mathfrak{H}$ into the direct sum of two infinite-dimensional subspaces such that, if the corresponding matrix for T is*

$$T = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix},$$

then both A_1 and D_1 are invertible.

Proof. To begin with, it is a simple matter to obtain via Lemma 2.1 a preliminary resolution $\mathfrak{H} = \mathfrak{K}_1 \oplus \mathfrak{K}_2$ with respect to which the matrix representation

$$\text{(I)} \qquad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

has the property that D is invertible. Indeed, we have only to choose for ϵ any positive number less than $|\lambda|$, and then choose $\mathfrak{K}_2$ to be any infinite-dimensional subspace whose orthocomplement $\mathfrak{K}_1$ is infinite-dimensional and contains the subspace $\mathfrak{K}$ of Lemma 2.1. Next, note that if U is a unitary operator on $\mathfrak{H}$ with $(U_{ij})_{i,j=1}^2$ as its matrix representation relative to the decomposition $\mathfrak{H} = \mathfrak{K}_1 \oplus \mathfrak{K}_2$, and if

$$\text{(II)} \qquad \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} U_{11}^* & U_{21}^* \\ U_{12}^* & U_{22}^* \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix},$$

then

$$\begin{pmatrix} U^*A_1U & U^*B_1U \\ U^*C_1U & U^*D_1U \end{pmatrix}$$

is the matrix representation for T relative to the decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, where $\mathfrak{H}_i = U^*(\mathfrak{K}_i)$, $i = 1, 2$. Thus the theorem will be proved if we can find a unitary operator U such that in equation (II), both A_1 and D_1 are invertible. Now the operator A is a compression of T, and therefore is also of the form $A = \lambda + K_1$, where K_1 is a compact operator on $\mathfrak{K}_1$. Hence, A is either invertible or has a non-trivial null space. In the former case, the proof is complete; in the latter case, the set of all those vectors $x \in \mathfrak{K}_1$ satisfying $A^k x = 0$ for some positive integer k form a non-trivial finite-dimensional subspace $\mathfrak{N}_1$ of $\mathfrak{K}_1$. Let $\dim(\mathfrak{N}_1) = n$, and define $\mathfrak{N}_2 = \mathfrak{K}_1 \ominus \mathfrak{N}_1$, so that $\mathfrak{K}_1 = \mathfrak{N}_1 \oplus \mathfrak{N}_2$. The subspace $\mathfrak{N}_1$ is invariant under A, and if we write N for the nilpotent operator in $\mathfrak{L}(\mathfrak{N}_1)$ defined by $N = A|\mathfrak{N}_1$, then the matrix representation for A relative to the decomposition $\mathfrak{K}_1 = \mathfrak{N}_1 \oplus \mathfrak{N}_2$ has the form

$$A = \begin{pmatrix} N & A_{12} \\ 0 & A_{22} \end{pmatrix}.$$

The advantage of this particular dissection of A is that the diagonal entry A_{22} is invertible. To see this, note that A_{22} is of the form $\lambda + K_2$, where $K_2 \in \mathfrak{L}(\mathfrak{N}_2)$ is compact. Thus, it suffices to show that A_{22} has trivial null space.

Suppose, accordingly, that $A_{22}y = 0$ with $y \in \mathfrak{N}_2$. Then $Ay \in \mathfrak{N}_1$, so that $A^k(Ay) = A^{k+1}y = 0$ for some $k > 0$. But then $y \in \mathfrak{N}_1$, and therefore $y = 0$.

Let now $\mathfrak{M}_1$ be an n-dimensional subspace of $\mathfrak{K}_2$, the precise determination of which will be made later, and write $\mathfrak{M}_2 = \mathfrak{K}_2 \ominus \mathfrak{M}_1$, so that

$$\mathfrak{H} = \mathfrak{N}_1 \oplus \mathfrak{N}_2 \oplus \mathfrak{M}_1 \oplus \mathfrak{M}_2.$$

The matrix representation for T corresponding to this decomposition may be written as

$$\text{(III)} \qquad T = \begin{pmatrix} N & A_{12} & B_{11} & B_{12} \\ 0 & A_{22} & B_{21} & B_{22} \\ C_{11} & C_{12} & D_{11} & D_{12} \\ C_{21} & C_{22} & D_{21} & D_{22} \end{pmatrix}.$$

We next consider unitary operators $U(\theta)$ on $\mathfrak{H}$ $(0 < \theta < \pi/2)$ whose matrices relative to this same decomposition of $\mathfrak{H}$ have the form

$$U(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta V & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta V^* & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where V is some arbitrary isometry mapping $\mathfrak{M}_1$ onto $\mathfrak{N}_1$. A brief calculation shows that in the representation of $U(\theta)TU^*(\theta)$ as a 2×2 matrix corresponding to the splitting $\mathfrak{H} = \mathfrak{K}_1 \oplus \mathfrak{K}_2$ (see (II) above), the entries A_1 and D_1 are given by

$$A_1(\theta) = \begin{pmatrix} \cos^2\theta N + \sin^2\theta V D_{11} V^* + \sin\theta\cos\theta(B_{11}V^* + VC_{11}) & \cos\theta A_{12} + \sin\theta V C_{12} \\ \sin\theta B_{21} V^* & A_{22} \end{pmatrix}$$

and

$$D_1(\theta) = \begin{pmatrix} \sin^2\theta V^* N V + \cos^2\theta D_{11} - \sin\theta\cos\theta(C_{11}V + V^*B_{11}) & -\sin\theta V^* B_{12} + \cos\theta D_{12} \\ -\sin\theta C_{21} V + \cos\theta D_{21} & D_{22} \end{pmatrix}.$$

Thus our task reduces to choosing the subspace $\mathfrak{M}_1$ and the angle θ in such a way that these operators are invertible. To this end, note that the entries of the matrix (III) are all bounded in norm by $\|T\|$, independently of how the subspace $\mathfrak{M}_1$ is selected. It follows that $\|D_1(\theta) - D\| \to 0$ as $\theta \to 0$, and that this convergence is uniform with respect to $\mathfrak{M}_1$. Since D is invertible, there exist angles $\theta > 0$ so small that $D(\theta)$ is invertible no matter how $\mathfrak{M}_1$ is chosen. We choose one such θ_0, hold it fixed, and proceed to adjust $\mathfrak{M}_1$ so as to make $A_1(\theta_0)$ invertible. That such a choice is possible may be seen most clearly as follows. Let $D_{11} = \lambda + K_3$, with λ and K_3 in $\mathfrak{L}(\mathfrak{M}_1)$. (The operator K_3 depends,

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of course, on the choice of $\mathfrak{M}_1$.) Also, write $A_1(\theta_0) = A_0 + \delta(\mathfrak{M}_1)$, where

$$A_0 = \begin{pmatrix} \cos^2\theta_0 N + \sin^2\theta_0\lambda & \cos\theta_0 A_{12} \\ 0 & A_{22} \end{pmatrix}$$

and

$$\delta(\mathfrak{M}_1) = \begin{pmatrix} \sin^2\theta_0 VK_3V^* + \sin\theta_0\cos\theta_0(B_{11}V^* + VC_{11}) & \sin\theta_0 VC_{12} \\ \sin\theta_0 B_{21}V^* & 0 \end{pmatrix}$$

so that A_0 is independent of the choice of $\mathfrak{M}_1$. Since N is nilpotent and $\sin^2\theta_0\lambda$ is a non-zero scalar, the entry $\cos^2\theta_0 N + \sin^2\theta_0\lambda$ of A_0 is invertible; since A_{22} is also known to be invertible, it follows that A_0 is invertible. On the other hand, according to Lemma 2.1, it is possible to choose $\mathfrak{M}_1$ in such a way so as to make B_{11}, B_{21}, C_{11}, C_{12}, and K_3 as small in norm as desired. Since $\|V\| = 1$, it follows that by appropriate choice of $\mathfrak{M}_1$, $\|\delta(\mathfrak{M}_1)\|$ can be made arbitrarily small. Hence $A_1(\theta_0)$ can be made arbitrarily close to A_0, and the result follows.

Summary. We have shown that if T is an arbitrary operator of the form $\lambda + K$ with $\lambda \neq 0$ and K compact, then T can be viewed, relative to some decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ of $\mathfrak{H}$, as a 2×2 matrix whose diagonal entries are invertible.

If we now identify $\mathfrak{H}_2$ with $\mathfrak{H}_1$ via a unitary isomorphism, then $\mathfrak{H}$ is identified with $\mathfrak{H}_1 \oplus \mathfrak{H}_1$, and T is identified with (is unitarily equivalent to) an operator $T_1 \in \mathfrak{L}(\mathfrak{H}_1 \oplus \mathfrak{H}_1)$. The advantage of this identification is that T_1 can be regarded as a 2×2 matrix all of whose entries act on the same space $\mathfrak{H}_1$; of course, the diagonal entries of T_1 remain invertible. The following lemma thus concludes the proof of our theorem in the separable case.

LEMMA 2.3. *If T is an operator on $\mathfrak{L}(\mathfrak{H} \oplus \mathfrak{H})$ whose 2×2 matrix over $\mathfrak{L}(\mathfrak{H})$ is*

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

where T_1 and T_4 are invertible operators, then there exist invertible operators P, A, and Q on $\mathfrak{H} \oplus \mathfrak{H}$ such that $T = PAQ - QAP$.

Proof. We define P, A, and Q by writing

$$P = \begin{pmatrix} -T_1 & 0 \\ 0 & T_4 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & 0 \\ 1 & A_2 \end{pmatrix}, \quad \text{and} \quad Q = \begin{pmatrix} Q_1 & 1 \\ 0 & Q_2 \end{pmatrix},$$

where the entries A_i, Q_i are to be determined. Note that if A_i and Q_i, $i = 1, 2$, are all invertible, then P, A, and Q are invertible also. A brief calculation reduces the matrix equation $PAQ - QAP = T$ to the system of equations

$$\text{(IV)} \qquad \begin{cases} Q_1A_1T_1 - T_1A_1Q_1 = 0 \\ T_1A_1 + A_2T_4 = -T_2 \\ T_4Q_1 + Q_2T_1 = T_3 \\ T_4A_2Q_2 - Q_2A_2T_4 = 0. \end{cases}$$

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That this system possesses invertible solutions A_1, A_2, Q_1, Q_2 when T_1 and T_4 are themselves both invertible may be seen as follows. If we agree to write

$$A_2 = \alpha Q_2^{-1} \quad \text{and} \quad Q_1 = \beta T_1$$

(where α and β denote positive parameters to be determined), then the first and last equations will be automatically satisfied, so that the problem reduces to solving the third equation

$$\beta T_4 T_1 + Q_2 T_1 = T_3$$

for Q_2 in such a way as to make Q_2 invertible, and then solving the second equation

$$T_1 A_1 + \alpha Q_2^{-1} T_4 = -T_2$$

for A_1 in such a way as to make it invertible. Clearly these requirements will be met if β is first chosen large enough to ensure the invertibility of $T_3 - \beta T_4 T_1$ and if α is then chosen large enough to make $T_2 + \alpha Q_2^{-1} T_4$ invertible.

3. The non-separable case. In this section we sketch a proof of the theorem in the case that $\dim(\mathfrak{H}) = \aleph > \aleph_0$. Let (K) denote the maximal proper norm-closed ideal in $\mathfrak{L}(\mathfrak{H})$. According to (**1**, Theorem 4), the non-commutators on $\mathfrak{H}$ are exactly the operators of the form $\lambda + K$, where $\lambda \neq 0$ and $K \in (K)$. Furthermore, just as above, it suffices to treat the non-commutators. Let $T = \lambda + K$ be such an operator. Then the lemma obtained from Lemma 2.1 above by replacing the phrase "finite-dimensional subspace $\mathfrak{K}$" by "subspace $\mathfrak{K}$ of dimension less than $\aleph$" is valid for T and is essentially contained in (**1**, Lemma 6.1) and (**2**, Lemma 4.1). Accordingly, in the notation of Lemma 2.1, let $\epsilon = |\lambda|/2$, let $\mathfrak{K}$ be the corresponding subspace of dimension less than $\aleph$, and let $\mathfrak{M}$ denote the smallest invariant subspace of T that contains $\mathfrak{K}$. An easy cardinality argument shows that $\mathfrak{M}$ has dimension equal to that of $\mathfrak{K}$. Since $\mathfrak{M}^\perp$ is orthogonal to $\mathfrak{K}$, the compression Z of $T - \lambda$ to $\mathfrak{M}^\perp$ has norm less than $\epsilon = |\lambda|/2$, and it follows that the matrix for T relative to the decomposition $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$ has the form

$$T = \begin{pmatrix} X & Y \\ 0 & Z + \lambda \end{pmatrix}.$$

Since $X \in \mathfrak{L}(\mathfrak{M})$ and $\dim \mathfrak{M} < \aleph$, we may assume by transfinite induction that the conclusion of the theorem holds for X. To see that the conclusion of the theorem also holds for $Z + \lambda$, write $\mathfrak{M}^\perp = \mathfrak{N}_1 \oplus \mathfrak{N}_2$, where $\dim \mathfrak{N}_1 = \dim \mathfrak{N}_2 = \dim \mathfrak{M}^\perp$. Then the matrix for $Z + \lambda$ relative to this resolution has the form

$$Z + \lambda = \begin{pmatrix} Z_1 + \lambda & Z_2 \\ Z_3 & Z_4 + \lambda \end{pmatrix},$$

and since $\|Z\| < \epsilon = |\lambda|/2$, $\|Z_1\|, \|Z_4\| < |\lambda|/2$, from which it follows that $Z_1 + \lambda$ and $Z_4 + \lambda$ are invertible. Thus Lemma 2.3, which is easily seen to be independent of the dimension of $\mathfrak{H}$, can be applied to yield the desired conclusion for $Z + \lambda$.

The proof in the non-separable case is completed by the following lemma.

LEMMA 3.1. *Suppose that the conclusion of the theorem holds for operators X and Z on Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively, and let Y be any operator from $\mathfrak{K}$ to $\mathfrak{H}$. Then the conclusion of the theorem also holds for the operator*

$$\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$$

on the space $\mathfrak{H} \oplus \mathfrak{K}$.

Proof. Choose invertible operators P_i, A_i, and Q_i ($i = 1, 2$) such that

$$P_1A_1Q_1 - Q_1A_1P_1 = X \quad \text{and} \quad P_2A_2Q_2 - Q_2A_2P_2 = Z.$$

Let $P \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{K})$ denote the operator

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix},$$

let

$$A_s = \begin{pmatrix} A_1 & 0 \\ 0 & sA_2 \end{pmatrix},$$

where s is a positive real parameter to be determined, and finally, let

$$Q_s(W) = \begin{pmatrix} Q_1 & W \\ 0 & s^{-1}Q_2 \end{pmatrix},$$

where W is an operator from $\mathfrak{K}$ to $\mathfrak{H}$ which also is to be determined. A simple calculation shows that

$$PA_sQ_s(W) - Q_s(W)A_sP = \begin{pmatrix} X & (P_1A_1)W - W(sA_2P_2) \\ 0 & Z \end{pmatrix}$$

so that, to complete the proof, it suffices to solve the equation

$$\text{(V)} \qquad (P_1A_1)W - W(sA_2P_2) = Y$$

for s and W. Now for fixed s, it is well known that this equation possesses a unique solution W provided only that the spectra of P_1A_1 and sA_2P_2 are disjoint. Furthermore, since A_2P_2 is invertible, it is obviously possible to make these spectra disjoint by choosing s sufficiently large.

REMARK 3.2. The complete story concerning (V) is as follows: the spectrum of the linear transformation

$$W \to BW - WC$$

is precisely the set of differences $\beta - \gamma$, where β and γ run over the spectra of B and C, respectively. The usual proof of this fact (see **3**) assumes that B, C, and W are all operators on the same Hilbert space, but the argument can easily

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be modified so as to apply to the case in which B and C act on different Hilbert spaces and W is a linear transformation from one Hilbert space to the other.

REMARK 3.3. A very short construction due to Paul Federbush shows that every operator T on an infinite-dimensional space can be written as $T = PAQ - QAP$ for P, A, Q not invertible. The argument goes as follows. Write $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$, where $\mathfrak{M}$ and $\mathfrak{M}^\perp$ are of the same dimension, and let P (Q) be an isometry with range $\mathfrak{M}$ $(\mathfrak{M}^\perp)$. If X is an arbitrary operator, then $X = PAQ - QAP$, where $A = P^*XQ^* - Q^*XP^*$. We are also indebted to Federbush for bringing (**4**, Theorem 3) to our attention.

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