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TABULARITY AND POST-COMPLETENESS IN TENSE LOGIC

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Abstract. A new characterization of tabularity in tense logic is established, namely, a tense logic L is tabular if and only if $\mathsf{tab}_n^T \in L$ for some $n \geq 1$. Two characterization theorems for the Post-completeness in tabular tense logics are given. Furthermore, a characterization of the Post-completeness in the lattice of all tense logics is established. Post numbers of some tense logics are shown.

§1. Introduction. Tabularity and Post-completeness in modal logic has been explored for a long time in the literature. Let us recall some notions and results from [4, 8]. We consider the monomodal language $\mathcal{L}_\Box$ in which $\Diamond$ is defined as the dual of $\Box$. From the semantic perspective, every class $\mathcal{C}$ of relational structures has its modal theory $Th(\mathcal{C})$ which consists of modal formulas true or valid in $\mathcal{C}$. Let K be the modal theory of the class of all frames. A *quasi-normal modal logic* is a set of modal formulas S containing K and closed under modus ponens and uniform substitution. A quasi-normal modal logic is *normal* if it is closed under the rule of necessitation. Let $\mathsf{Ext}(S)$ and $\mathsf{NExt}(S)$ denote lattices of all quasi-normal and normal extensions of S respectively.

A modal logic S is *tabular* if $S = Th(\mathfrak{F})$ for some finite frame $\mathfrak{F}$. It is well-known that the tabularity in quasi-normal modal logics can be characterized by peculiar modal formulas (cf., e.g., [4, p. 417]). A quasi-normal modal logic $S \in \mathsf{Ext}(\mathsf{K})$ is tabular if and only if $\mathsf{tab}_n \in S$ for some natural number $n \in \omega$, where $\mathsf{tab}_n = \psi_n \wedge \chi_n$ is the formula defined as follows:

$$\psi_n = \neg(\varphi_1 \wedge \Diamond(\varphi_2 \wedge \Diamond(\varphi_2 \wedge \cdots \wedge \Diamond\varphi_n)\ldots)),$$

$$\chi_n = \bigwedge_{m<n} \neg\Diamond^m(\Diamond\varphi_1 \wedge \cdots \wedge \Diamond\varphi_n),$$

$$\varphi_i = p_1 \wedge \cdots \wedge p_{i-1} \wedge \neg p_i \wedge p_{i+1} \wedge \cdots \wedge p_n \text{ for } 1 \leq i \leq n.$$

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As a corollary, a normal modal logic $S \in \mathsf{NExt}(\mathsf{K})$ is tabular if and only if $\mathsf{alt}_n \wedge \mathsf{tra}_n \in S$ for some $n \in \omega$, where $\mathsf{alt}_n \wedge \mathsf{tra}_n$ is the formula defined as follows:

$$\mathsf{alt}_n = \Box p_0 \vee \Box(p_0 \to p_1) \vee \cdots \vee \Box(p_0 \wedge \cdots \wedge p_{n-1} \to p_n),$$
$$\mathsf{tra}_n = \bigwedge_{i \leq n} \Box^i p \to \Box^{n+1} p.$$

Every tabular modal logic has finitely many extensions and all of them are tabular, and all tabular modal logics are finitely axiomatizable (cf., e.g., [4, 24, 25]).

A consistent modal logic S is *Post-complete* if there exists no consistent proper extension of S. This property in lattices of quasi-normal modal logics is quite complicated (cf., e.g., [4, 5, 11, 14, 15]). There are $2^{\aleph_0}$ Post-complete logics in $\mathsf{Ext}(\mathsf{K4})$ (cf. [4, Theorem 13.15]). However, there are exactly two Post-complete logics in $\mathsf{NExt}(\mathsf{K})$ (cf. [4, 8, 10]). Post-completeness is also related to tabularity. A consistent logic in $\mathsf{Ext}(\mathsf{K})$ is *anti-tabular* if it has no finite models. A logic $S \in \mathsf{Ext}(\mathsf{K})$ is anti-tabular if and only if all Post-complete extensions of S are not tabular. If $S \in \mathsf{Ext}(\mathsf{K4})$ has infinitely many Post-complete extensions, it has an anti-tabular extension (cf. [4, Theorem 13.22]).

The present work is going to study tabularity and Post-completeness in tense logics. The tense language extends the modal language with a past modality $\blacklozenge$ such that $(\blacklozenge, \Box)$ forms an adjoint pair. Dually we get the adjoint pair of operators $(\Diamond, \blacksquare)$. Let $\mathscr{L}_\Box$ and $\mathscr{L}_t$ be the sets of all modal and tense formulas respectively. The least tense logic is denoted by K_t. Let $\Lambda(L)$ be the lattice of extensions of a tense logic L (cf. Definition 2.5). Normal modal and tense logics are correlated. Given a normal modal logic S and tense logic L, let S^+ be the smallest tense logic containing S, and L_+ be the normal modal logic $L \cap \mathscr{L}_\Box$. We say that S^+ is the *minimal tense extension* of S and L_+ the *modal restriction* of L. This gives maps $(.)^+ : \mathsf{NExt}(\mathsf{K}) \to \Lambda(\mathsf{K}_t)$ and $(.)_+ : \Lambda(\mathsf{K}_t) \to \mathsf{NExt}(\mathsf{K})$. If S is Kripke-complete, then $(S^+)_+ = S$. However, the map $((.)^+)_+$ is in general not injective, namely, there exists $S \in \mathsf{NExt}(\mathsf{K})$ with $(S^+)_+ \neq S$ (cf. [17, p. 84]).

There are many results on lattices of tense logics in the literature. A tense logic L has *codimension n* with $n \in \omega$, if there exists a chain $L = L_0 \subset L_1 \subset \cdots \subset L_n = \mathscr{L}_t$ which cannot be refined (cf. [8, 13]). Post-complete tense logics are exactly those of codimension 1. Thomason [16] gives a Kripke-incomplete tense logic of codimension 1. Rautenburg [13] gives a characterization of Post-completeness in the set of all tabular tense logics and describes the splittings of $\Lambda(\mathsf{K}_t)$. Kracht [7] proves results on the splittings of lattices $\Lambda(\mathsf{K}_t)$, $\Lambda(\mathsf{K4}^+)$ and $\Lambda(\mathsf{S4}^+)$. As far as the notion of codimension concerned, Kracht [7] proves that for $n \geq 3$ there are infinitely many logics of codimension n in the lattice $\Lambda(\mathsf{Ga})$ (cf. [8, Section 7.9]). It is remarkable that many properties like completeness, finite model property, decidability and finitely axiomatizability are thoroughly investigated in a series of works by Wolter (cf. [17–23]).

A characterization of tabularity in $\Lambda(\mathsf{K}_t)$ has been given by Chagrov and Shehtman [3]. It says that a tense logic L is tabular if and only if $\alpha_n \wedge \beta_n \in L$ for some $n \in \omega$. Details are found in Remark 3.10. We shall give a new criterion of tabularity in $\Lambda(\mathsf{K}_t)$ by defining formulas tab_n^T with $n \geq 1$ (Theorem 3.7). As far as the Post-completeness in $\Lambda(\mathsf{K}_t)$ is concerned, it is known that there are infinitely many Post-complete tense logics (cf., e.g., [9, 21]). In the present work, we give three characterization theorems for the Post-completeness in $\Lambda(\mathsf{K}_t)$: (i) the first theorem gives three equivalent conditions

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for the Post-completeness of a tense logic $Th(\mathfrak{F})$ where $\mathfrak{F}$ is a finite point-generated frame (Theorem 4.9), and Rautenburg's characterization in [13] follows from this result; (ii) a tabular tense logic L is Post-complete if and only if L has only one point-generated frame up to isomorphism (Theorem 4.12); and (iii) a consistent tense logic L is Post-complete if and only if it satisfies two conditions on constant formulas (Theorem 5.3). Using these results, we give the Post numbers of some tense logics. It is worth mentioning that there exist $2^{\aleph_0}$ Post-complete extensions of a bimodal logic (cf. [6, 8]). Note that tense logics discussed in the present work are bimodal logics with adjointness between modal operators.

This paper is structured as follows. Section 2 gives some preliminaries on tense logic. Section 3 gives a new characterization of tabularity in $\Lambda(\mathsf{K}_t)$. Section 4 proves two characterization theorems on the Post-completeness in the set of all tabular tense logics. Section 5 gives a characterization theorem for the Post-completeness in $\Lambda(\mathsf{K}_t)$. Section 6 gives some concluding remarks.

§2. Preliminaries. We recall some preliminaries on tense logic from [9]. The cardinality of a set X is denoted by $|X|$. Boolean operations of union $\cup$, intersection $\cap$ and complement $\overline{(.)}$ on the powerset $\mathcal{P}(X)$ shall be used. Let $\aleph_0$ be the least infinite cardinal number. The language of tense logic consists of a denumerable set of variables $\Phi = \{p_i : i \in \omega\}$, connectives $\bot$ (falsum) and $\to$ (implication), and tense operators $\Diamond$ (the future) and $\blacklozenge$ (the past).

Definition 2.1. *The set of all formulas $\mathcal{L}_t$ is defined inductively as follows*:

$$\mathcal{L}_t \ni \varphi ::= p \mid \bot \mid (\varphi_1 \to \varphi_2) \mid \Diamond\varphi \mid \blacklozenge\varphi,$$

where $p \in \Phi$. Let $var(\varphi)$ be the set of all variables appearing in φ. A formula φ is constant, *if $var(\varphi) = \varnothing$. Let $\mathcal{L}_t^0$ be the set of all constant formulas.*

We shall use abbreviations $\neg\varphi := \varphi \to \bot$ (negation), $\top := \neg\bot$ (true), $\varphi \wedge \psi := \neg(\varphi \to \neg\psi)$ (conjunction), $\varphi \vee \psi := \neg\varphi \to \psi$ (disjunction) and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$ (equivalence). Let $\Box\varphi := \neg\Diamond\neg\varphi$ (future necessity) and $\blacksquare\varphi := \neg\blacklozenge\neg\varphi$ (past necessity). For every finite set Γ of formulas, let $\bigvee\Gamma$ be the conjunction of all formulas in Γ. In particular, let $\bigvee\varnothing = \bot$.

Note that $\mathcal{L}_t$ forms the formula algebra, and $\mathcal{L}_t^0$ forms the constant formula algebra. A *substitution* is a homomorphism $s : \mathcal{L}_t \to \mathcal{L}_t$. Let φ^s be the formula obtained from φ by the substitution s. A *constant substitution* is a homomorphism $s : \mathcal{L}_t \to \mathcal{L}_t^0$. Let CS be the set of all constant substitutions.

Definition 2.2. *A* frame *is a pair $\mathfrak{F} = (W, R)$ where $W \neq \varnothing$ and $R \subseteq W \times W$. The* inverse *of R is defined as $\breve{R} = \{(v, w) : wRv\}$. For every $w \in W$, one defines*

$$R(w) = \{u \in W : wRu\} \text{ and } \breve{R}(w) = \{u \in W : uRw\}.$$

For every $X \subseteq W$, let $R(X) = \bigcup_{w\in X} R(w)$ and $\breve{R}(X) = \bigcup_{w\in X} \breve{R}(w)$. The unary operations $\Diamond_R$ and $\blacklozenge_R$ on $\mathcal{P}(W)$ are defined by setting

$$\Diamond_R X = \{w \in W : R(w) \cap X \neq \varnothing\} \text{ and } \blacklozenge_R X = \{w \in W : \breve{R}(w) \cap X \neq \varnothing\}.$$

Let $\Box_R X = \overline{\Diamond_R \overline{X}}$ and $\blacksquare_R X = \overline{\blacklozenge_R \overline{X}}$. The cardinality of a frame $\mathfrak{F} = (W, R)$ is defined as the cardinality $|W|$ and denoted by $|\mathfrak{F}|$. A frame $\mathfrak{F}$ is finite, *if $|\mathfrak{F}| < \aleph_0$.*

A model *is a triple* $\mathfrak{M} = (W, R, V)$ *where* (W, R) *is a frame and* $V : \Phi \to \mathcal{P}(W)$ *is a function which is called a* valuation *in* (W, R). *A valuation* V *is extended to the set of all formulas* $\mathcal{L}_t$ *by the following rules:*

$$V(\bot) = \varnothing, \qquad V(\varphi \to \psi) = \overline{V(\varphi)} \cup V(\psi),$$
$$V(\Diamond\varphi) = \Diamond_R V(\varphi), \qquad V(\blacklozenge\varphi) = \blacklozenge_R V(\varphi).$$

A formula φ *is* true *at* w *in* $\mathfrak{M}$ *(notation:* $\mathfrak{M}, w \models \varphi$*) if* $w \in V(\varphi)$.

A formula φ *is* valid *at* w *in* $\mathfrak{F}$ *(notation:* $\mathfrak{F}, w \models \varphi$*) if* $w \in V(\varphi)$ *for every valuation* V *in* $\mathfrak{F}$. *A formula* φ *is* valid *in* $\mathfrak{F}$, *notation* $\mathfrak{F} \models \varphi$, *if* $\mathfrak{F}, w \models \varphi$ *for every* $w \in W$. *A formula* φ *is* valid *in a class* $\mathcal{K}$ *of frames, notation* $\mathcal{K} \models \varphi$, *if* $\mathfrak{F} \models \varphi$ *for every* $\mathfrak{F} \in \mathcal{K}$. *The* theory *of a class* $\mathcal{K}$ *of frames is defined as the set*

$$Th(\mathcal{K}) = \{\varphi : \mathcal{K} \models \varphi\}.$$

If $\mathcal{K} = \{\mathfrak{F}\}$, *we write* $Th(\mathfrak{F})$. *A frame* $\mathfrak{F}$ *is called* a frame for a set Γ of formulas *(notation:* $\mathfrak{F} \models \Gamma$*) if* $\mathfrak{F} \models \varphi$ *for all* $\varphi \in \Gamma$. *Let* $\mathsf{Fr}(\Gamma) = \{\mathfrak{F} : \mathfrak{F} \models \Gamma\}$ *and* $\mathsf{Fr}^{<\aleph_0}(\Gamma) = \{\mathfrak{F} : \mathfrak{F} \models \Gamma \;\&\; |\mathfrak{F}| < \aleph_0\}$. *If* $\Gamma = \{\varphi\}$, *we write* $\mathsf{Fr}(\varphi)$ *and* $\mathsf{Fr}^{<\aleph_0}(\varphi)$.

Definition 2.3. *Let* $\mathfrak{F} = (W, R)$ *be a frame,* $w \in W$ *and* $\varnothing \neq X \subseteq W$. *Let*

$$R(X) = \bigcup_{w \in X} R(w) \text{ and } \breve{R}(X) = \bigcup_{w \in X} \breve{R}(w).$$

For $k \geq 0$, *we define* $S^k(\mathfrak{F}, w)$ *and* $S^\omega(\mathfrak{F}, w)$ *as follows*:

$$S^0_*(\mathfrak{F}, w) = \{w\}, \qquad S^{k+1}_*(\mathfrak{F}, w) = R(S^k(\mathfrak{F}, w)) \cup \breve{R}(S^k(\mathfrak{F}, w)),$$
$$S^k(\mathfrak{F}, w) = \bigcup_{m \leq k} S^m_*(\mathfrak{F}, w), \qquad S^\omega(\mathfrak{F}, w) = \bigcup_{k \geq 0} S^k(\mathfrak{F}, w).$$

Let $S(\mathfrak{F}, w) = S^1(\mathfrak{F}, w)$ *and* $S^\omega(\mathfrak{F}, X) = \bigcup_{w \in X} S^\omega(\mathfrak{F}, w)$. *The* subframe of $\mathfrak{F}$ induced by X *is defined as* $\mathfrak{F}{\restriction}X = (X, R{\restriction}X)$ *where* $R{\restriction}X = R \cap (X \times X)$. *The* subframe of $\mathfrak{F}$ generated by X *is defined as* $\mathfrak{F}{\restriction}S^\omega(\mathfrak{F}, X)$. *A frame* $\mathfrak{G}$ *is a* generated subframe *of* $\mathfrak{F} = (W, R)$, *if* $\mathfrak{G} = \mathfrak{F}{\restriction}S^\omega(\mathfrak{F}, X)$ *for some* $\varnothing \neq X \subseteq W$. *We write* $\mathfrak{F}_w = \mathfrak{F}{\restriction}S^\omega(\mathfrak{F}, w)$. *A frame* $\mathfrak{F}$ *is* point-generated, *if* $\mathfrak{F} = \mathfrak{F}_w$ *for some* w *in* $\mathfrak{F}$.

We use $\mathsf{Fr}_g(\Gamma)$ to denote the class of all point-generated frames for a set Γ of formulas. Let $\mathsf{Fr}_g^{<\aleph_0}(\Gamma) = \mathsf{Fr}_g(\Gamma) \cap \mathsf{Fr}^{<\aleph_0}(\Gamma)$. By [9, Proposition 2.3], if $\mathfrak{G}$ is a generated subframe of $\mathfrak{F}$, then $Th(\mathfrak{F}) \subseteq Th(\mathfrak{G})$.

Definition 2.4. *Let* $\mathfrak{F} = (W, R)$ *and* $\mathfrak{F}' = (W', R')$ *be frames. We use* $\mathfrak{F} \cong \mathfrak{F}'$ *to denote that* $\mathfrak{F}$ *is* isomorphic *to* $\mathfrak{F}'$, *i.e., there exists a bijective function* $f : W \to W'$ *such that, for all* $w, u \in W$, wRu *if and only if* $f(w)R'f(u)$.

A mapping $f : W \to W'$ *is a* bounded morphism *from* $\mathfrak{F}$ *to* $\mathfrak{F}'$, *notation* $f : \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$, *if for all* $w, v \in W$, $v' \in W'$, *the following conditions hold*:

1. *If* wRv, *then* $f(w)R'f(v)$.
2. *If* $f(w)R'v'$, *there exists* $v \in R(w)$ *with* $f(v) = v'$.
3. *If* $f(w)\breve{R}'v'$, *there exists* $v \in \breve{R}(w)$ *with* $f(v) = v'$.

A frame $\mathfrak{F}'$ *is a* bounded morphic image *of* $\mathfrak{F}$, *if there exists a surjective bounded morphism from* $\mathfrak{F}$ *to* $\mathfrak{F}'$.

For a class $\mathcal{K}$ of frames, let $|\mathcal{K}|$ be the cardinality of $\mathcal{K}$ up to isomorphism. By [9, Proposition 2.3], if $\mathfrak{G}$ is a bounded morphic image of $\mathfrak{F}$, then $Th(\mathfrak{F}) \subseteq Th(\mathfrak{G})$.

DEFINITION 2.5. *A* tense logic *is a set of formulas $L \subseteq \mathcal{L}_t$ such that*

(Tau) *L contains all instances of classical propositional tautologies;*
(Dual) $\Diamond p \leftrightarrow \neg\Box\neg p \in L$*;*
(Adj) $\blacklozenge\varphi \to \psi \in L$ *if and only if* $\varphi \to \Box\psi \in L$*;*
(MP) *if* $\varphi, \varphi \to \psi \in L$*, then* $\psi \in L$*;*
(Sub) *if* $\varphi \in L$*, then* $\varphi^s \in L$ *for every substitution* s.

The least tense logic is denoted by K_t. For every tense logic L and a set Σ of formulas, let $L \oplus \Sigma$ denote the smallest tense logic containing $L \cup \Sigma$. A tense logic L is consistent *if $\bot \notin L$. A tense logic L is* finitely axiomatizable*, if there is a finite set Σ of formulas such that $L = \mathsf{K}_t \oplus \Sigma$. A tense logic L_2 is an* extension *of L_1, if $L_1 \subseteq L_2$; and L_2 is a* proper extension *of L_1 (notation: $L_1 \subset L_2$), if $L_1 \subseteq L_2$ and $L_1 \neq L_2$.*

A formula φ is deducible *in a tense logic L from a set Γ of formulas (notation: $\Gamma \vdash_L \varphi$), if $\varphi \in L$ or there exist $\psi_1, \dots, \psi_n \in \Gamma$ with $(\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi \in L$. A set Γ of formulas is L-*consistent *if $\Gamma \nvdash_L \bot$. A set Γ of formulas is* maximal L*-consistent if Γ is L-consistent and it has no L-consistent proper extension.*

REMARK 2.6. *Let L be a tense logic. We can show that the following dual statement of (Adj) holds for L:*

$$(\mathrm{Adj}^{\partial})\Diamond\varphi \to \psi \in L \textit{ if and only if } \varphi \to \blacksquare\psi \in L.$$

Assume $\Diamond\varphi \to \psi \in L$. Then $\neg\psi \to \Box\neg\varphi \in L$. By (Adj), $\blacklozenge\neg\psi \to \neg\varphi \in L$ and so $\varphi \to \blacksquare\psi \in L$. The other direction is similar. Now we show that L is normal, i.e., if $\varphi \in L$, then $\Box\varphi, \blacksquare\varphi \in L$. Assume $\varphi \in L$. Then $\blacklozenge\top \to \varphi, \Diamond\top \to \varphi \in L$. By (Adj) and (Adj$^{\partial}$)*, we have $\top \to \Box\varphi, \top \to \blacksquare\varphi \in L$. Hence $\Box\varphi, \blacksquare\varphi \in L$.*

Let $\Lambda(L)$ be the set of all extensions of a tense logic L. Note that $\Lambda(L)$ is closed under the operations $\cap$ and $\oplus$. Indeed, $(\Lambda(L), \cap, \oplus)$ forms a lattice with top $\mathcal{L}_t$ and bottom L. A tense logic L is *consistent* if $\bot \notin L$. It is obvious that $\mathcal{L}_t$ is the unique inconsistent tense logic.

DEFINITION 2.7. *A tense logic L is* Kripke-complete *if $L = Th(\mathsf{Fr}(L))$. A tense logic L is* tabular *if $L = Th(\mathfrak{F})$ for some finite frame $\mathfrak{F}$. Let* TAB *be the set of all tabular tense logics. A consistent tense logic L is* Post-complete *if there is no consistent proper extension of L. Let* PC *be the set of all Post-complete tense logics. The* Post number *of a tense logic L is the cardinality* $\mathsf{PN}(L) = |\Lambda(L) \cap \mathsf{PC}|$.

If a tense logic L is Kripke-complete, then $L = Th(\mathsf{Fr}_g(L))$. If $L = Th(\mathcal{K})$ for some finite set $\mathcal{K}$ of finite frames, then L is tabular. Every tabular tense logic is obviously Kripke-complete. These results can be found in, e.g., [9]. The *canonical model* for a tense logic L is defined as $\mathfrak{M}^L = (W^L, R^L, V^L)$ where (i) W^L is the set of all maximal L-consistent sets of formulas; (ii) $R^L = \{(w, v) \in W^L \times W^L : \Diamond\varphi \in w \text{ for all } \varphi \in v\}$; and (iii) $V^L(p) = \{w \in W^L : p \in w\}$ for each $p \in \Phi$. The *canonical frame* for L is defined as $\mathfrak{F}^L = (W^L, R^L)$. Some Kripke-completeness results are obtained by the canonical model method (cf. [9]).

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DEFINITION 2.8. *A* general frame *is a pair* $\mathbb{F} = (\mathfrak{F}, A)$ *where* $\mathfrak{F} = (W, R)$ *is a frame and* $A \subseteq \mathcal{P}(W)$ *satisfies the following conditions*:

(1) $\varnothing \in A$.
(2) *If* $X, Y \in A$, *then* $X \cup Y \in A$.
(3) *If* $X \in A$, *then* $\overline{X}, \blacklozenge_R X, \lozenge_R X \in A$.

For a general frame $\mathbb{F} = (\mathfrak{F}, A)$, *let* $\kappa\mathbb{F} = \mathfrak{F}$. *A valuation* V *in* $\mathfrak{F}$ *is* admissible *for* $\mathbb{F}$, *if* $V(p) \in A$ *for every* $p \in \Phi$. *A* general model *is a triple* $\mathbb{M} = (\mathfrak{F}, A, V)$ *where* $(\mathfrak{F}, A)$ *is a general frame and* V *is an admissible valuation.*

Validity in general frames is defined as in Definition 2.2 by replacing valuation with admissible valuation. Let $Th(\mathbb{K})$ be the theory of a class $\mathbb{K}$ of general frames. Descriptive frames are defined as in [9, Definition 3.5]. Let $DF(\Gamma)$ be the class of all descriptive general frames for a set Γ of formulas.

For every general frame $\mathbb{F}$ the theory $Th(\mathbb{F})$ is a tense logic. The *general canonical frame* for a tense logic L is defined as $\mathbb{F}^L = (\mathfrak{F}^L, A^L)$ where $A^L = \{V^L(\varphi) : \varphi \in \mathcal{L}_t\}$. Obviously $\mathbb{F}^L$ is descriptive. By [9, Lemma 3.6], $L = Th(\mathbb{F}^L)$. Moreover, by [9, Theorem 3.7], $DF(L) \neq \varnothing$ and $L = Th(DF(L))$.

DEFINITION 2.9. *Let* $\mathbb{F} = (\mathfrak{F}, A)$ *and* $\mathbb{F}' = (\mathfrak{F}', A')$ *be general frames where* $\mathfrak{F} = (W, R)$ *and* $\mathfrak{F}' = (W', R')$. *For every* $\varnothing \neq X \subseteq W$, *the* subframe of $\mathfrak{F}$ induced by X *is defined as* $\mathbb{F}{\restriction}X = (\mathfrak{F}{\restriction}X, A{\restriction}X)$ *where* $\mathfrak{F}{\restriction}X$ *is the subframe of* $\mathfrak{F}$ *induced by* X *and* $A{\restriction}X = \{Y \cap X : Y \in A\}$. *The* general subframe of $\mathbb{F}$ generated by X *is defined as* $\mathbb{F}{\restriction}S^\omega(\mathfrak{F}, X)$. *We say that* $\mathbb{F}$ *is a* generated general subframe *of* $\mathbb{F}'$, *if* $\mathbb{F} = \mathbb{F}'{\restriction}S^\omega(\mathfrak{F}, X)$ *for some* $\varnothing \neq X \subseteq W$. *We write* $\mathbb{F}_w$ *for* $\mathbb{F}{\restriction}S^\omega(\mathfrak{F}, \{w\})$. *A general frame* $\mathbb{F}$ *is* point-generated *if* $\mathbb{F} = \mathbb{F}_w$ *for some* $w \in W$.

Generated general subframe preserves validity (cf. [9]), i.e., if $\mathbb{G}$ is a generated general subframe of $\mathbb{F}$, then $Th(\mathbb{F}) \subseteq Th(\mathbb{G})$.

DEFINITION 2.10. *Let* $\mathbb{F} = (\mathfrak{F}, A)$ *and* $\mathbb{F}' = (\mathfrak{F}', A')$ *be general frames where* $\mathfrak{F} = (W, R)$ *and* $\mathfrak{F}' = (W', R')$. *A map* $f : W \to W'$ *is a* bounded morphism *from* $\mathbb{F}$ *to* $\mathbb{F}'$ *if* f *is a bounded morphism from* $\mathfrak{F}$ *to* $\mathfrak{F}'$ *such that* $f^{-1}(a') \in A$ *for every* $a' \in A'$. *We say that* $\mathbb{F}'$ *is a* bounded morphic image *of* $\mathbb{F}$ *if there is a bounded morphism from* $\mathbb{F}$ *to* $\mathbb{F}'$.

Bounded morphic image of general frame also preserves validity (cf. [9]), i.e., if $\mathbb{G}$ is a bounded morphic image of $\mathbb{F}$, then $Th(\mathbb{F}) \subseteq Th(\mathbb{G})$.

§3. A new characterization of tabularity in $\Lambda(\mathsf{K}_t)$. In this section, we present a new characterization of tabularity in tense logics. We give some required notions first. For every $n \in \omega$, let $\mathsf{P}(n) = \{\lozenge, \blacklozenge\}^n$ be the set of all n-tuples of tense operators in $\{\lozenge, \blacklozenge\}$. Let $\mathsf{P}(0) = \{\varepsilon\}$ and $\mathsf{P}(\omega) = \bigcup_{n<\omega} \mathsf{P}(n)$. A *possibility* is an element in $\mathsf{P}(\omega)$. We use π with or without subscripts for possibilities. The *length* of a possibility π, denoted by $|\pi|$, is the number of occurrences of tense operators in π. Let $|\varepsilon| = 0$. For every possibility π and formula φ, let $\pi\varphi$ be the formula obtained by putting the sequence of operators π in front of φ.

DEFINITION 3.1. *For every* $n \geq 1$ *and* $\varphi \in \mathcal{L}_t$, *the formula* Δ_n *is defined as*

$$\Delta_n\varphi = \bigvee\{\pi\varphi : \pi \in \mathsf{P}(\omega), |\pi| \leq n\}.$$

Let $\nabla_n\varphi := \neg\Delta_n\neg\varphi$. *The formula* tab_n^T *is defined as*

$$\mathsf{tab}_n^T = \neg(\Delta_n\psi_0 \wedge \cdots \wedge \Delta_n\psi_n),$$

where $\psi_i = \neg p_0 \wedge \cdots \wedge \neg p_{i-1} \wedge p_i$ *for each* $i \leq n$. *Note that* $\psi_0 = p_0$.

For example, $\Delta_1 p = p \vee \Diamond p \vee \blacklozenge p$ and $\Delta_2 p = p \vee \Diamond p \vee \blacklozenge p \vee \Diamond\Diamond p \vee \Diamond\blacklozenge p \vee \blacklozenge\Diamond p \vee \blacklozenge\blacklozenge p$, and $\mathsf{tab}_1^T = \neg((p_0 \vee \Diamond p_0 \vee \blacklozenge p_0) \wedge ((\neg p_0 \wedge p_1) \vee \Diamond(\neg p_0 \wedge p_1) \vee \blacklozenge(\neg p_0 \wedge p_1)))$.

Let $\mathfrak{F} = (W, R)$ be a frame and $w, u \in W$. For every $n \geq 1$, a finite sequence $\langle v_1, \dots, v_n\rangle \in W^n$ is called *a route* of length n between w and u, if $w = v_1$, $u = v_n$ and $v_i R v_{i+1}$ or $v_i \breve{R} v_{i+1}$ for all $i \in \{1, 2, \dots, n-1\}$. Let $\mathsf{R}_n(w, u)$ be the set of all routes of length n between w and u. Let $\mathsf{R}(w, u) = \bigcup_{n\geq 1} \mathsf{R}_n(w, u)$ whose elements are called *routes* between w and u. We use ρ with or without subscripts for a route. The length of a route ρ, denoted by $|\rho|$, is the number of occurrences of elements in the sequence.

LEMMA 3.2. *Let* $\mathfrak{F} = (W, R)$ *be a frame. Then* (1) *for every* $n \geq 1$, $w \in W$ *and* $u \in S^n(\mathfrak{F}, w)$, *there exists* $\rho \in \mathsf{R}(w, u)$ *with* $|\rho| \leq n + 1$; *and* (2) *if* $\mathfrak{F}$ *is point-generated, then* $\mathsf{R}(w, u) \neq \varnothing$ *for all* $w, u \in W$.

Proof. Clearly (1) follows from Definition 2.3. For (2), let $\mathfrak{F} = \mathfrak{F}_v$ for some $v \in W$. Let $w, u \in W$. Then $w \in S^k(\mathfrak{F}, v)$ and $u \in S^l(\mathfrak{F}, v)$ for some $k, l < \omega$. By Definition 2.3, there exists a route between w and u. □

LEMMA 3.3. *Let* $\mathfrak{F} = (W, R)$ *be a frame,* $\mathfrak{M} = (\mathfrak{F}, V)$ *be a model and* $w, u \in W$. *If* $\mathsf{R}_{n+1}(w, u) \neq \varnothing$, *there exists* $\pi \in \mathsf{P}(n)$ *such that for all* $\varphi \in \mathcal{L}_t$ *and* $\psi \in \mathcal{L}_t^0$, (1) *if* $\mathfrak{M}, u \models \varphi$, *then* $\mathfrak{M}, w \models \pi\varphi$; *and* (2) *if* $\mathfrak{F}, u \models \psi$, *then* $\mathfrak{F}, w \models \pi\psi$.

Proof. Clearly (2) follows from (1). The proof proceeds by induction on n. Assume $n = 0$. Suppose $\mathsf{R}_1(w, u) \neq \varnothing$. Then $w = u$ and so $\pi = \varepsilon$ is required. Let $n > 0$. Suppose $\langle v_1, \dots, v_n, v_{n+1}\rangle \in \mathsf{R}_{n+1}(w, u)$. Then $v_1 = w$, $v_{n+1} = u$ and $\langle v_1, \dots, v_n\rangle \in \mathsf{R}_n(w, v_n)$. By induction hypothesis, there exists a possibility $\pi \in \mathsf{P}(n)$ such that for all $\varphi \in \mathcal{L}_t$, if $\mathfrak{M}, v_n \models \varphi$, then $\mathfrak{M}, w \models \pi\varphi$. Suppose $v_n R u$. If $\varphi \in \mathcal{L}_t$ and $\mathfrak{M}, u \models \varphi$, then $\mathfrak{M}, v_n \models \Diamond\varphi$ and so $\mathfrak{M}, w \models \Diamond\pi\varphi$. Hence $\Diamond\pi$ is required. Similarly, if $v_n \breve{R} u$, then $\blacklozenge\pi$ is required. □

LEMMA 3.4. *For every* $n \in \omega$, $\mathfrak{F}, w \models \mathsf{tab}_n^T$ *if and only if* $|S^n(\mathfrak{F}, w)| \leq n$.

Proof. Let $\mathfrak{F} = (W, R)$ be a frame and $w \in W$. Assume $|S^n(\mathfrak{F}, w)| > n$. Then there exists $X = \{w_0, \dots, w_n\} \subseteq S^n(\mathfrak{F}, w)$ with $|X| = n + 1$. Let V be a valuation on $\mathfrak{F}$ such that $V(p_i) \cap X = \{w_i\}$ for all $i \leq n$. By Lemmas 3.2 and 3.3, we have $\mathfrak{F}, V, w \models \Delta_n\psi_i$ for each $i \leq n$. Hence $\mathfrak{F}, w \not\models \mathsf{tab}_n^T$. Assume $\mathfrak{F}, w \not\models \mathsf{tab}_n^T$. Then $\mathfrak{M}, w \models \neg\mathsf{tab}_n^T$ for some model $\mathfrak{M} = (\mathfrak{F}, V)$. Then for each $i \leq n$, $\mathfrak{M}, w \models \Delta_n\psi_i$, and so $\mathfrak{M}, w_i \models \psi_i$ for some $w_i \in W$. Clearly $\{w_0, \dots, w_n\} \subseteq S^n(\mathfrak{F}, w)$ and $w_i \neq w_j$ for $i \neq j \leq n$. Hence $|S^n(\mathfrak{F}, w)| > n$. □

LEMMA 3.5. *For every* $n \in \omega$, $S^n(\mathfrak{F}, w) \neq S^{n+1}(\mathfrak{F}, w)$ *if and only if* $S^n(\mathfrak{F}, w) \neq S^\omega(\mathfrak{F}, w)$.

Proof. The left-to-right direction is trivial. Assume $S^n(\mathfrak{F}, w) = S^{n+1}(\mathfrak{F}, w)$. By Definition 2.3, $S^n(\mathfrak{F}, w) = S^\omega(\mathfrak{F}, w)$. □

LEMMA 3.6. *For every* $n \geq 1$, *if* $\mathfrak{F}, w \models \mathsf{tab}_n^T$, *then* $|S^\omega(\mathfrak{F}, w)| \leq n$.

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Proof. Assume $\mathfrak{F}, w \models \mathsf{tab}_n^T$. By Lemma 3.4, $|S^n(\mathfrak{F}, w)| \leq n$. Suppose $S^n(\mathfrak{F}, w) \neq S^\omega(\mathfrak{F}, w)$. Then $S^i(\mathfrak{F}, w) \neq S^\omega(\mathfrak{F}, w)$ for all $i \leq n$. By Lemma 3.5, $1 = |S^0(\mathfrak{F}, w)| < |S^1(\mathfrak{F}, w)| < \cdots < |S^n(\mathfrak{F}, w)|$. Then $|S^n(\mathfrak{F}, w)| > n$ which contradicts $|S^n(\mathfrak{F}, w)| \leq n$. Hence $S^n(\mathfrak{F}, w) = S^\omega(\mathfrak{F}, w)$ and $|S^\omega(\mathfrak{F}, w)| \leq n$. □

For a new characterization of tabularity, we recall finitely alternative tense logics from [9]. For every $n, m \in \omega$, the (n, m)-*alternative tense logic* is the tense logic $T_{m,n} = \mathsf{K}_t \oplus \{\mathrm{Alt}_n^F, \mathrm{Alt}_m^P\}$ where

$$\mathrm{Alt}_n^F = \Box p_0 \vee \Box(p_0 \to p_1) \vee \cdots \vee \Box(p_0 \wedge \cdots \wedge p_{n-1} \to p_n),$$
$$\mathrm{Alt}_m^P = \blacksquare p_0 \vee \blacksquare(p_0 \to p_1) \vee \cdots \vee \blacksquare(p_0 \wedge \cdots \wedge p_{m-1} \to p_m).$$

For every frame $\mathfrak{F} = (W, R)$ and $w \in W$, (i) $\mathfrak{F}, w \models \mathrm{Alt}_n^F$ if and only if $|R(w)| \leq n$; and (ii) $\mathfrak{F}, w \models \mathrm{Alt}_n^P$ if and only if $|\breve{R}(w)| \leq n$. By [9, Theorem 3.10], every consistent tense logic $L \in \Lambda(T_{n,m})$ is Kripke-complete and so $L = Th(\mathsf{Fr}_g(L))$.

THEOREM 3.7. *For every consistent logic $L \in \Lambda(\mathsf{K}_t)$, $L \in \mathsf{TAB}$ if and only if $\mathsf{tab}_n^T \in L$ for some $n \geq 1$.*

Proof. Assume $L \in \mathsf{TAB}$. Then $L = Th(\mathfrak{F})$ for some frame $\mathfrak{F}$ with $|\mathfrak{F}| = n \in \omega$. By Lemma 3.4, $\mathfrak{F} \models \mathsf{tab}_n^T$, i.e., $\mathsf{tab}_n^T \in L$. Assume $\mathsf{tab}_n^T \in L$ for some $n \geq 1$. Let $\mathfrak{F} = (W, R)$ be a point-generated frame for L and $w \in W$. Then $\mathfrak{F}, w \models \mathsf{tab}_n^T$. By Lemma 3.6, $|W| = |S^\omega(\mathfrak{F}, w)| = |S^n(\mathfrak{F}, w)| \leq n$. It follows that $T_{nn,\subseteq} L$. By [9, Theorem 3.10], $L = Th(\mathsf{Fr}_g(L))$. Clearly $|\mathfrak{F}| \leq n$ for all $\mathfrak{F} \in \mathsf{Fr}_g(L)$. Then $|\mathsf{Fr}_g(L)| < \aleph_0$. By [9, Proposition 3.4], $L \in \mathsf{TAB}$. □

LEMMA 3.8. *If $L \in \mathsf{TAB}$, then all consistent logics in $\Lambda(L)$ are Kripke-complete.*

Proof. Let $L = Th(\mathfrak{F})$ for some frame $\mathfrak{F}$ with $|\mathfrak{F}| = k \in \omega$. Obviously $\mathfrak{F} \models \mathrm{Alt}_k^F \wedge \mathrm{Alt}_k^P$. Then $T_{kk,\subseteq} L$. It follows that $\Lambda(L) \subseteq \Lambda(T_{kk,})$. By [9, Theorem 3.10], all consistent logics in $\Lambda(L)$ are Kripke-complete. □

THEOREM 3.9. *If $L \in \mathsf{TAB}$, then* (1) Λ (L) *is finite and every consistent logic in* $\Lambda(L)$ *is tabular*; *and* (2) L *is finitely axiomatizable.*

Proof. Let $L \in \mathsf{TAB}$ and $L' \in \Lambda(L)$. For (1), by Theorem 3.7, $\mathsf{tab}_n^T \in L \subseteq L'$. By the proof of Theorem 3.7, L' is tabular. Let $f : \Lambda(L) \to \mathcal{P}(\mathsf{Fr}_g(L))$ be the function given by $f(L') = Fr_g(L')$. Clearly $Fr_g(L') \subseteq Fr_g(L)$. By Lemma 3.8, every consistent tense logic in $\Lambda(L)$ is Kripke-complete. Then f is injective and $|\Lambda(L)| \leq 2^{|\mathsf{Fr}_g(L)|}$. By the proof of Theorem 3.7, $|\mathsf{Fr}_g(L)| < \aleph_0$. For (2), it is well-known that every tabular logic is finitely axiomatizable (cf., e.g., [3, 4, 8]). □

REMARK 3.10. *The characterization of tabularity given by Chagrov and Shehtman* [3] *uses formulas α_n and β_n with $n \geq 1$ which are defined as follows*:

(1) α_n *is the conjunction of all formulas of the form*

$$\neg(\gamma_1 \wedge M_1(\gamma_2 \wedge M_2(\gamma_3 \wedge \cdots \wedge M_{n-1}\gamma_n)) \cdots),$$

(2) β_n *is the conjunction of all formulas of the form*

$$\neg M_1 \cdots M_s(M_{s+1}\gamma_1 \wedge \cdots \wedge M_{s+n}\gamma_n),$$

where $s < n$, each $M_i \in \{\Diamond, \blacklozenge\}$ with $1 \leq i \leq s + n$, and for each $1 \leq i \leq n$, $\gamma_i = p_1 \wedge \cdots \wedge p_{i-1} \wedge \neg p_i \wedge p_{i+1} \wedge \cdots \wedge p_n$. For every frame $\mathfrak{F} = (W, R)$ and $w \in W$,

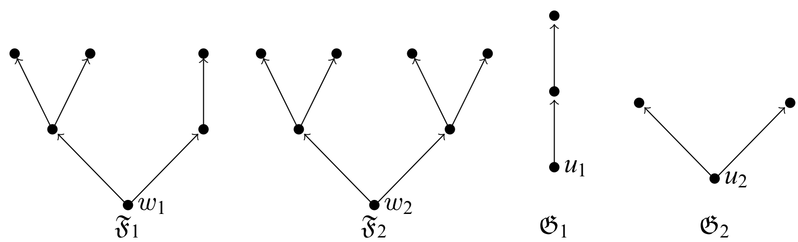

Fig. 1. Frames $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{G}_1$ and $\mathfrak{G}_2$.

(i) $\mathfrak{F}, w \not\models \alpha_n$ if and only if there exists a route $\langle w_0, \ldots, w_{n-1}\rangle$ with $w = w_0$ and $w_i \neq w_j$ for all $i \neq j < n$; and (ii) $\mathfrak{F}, w \not\models \beta_n$ if and only if there exist $m < n$ and $u \in S^m(\mathfrak{F}, w)$ with $|R(u) \cup \breve{R}(u)| \geq n$. It follows that

$$\text{if } \mathfrak{F}_w, w \models \alpha_n \wedge \beta_n, \text{ then } |\mathfrak{F}_w| < f(n) = \sum_{k=0}^{n-1}(n-1)^k. \qquad (\dagger)$$

By these results, the tabularity in tense logic is characterized as follows [3]:

$$A \text{ consistent logic } L \in \mathsf{TAB} \text{ if and only if } \alpha_n \wedge \beta_n \in L \text{ for some } n \geq 1. \qquad (\ddagger)$$

Theorem 3.9 also follows from $(\ddagger)$. For every $n \geq 1$, we can show the difference between tab_n^T and $\alpha_n \wedge \beta_n$. Consider frames $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{G}_1$ and $\mathfrak{G}_2$ in Figure 1. Clearly $\mathfrak{F}_1, w_1 \models \alpha_n \wedge \beta_m$ if and only if $\mathfrak{F}_2, w_2 \models \alpha_n \wedge \beta_m$ for each $n, m \geq 1$. It follows that $\alpha_n \wedge \beta_n$ for all $n \geq 1$ cannot distinguish $(\mathfrak{F}_1, w_1)$ from $(\mathfrak{F}_2, w_2)$. However $\mathfrak{F}_1, w_1 \models \mathsf{tab}_7^T$ and $\mathfrak{F}_2, w_2 \not\models \mathsf{tab}_7^T$. On the other hand, for all $n \geq 1$, $\mathfrak{G}_1 \models \mathsf{tab}_n^T$ if and only if $\mathfrak{G}_2 \models \mathsf{tab}_n^T$. However $\mathfrak{G}_1, u_1 \models \alpha_3 \wedge \beta_3$ and $\mathfrak{G}_2, u_2 \not\models \alpha_3 \wedge \beta_3$. In general, by Lemma 3.6, we can replace the function $f(n)$ in $(\dagger)$ with n *when the cardinality of a frame is concerned. For every point-generated frame $\mathfrak{F}$, $\alpha_n \wedge \beta_n$ just gives the boundary $f(n)$ for the cardinality $|\mathfrak{F}|$, while tab_n^T tells the exact cardinality of $\mathfrak{F}$ since $|\mathfrak{F}| = n$ if and only if $\mathfrak{F} \models \mathsf{tab}_n^T$ and $\mathfrak{F} \not\models \mathsf{tab}_{n-1}^T$.*

§4. Post-completeness in TAB. A characterization of Post-completeness in TAB has been given in [13, Proposition 2] without giving a proof. In this section, we present a new characterization which is called *the first Post-completeness theorem* (Theorem 4.9). We recall the notion of contraction from [13]. A *partition* of a nonempty set W is a subset $\delta \subseteq \mathcal{P}(W)$ such that $\varnothing \notin \delta$, $\bigcup \delta = W$ and $A_1 \cap A_2 = \varnothing$ for all $A_1, A_2 \in \delta$. We use $\mathrm{Part}(W)$ for the set of all partitions of W. For $\delta \in \mathrm{Part}(W)$ and $w \in W$, we write $\delta(w)$ for δ if $w \in \delta$, and call δ the *block* of w. The *trivial partition* of W is $Id_W = \{\{w\} : w \in W\}$. A partition δ_1 is a *refinement* of δ_2, if for every $A \in \delta_1$ there exists $B \in \delta_2$ with $A \subseteq B$.

Definition 4.1. *Let $\mathfrak{F} = (W, R)$ be a frame. A partition δ of W is called a* contraction, *if for all $w, u \in W$, the following conditions hold:*

(C1) *If wRu and $w' \in \delta(w)$, there exists $u' \in \delta(u)$ with $w'Ru'$.*
(C2) *If uRw and $w' \in \delta(w)$, there exists $u' \in \delta(u)$ with $u'Rw'$.*

Let $Ctr(\mathfrak{F})$ be the set of all contraction of $\mathfrak{F}$.

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Obviously, for every frame $\mathfrak{F} = (W, R)$, the trivial partition Id_W belongs to $Ctr(\mathfrak{F})$. Then a frame $\mathfrak{F}$ has no nontrivial contraction if and only if $|Ctr(\mathfrak{F})| = 1$. It is mentioned in [13] that a tabular tense logic L is Post-complete if and only if $L = Th(\mathfrak{F})$ for some finite point-generated frame $\mathfrak{F}$ with $|Ctr(\mathfrak{F})| = 1$. Now we give a new characterization of Post-completeness in TAB utilizing constant formulas. In what follows, for each frame $\mathfrak{F} = (W, R)$ and $w \in W$, the *constant theory* of w in $\mathfrak{F}$ is defined as the set $C_{\mathfrak{F}}(w) = \{\varphi \in \mathcal{L}_t^0 : \mathfrak{F}, w \models \varphi\}$.

Definition 4.2. *Let $\mathfrak{F} = (W, R)$ be a frame. The* constant filtration *of $\mathfrak{F}$ is defined as the frame $\mathfrak{F}^c = (W^c, R^c)$ where*

(1) $W^c = \{C_{\mathfrak{F}}(w) : w \in W\}$,
(2) $C_{\mathfrak{F}}(w) R^c C_{\mathfrak{F}}(u)$ *if and only if* $\Diamond\varphi \in C_{\mathfrak{F}}(w)$ *for every* $\varphi \in C_{\mathfrak{F}}(u)$.

For each $w \in W$, let $[w]_c = \{u \in W : C_{\mathfrak{F}}(w) = C_{\mathfrak{F}}(u)\}$. Let $\delta_{\mathfrak{F}}^c = \{[w]_c : w \in W\}$.

Lemma 4.3. *Let $\mathfrak{F} = (W, R)$ be a finite point-generated frame and $\delta \in Ctr(\mathfrak{F})$. Then (1)$u \in \delta(w)$ implies $C_{\mathfrak{F}}(w) = C_{\mathfrak{F}}(u)$; and (2)$\delta_{\mathfrak{F}}^c \in$ Part(W) and δ is a refinement of $\delta_{\mathfrak{F}}^c$.*

Proof. For (1), note that contraction is a particular case of bisimulation (cf. [1]). By (C1) and (C2), for every formula $\varphi \in \mathcal{L}_t^0$ and $w, u \in W$, if $u \in \delta(w)$, then $\mathfrak{F}, w \models \varphi$ if and only if $\mathfrak{F}, u \models \varphi$. For (2), clearly $\delta_{\mathfrak{F}}^c \in$ Part(W). Let $\delta(w) \in \delta$. Assume $u \in \delta(w)$. By (1), $C_{\mathfrak{F}}(w) = C_{\mathfrak{F}}(u)$ which implies $u \in [w]_c$. Then $\delta(w) \subseteq [w]_c$. Hence δ is a refinement of $\delta_{\mathfrak{F}}^c$. □

Lemma 4.4. *If $\mathfrak{F} = (W, R)$ is a finite point-generated frame, then $\delta_{\mathfrak{F}}^c \in Ctr(\mathfrak{F})$.*

Proof. Let $\mathfrak{F} = (W, R)$ be finite point-generated. By Lemma 4.3(2), $\delta_{\mathfrak{F}}^c \in$ Part(W). Then $\delta_{\mathfrak{F}}^c$ is finite. Let $\delta_{\mathfrak{F}}^c = \{A_i : i \leq n\}$. Then there exists a finite set of formulas $\{\varphi_i \in \mathcal{L}_t^0 : i \leq n\}$ such that, for every $i \leq n$ and $v \in W$, $\mathfrak{F}, v \models \varphi_i$ if and only if $v \in A_i$. Assume wRu and $v \in \delta(w)$. Let $w \in A_i$ and $u \in A_j$ with $i, j \leq n$. Then $\mathfrak{F}, w \models \Diamond\varphi_j$. By Lemma 4.3(1), $\mathfrak{F}, v \models \Diamond\varphi_j$. Thus there exists $v' \in R(v)$ with $\mathfrak{F}, v' \models \varphi_j$. Then $v' \in A_j$. Hence (C1) holds. Similarly we can prove (C2). It follows that $\delta_{\mathfrak{F}}^c \in Ctr(\mathfrak{F})$. □

Theorem 4.5. *If $\mathfrak{F} = (W, R)$ is a finite point-generated frame, then $|Ctr(\mathfrak{F})| = 1$ if and only if $\delta_{\mathfrak{F}}^c = Id_W$.*

Proof. Let $\mathfrak{F} = (W, R)$ be finite point-generated. By Lemma 4.4, $\delta_{\mathfrak{F}}^c \in Ctr(\mathfrak{F})$. Hence $|Ctr(\mathfrak{F})| = 1$ which implies $\delta_{\mathfrak{F}}^c = Id_W$. Suppose $\delta \in Ctr(\mathfrak{F})$ and $\delta \neq Id_W$. By Lemma 4.3(2), we have $\delta_{\mathfrak{F}}^c \neq Id_W$. □

Definition 4.6. *Let $\mathfrak{F} = (W, R)$ be a frame and $X \subseteq W$. A set of constant formulas Σ is* satisfiable *in X, if there exists $w \in X$ with $\mathfrak{F}, w \models \varphi$ for all $\varphi \in \Sigma$; and Σ is* finitely satisfiable *in X, if every finite subset of Σ is satisfiable in X.*

A frame $\mathfrak{F}$ is called 0-saturated, *if for every $w \in W$ and $\Sigma \subseteq \mathcal{L}_t^0$, the following conditions hold: (i) if Σ is finitely satisfiable in $R(w)$, then Σ is satisfiable in $R(w)$; and (ii) if Σ is finitely satisfiable in $\breve{R}(w)$, then Σ is satisfiable in $\breve{R}(w)$.*

Lemma 4.7. *Let $\mathfrak{F} = (W, R)$ be* 0*-saturated and $f : W \to W^c$ be the function with $f(w) = C_{\mathfrak{F}}(w)$ for all $w \in W$. Then $f : \mathfrak{F} \twoheadrightarrow \mathfrak{F}^c$.*

Proof. Clearly f is surjective. Suppose wRu. Then $C_{\mathfrak{F}}(w) R^c C_{\mathfrak{F}}(u)$. Assume $C_{\mathfrak{F}}(w) R^c C_{\mathfrak{F}}(u)$. It suffices to to show that there exists $v \in R(w)$ with $C_{\mathfrak{F}}(u) = C_{\mathfrak{F}}(v)$.

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Let $\Theta = \{\varphi_1, \dots, \varphi_n\} \subseteq C_{\mathfrak{F}}(u)$. Then $\Diamond(\varphi_1 \wedge \cdots \wedge \varphi_n) \in f(w)$, i.e., $\mathfrak{F}, w \models \Diamond(\varphi_1 \wedge \cdots \wedge \varphi_n)$. Then Θ is satisfiable in $R(w)$. Hence $C_{\mathfrak{F}}(u)$ is finitely satisfiable in $R(w)$. Since $\mathfrak{F}$ is 0-saturated, there exists $v \in R(w)$ with $\mathfrak{F}, v \models \psi$ for all $\psi \in C_{\mathfrak{F}}(u)$. Clearly $C_{\mathfrak{F}}(u) \subseteq C_{\mathfrak{F}}(v)$. Suppose $\chi \notin C_{\mathfrak{F}}(u)$. Then $\neg\chi \in C_{\mathfrak{F}}(u)$. Hence $\neg\chi \in C_{\mathfrak{F}}(v)$, i.e., $\chi \notin C_{\mathfrak{F}}(v)$. Hence $C_{\mathfrak{F}}(u) = C_{\mathfrak{F}}(v)$. Similarly $C_{\mathfrak{F}}(u) R^c C_{\mathfrak{F}}(w)$ implies $C_{\mathfrak{F}}(v) = C_{\mathfrak{F}}(u)$ for some $v \in \breve{R}(w)$. □

LEMMA 4.8. *If $\mathfrak{F} = (W, R)$ is a finite point-generated frame, then $\mathfrak{F}$ is 0-saturated and $Th(\mathfrak{F}) \subseteq Th(\mathfrak{F}^c)$.*

Proof. Let $\mathfrak{F} = (W, R)$ be finite point-generated, $w \in W$ and $\Sigma \subseteq \mathcal{L}_t^0$. Let $R(w) = \{w_0, \dots, w_n\}$ and $|R(w)| = n + 1$. Suppose that Σ is not satisfiable in $R(w)$. Then for every $i \leq n$, there exists $\varphi_i \in \Sigma$ with $\mathfrak{F}, w_i \models \neg\varphi_i$. Hence $\mathfrak{F}, w \models \Box\neg(\varphi_0 \wedge \cdots \wedge \varphi_n)$. Let $\Theta = \{\varphi_0, \dots, \varphi_n\}$. Then Θ is not satisfiable in $R(w)$, i.e., Σ is not finitely satisfiable in $R(w)$. Similarly, if Σ is finitely satisfiable in $\breve{R}(w)$, then Σ is satisfiable in $\breve{R}(w)$. It follows that $\mathfrak{F}$ is 0-saturated. By Lemma 4.7, $\mathfrak{F} \twoheadrightarrow \mathfrak{F}^c$. By [9, Proposition 2.6], $Th(\mathfrak{F}) \subseteq Th(\mathfrak{F}^c)$. □

THEOREM 4.9 (The first Post-completeness theorem). *Let $\mathfrak{F} = (W, R)$ be a finite point-generated frame. The following are equivalent:*

(1) *$Th(\mathfrak{F})$ is Post-complete.*
(2) *$\mathfrak{F} \cong \mathfrak{F}^c$.*
(3) *For every $w, u \in W$, $C_{\mathfrak{F}}(w) = C_{\mathfrak{F}}(u)$ if and only if $w = u$.*
(4) *$|Ctr(\mathfrak{F})| = 1$.*

Proof. By Lemma 4.5, (3) is equivalent to (4). Clearly (3) implies (2). Now we show that (1) implies (3). Assume $Th(\mathfrak{F}) \in \mathsf{PC}$. For a contradiction, suppose $C_{\mathfrak{F}}(w) = C_{\mathfrak{F}}(u)$ and $w \neq u$. Recall that $[w]_c = \{v \in W : C_{\mathfrak{F}}(v) = C_{\mathfrak{F}}(w)\}$. Since $\mathfrak{F}$ is finite, there exists $\varphi_w \in C_{\mathfrak{F}}(w)$ such that $\mathfrak{F}, v \models \neg\varphi_w$ for all $v \notin [w]_c$. Since $\mathfrak{F}$ is point-generated, by Lemma 3.2(2), $\mathsf{R}(w, u) \neq \varnothing$. Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a model with $V(p) = \{w\}$. By Lemma 3.3, there exists $\pi \in \mathsf{P}(\omega)$ such that $\mathfrak{M}, u \models \varphi$ implies $\mathfrak{M}, w \models \pi\varphi$ for all $\varphi \in \mathcal{L}_t$. Thus $\mathfrak{M}, w \not\models (\varphi_w \wedge p) \to \neg\pi\neg(\varphi_w \to p)$. Hence $\mathfrak{F} \not\models (\varphi_w \wedge p) \to \neg\pi\neg(\varphi_w \to p)$. Let $\mathfrak{F}^c$ be the constant filtration of $\mathfrak{F}$. Since $\mathfrak{F}$ is finite point-generated, by Lemma 4.8, $Th(\mathfrak{F}) \subseteq Th(\mathfrak{F}^c)$. Note that $C_{\mathfrak{F}}(w)$ is the only point in W^c validating φ_w. Then $\mathfrak{F}^c \models (\varphi_w \wedge p) \to \neg\pi\neg(\varphi_w \to p)$. Hence $Th(\mathfrak{F}) \subsetneq Th(\mathfrak{F}^c)$ which contradicts the assumption $Th(\mathfrak{F}) \in \mathsf{PC}$.

Now we show that (2) implies (1). Suppose $\mathfrak{F} \cong \mathfrak{F}^c$ and $Th(\mathfrak{F}) \subseteq L$ where L is a consistent tense logic. Let $W = \{w_0, \dots, w_n\}$ and $|W| = n + 1$. Since $\mathfrak{F} \cong \mathfrak{F}^c$, for every $i \neq j \leq n$, $C_{\mathfrak{F}}(w_i) \neq C_{\mathfrak{F}}(w_j)$. For each $i \leq n$, there exists $\varphi_i \in \mathcal{L}_t^0$ such that $\mathfrak{F}, w \models \varphi_i$ if and only if $w = w_i$. For every $\psi \in \mathcal{L}_t^0$ and $i \leq n$, $\mathfrak{F}, w_i \models \psi$ implies $\mathfrak{F} \models \varphi_i \to \psi$. Hence $\{\varphi_i \to \psi \in \mathcal{L}_t^0 : \mathfrak{F}, w_i \models \psi\} \subseteq Th(\mathfrak{F})$ for all $i \leq n$. Since $\mathfrak{F}$ is point-generated, by Lemmas 3.2 and 3.3, for all $i, j \leq n$, there exists $\pi_{i,j} \in \mathsf{P}(\omega)$ with $\mathfrak{F}, w_i \models \pi_{i,j}\varphi_j$. By Lemma 3.8, L is Kripke-complete. Then $\mathsf{Fr}_g(L) \neq \varnothing$. Let $\mathfrak{F}' = (W', R')$ be a point-generated frame for L. Now we show $W'^c = W^c$ as follows:

(i) Assume $u \in W'$. Clearly $\bigvee_{i \leq n} \varphi_i \in Th(\mathfrak{F}) \subseteq L$. Then $\mathfrak{F}', u \models \varphi_{i_u}$ for some $i_u \leq n$. Clearly $\{\varphi_{i_u} \to \psi \in \mathcal{L}_t^0 : \mathfrak{F}, w_{i_u} \models \psi\} \subseteq Th(\mathfrak{F})$. Suppose $\psi \in C_{\mathfrak{F}}(w_{i_u})$. Then $\varphi_{i_u} \to \psi \in Th(\mathfrak{F}) \subseteq L$. Then $\mathfrak{F}' \models \varphi_{i_u} \to \psi$. By $\mathfrak{F}', u \models \varphi_{i_u}$, we have $\mathfrak{F}', u \models \psi$, i.e., $\psi \in C_{\mathfrak{F}'}(u)$. Then $C_{\mathfrak{F}}(w_{i_u}) \subseteq C_{\mathfrak{F}'}(u)$. If $\chi \notin C_{\mathfrak{F}}(w_{i_u})$, then

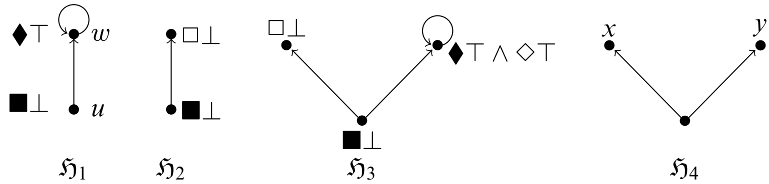

Fig. 2. Frames $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3$ and $\mathfrak{H}_4$.

$\neg\chi \in C_{\mathfrak{F}}(w_{i_u})$ and $\neg\chi \in C_{\mathfrak{F}'}(u)$, i.e., $\chi \notin C_{\mathfrak{F}'}(u)$. Hence $C_{\mathfrak{F}'}(u) \subseteq C_{\mathfrak{F}}(w_{i_u})$. Then $C_{\mathfrak{F}'}(u) = C_{\mathfrak{F}}(w_{i_u})$. Hence $W'^c \subseteq W^c$.

(ii) Assume $w_k \in W$. Take any $u \in W'$. By (i), $C_{\mathfrak{F}'}(u) = C_{\mathfrak{F}}(w_{i_u})$ for some $i_u \leq n$. Clearly $\mathfrak{F}, w_{i_u} \models \pi_{i_u,k}\varphi_k$. Then $\mathfrak{F}', u \models \pi_{i_u,k}\varphi_k$. There exists $u_k \in S^\omega(\mathfrak{F}', u) = W'$ with $\mathfrak{F}', u_k \models \varphi_k$. Clearly $\{\varphi_k \to \psi \in \mathcal{L}_t^0 : \mathfrak{F}, w_k \models \psi\} \subseteq Th(\mathfrak{F})$. Like the proof (i), we have $C_{\mathfrak{F}'}(u_k) = C_{\mathfrak{F}}(w_k)$. Hence $W^c \subseteq W'^c$.

Then $\mathfrak{F}'^c = \mathfrak{F}^c \cong \mathfrak{F}$. By $|\mathfrak{F}| = n+1$ and Lemma 3.4, $\mathfrak{F} \models \mathsf{tab}_{n+1}^T$. Then $\mathfrak{F}' \models \mathsf{tab}_{n+1}^T$. Since $\mathfrak{F}'$ is point-generated, by Lemma 3.6, $\mathfrak{F}'$ is finite. By Lemma 4.8, $L \subseteq Th(\mathfrak{F}') \subseteq Th(\mathfrak{F}'^c) = Th(\mathfrak{F}) \subseteq L$. Then $Th(\mathfrak{F}) = L$. Hence $Th(\mathfrak{F}) \in \mathsf{PC}$. □

Corollary 4.10. *The Post-completeness of the tense logic $Th(\mathfrak{F})$ for a given finite frame $\mathfrak{F}$ is decidable.*

Proof. Let $\mathfrak{F} = (W, R)$ be a finite frame. Then $Ctr(\mathfrak{F})$ is finite. By Theorem 4.9, it suffices to check if there exists a non-trivial contraction in $Ctr(\mathfrak{F})$. This is done in finitely many steps. □

By Theorem 4.9, we can check the Post-completeness of $Th(\mathfrak{F})$ for certain finite frames $\mathfrak{F}$. For example, consider frames $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$, and $\mathfrak{H}_4$ in Figure 2 and their theories L_1, L_2, L_3 and L_4 respectively. For $\mathfrak{H}_1, \mathfrak{H}_2$ and $\mathfrak{H}_3$, we can distinguish different points by constant formulas as in Figure 2. For $\mathfrak{H}_4$, clearly $C_{\mathfrak{H}_4}(x) = C_{\mathfrak{H}_4}(y)$ while $x \neq y$. By Theorem 4.9, L_1, L_2 and L_3 are Post-complete but L_4 is not. Note that L_1 was incorrectly claimed not to be Post-complete in [13].

Lemma 4.11. *Let $\mathfrak{F} = (W, R)$ be a finite point-generated frame. If $L = Th(\mathfrak{F}) \in \mathsf{PC}$, then $|\mathsf{Fr}_g(L)| = 1$.*

Proof. Let $\mathfrak{F} = (W, R)$ be finite point-generated and $W = \{w_0, \dots, w_{n-1}\}$. Assume $L = Th(\mathfrak{F}) \in \mathsf{PC}$. By Theorem 4.9, for all $w, u \in W$, $C_{\mathfrak{F}}(w) = C_{\mathfrak{F}}(u)$ if and only if $w = u$. Then there exist $\varphi_0, \dots, \varphi_{n-1} \in \mathcal{L}_t^0$ such that

$$\text{for all } u \in W \text{ and } i < n, \mathfrak{F}, u \models \varphi_i \text{ if and only if } u = w_i. \qquad (\sharp)$$

Indeed, we have the following claims:

(i) $\mathsf{tab}_n^T \in L$. Since $\mathfrak{F}$ is finite point-generated, by Lemma 3.4, $\mathfrak{F} \models \mathsf{tab}_n^T$.

(ii) $\Delta_n\varphi_i \in L$ for every $i < n$. Since $\mathfrak{F}$ is point-generated, for every $u \in W$, we obtain $S^n(\mathfrak{F}, u) = W$. By Lemma 3.3, $\mathfrak{F} \models \Delta_n\varphi_i$ for every $i < n$.

(iii) $\varphi_i \wedge \varphi_j \to \bot \in L$ whenever $i \neq j < n$. This follows from ($\sharp$).

(iv) If $w_i R w_j$, then $\varphi_i \to \Diamond\varphi_j \in L$. This follows from ($\sharp$).

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Suppose $\mathfrak{G} = (U, T) \in \mathsf{Fr}_g(L)$. By (i) and Lemma 3.6, $|U| \leq n$. By (ii) and (iii), for every $i < n$, there exists $u \in U$ with $\mathfrak{G}, u \models \varphi_i \wedge \bigwedge_{j \neq i} \neg\varphi_j$. Hence $|U| = n$. Let $U = \{u_0, \dots, u_{n-1}\}$. Without loss of generality, let $\mathfrak{G}, u_i \models \varphi_i$ for each $i < n$. Let $f : W \to U$ be the function with $f(w_i) = u_i$ for each $i < n$. Clearly f is bijective. Assume $w_i R w_j$. By (iv), $\varphi_i \to \Diamond\varphi_j \in L$. Since φ_i holds only in u_i and φ_j holds only in u_j, we have $u_i T u_j$. Similarly $u_i T u_j$ implies $w_i R w_j$. Then $\mathfrak{F} \cong \mathfrak{G}$. Hence $|\mathsf{Fr}_g(L)| = 1$. □

THEOREM 4.12 (The second Post-completeness theorem). *Let $L \in \mathsf{TAB}$. Then L is Post-complete if and only if $|\mathsf{Fr}_g(L)| = 1$.*

Proof. Let $L \in \mathsf{TAB}$. Assume $|\mathsf{Fr}_g(L)| = 1$. Suppose $L \subseteq L'$ where L' is a consistent tense logic. By Lemma 3.9(1), L' is tabular. By Lemma 3.8, L' is Kripke-complete and so $L' = Th(\mathsf{Fr}_g(L'))$. Clearly $\mathsf{Fr}_g(L') \subseteq \mathsf{Fr}_g(L)$. Then $L = Th(\mathsf{Fr}_g(L)) \subseteq Th(\mathsf{Fr}_g(L')) = L'$. Hence $L = L'$. It follows that $L \in \mathsf{PC}$. The other way round, assume $L \in \mathsf{PC}$. Since $L \in \mathsf{TAB}$, there exists a finite frame $\mathfrak{F} = (W, R)$ with $L = Th(\mathfrak{F})$. Let $w \in W$. By [9, Proposition 2.3], $Th(\mathfrak{F}) \subseteq Th(\mathfrak{F}_w)$. Since $L \in \mathsf{PC}$, $L = Th(\mathfrak{F}) = Th(\mathfrak{F}_w)$. By Lemma 4.11, $|\mathsf{Fr}_g(L)| = 1$. □

The second Post-completeness theorem gives a new characterization of Post-completeness in TAB. Consider frames in Figure 2 and their tense logics which are tabular. Note that $\mathsf{Fr}_g(L_i) = \{\mathfrak{H}_i\}$ for $i = 1, 2, 3$, and $\mathsf{Fr}_g(L_4) = \{\mathfrak{H}_2, \mathfrak{H}_4\}$. Then $L_1, L_2, L_3 \in \mathsf{PC}$ and $L_4 \notin \mathsf{PC}$.

§5. Post-completeness in $\Lambda(\mathsf{K}_t)$.

In this section, we explore the Post-completeness in the lattice $\Lambda(\mathsf{K}_t)$ and prove *the third Post-completeness theorem* (Theorem 5.3). Note that Theorem 4.9 gives a characterization of the Post-completeness of $Th(\mathfrak{F})$ where $\mathfrak{F}$ is a finite point-generated frame. We first show that there exists a Post-complete tense logic which has no finite frames. In what follows, the 0-general frame based on a frame $\mathfrak{F} = (W, R)$ is defined as $\mathfrak{F}^\clubsuit = (W, R, A^\clubsuit)$ where $A^\clubsuit = \{V(\varphi) : \varphi \in \mathcal{L}_t^0\}$ for arbitrary valuation V in $\mathfrak{F}$. The definition of $A^\clubsuit$ does not depend on the choice of valuation V.

PROPOSITION 5.1. *Let $\mathfrak{N} = (\omega, <)$ where $<$ is the strict natural order on ω. Let $L = Th(\mathfrak{N}^\clubsuit)$. Then $\mathsf{Fr}^{<\aleph_0}(L) = \varnothing$ and L is Post-complete.*

Proof. Assume $\mathfrak{F} \models L$ and $\mathfrak{F} = (W, R)$. Clearly $\blacklozenge\blacksquare\bot \vee \blacksquare\bot \in L$. It is easy to verify that $\mathfrak{N}^\clubsuit, 0 \models \Diamond(\blacklozenge^n\top \wedge \blacksquare^{n+1}\bot)$ for all $n \in \omega$, and $\mathfrak{N}^\clubsuit, m \not\models \blacksquare\bot$ for all $m > 0$. Then $\{\blacksquare\bot \to \Diamond(\blacklozenge^n\top \wedge \blacksquare^{n+1}\bot) : n \in \omega\} \subseteq L$. By $\mathfrak{F} \models L$, we have $\mathfrak{F} \models \blacklozenge\blacksquare\bot \vee \blacksquare\bot$. Then for every $n \in \omega$, $\{w \in W : \mathfrak{F}, w \models \blacklozenge^n\top \wedge \blacksquare^{n+1}\bot\} \neq \varnothing$. We choose a set $\{w_n : n \in \omega\} \subseteq W$ such that (i) $\mathfrak{F}, w_n \models \blacklozenge^n\top \wedge \blacksquare^{n+1}\bot$ for each $n \in \omega$; and (ii) for every $i, j \in \omega$, $w_i = w_j$ if and only if $i = j$. Then $\aleph_0 = |\{w_n : n \in \omega\}| \leq |W|$, i.e., $\mathfrak{F}$ is infinite. It follows that $\mathsf{Fr}^{<\aleph_0}(L) = \varnothing$.

Assume $\varphi \notin L$. Then $\mathfrak{N}^\clubsuit, V, w \models \neg\varphi$ for some admissible valuation V in $\mathfrak{N}^\clubsuit$ and $w \in W$. Let $var(\varphi) = \{p_1, \dots, p_n\}$. For every $p \in var(\varphi)$, we choose a constant formula $\psi_p \in \mathcal{L}_t^0$ with $V(p) = V(\psi_p)$. Let s be a substitution with $s(p_k) = \psi_{p_k}$ for each $1 \leq k \leq n$. Then $\mathfrak{N}^\clubsuit, w \models \neg\varphi^s$ and $\blacksquare\bot \to (\Diamond\neg\varphi^s \vee \neg\varphi^s) \in L$. Suppose that $L \oplus \varphi$ is consistent. Then $\mathbb{F} \models L \oplus \varphi$ for some descriptive frame $\mathbb{F}$. By $\mathbb{F} \models L$, we have $\mathbb{F}, w_0 \models \blacksquare\bot$ for some $w_0 \in W$. By $\blacksquare\bot \to (\Diamond\neg\varphi^s \vee \neg\varphi^s) \in L$, we have $\mathbb{F}, u \models \neg\varphi^s$ for some $u \in W$. Then $\mathbb{F} \not\models \varphi$ which contradicts $\mathbb{F} \models L \oplus \varphi$. Then $L \oplus \varphi$ is not consistent. Hence $L \in \mathsf{PC}$. □

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LEMMA 5.2. *Let $\mathfrak{F} = (W, R)$ be a frame and $L = Th(\mathfrak{F}^{\clubsuit})$. For every formula $\varphi \in \mathcal{L}_t$, if $\varphi \notin L$, then $\varphi^s \notin L$ for some $s \in \mathsf{CS}$.*

Proof. Assume $\varphi \notin L$. Then $\mathfrak{F}^{\clubsuit}, V, w \models \neg\varphi$ for some $w \in W$ and admissible valuation V in $\mathfrak{F}^{\clubsuit}$. For each $p \in \Phi$, we choose $\psi_p \in \mathcal{L}_t^0$ with $V(p) = V(\psi_p)$. Let s be the constant substitution with $s(p) = \psi_p$ for each $p \in \Phi$. Then $\mathfrak{F}^{\clubsuit}, w \models \neg\varphi^s$. By $\mathfrak{F}^{\clubsuit} \models L$, we have $\varphi^s \notin L$. $\square$

THEOREM 5.3 (The third Post-completeness theorem). *A consistent tense logic L is Post-complete if and only if the following conditions hold:*

(1) *For every $\psi \in \mathcal{L}_t^0$, if $\neg\psi \notin L$, then $\Delta_n\psi \in L$ for some $n \in \omega$.*
(2) *For every $\varphi \in \mathcal{L}_t$, if $\varphi \notin L$, then $\varphi^s \notin L$ for some $s \in \mathsf{CS}$.*

Proof. Assume $L \in \mathsf{PC}$. For (1), suppose that there exists $\psi \in \mathcal{L}_t^0$ such that $\neg\psi \notin L$ and $\Delta_n\psi \notin L$ for all $n \in \omega$. Now we show that $\Sigma = \{\nabla_n\neg\psi : n \in \omega\}$ is L-consistent. Suppose not. There exist $n_1, \dots, n_k \in \omega$ with $\neg(\nabla_{n_1}\neg\psi \wedge \dots \wedge \nabla_{n_k}\neg\psi) \in L$, i.e., $\Delta_{n_1}\psi \vee \dots \vee \Delta_{n_k}\psi \in L$. Clearly, if $1 \leq i \leq j \leq k$, then $\Delta_{n_i}\psi \to \Delta_{n_j}\psi \in L$. Let $h = max\{n_1, \dots, n_k\}$. Then $\Delta_h\psi \in L$ which contradicts $\Delta_h\psi \notin L$. Hence Σ is L-consistent. Let w be a maximal L-consistent set with $\Sigma \subseteq w$. Then $\mathbb{F}_w^L \models L$. By $\mathbb{F}_w^L \models \Sigma$, we have $\mathbb{F}_w^L \models \neg\psi$. Hence $\mathbb{F}_w^L \models L \oplus \neg\psi$. Then $L \subsetneq Th(\mathbb{F}_w^L)$ which contradicts $L \in \mathsf{PC}$. For (2), $Th(\mathbb{F}^L) \subseteq Th((\mathfrak{F}^L)^{\clubsuit})$. By $L \in \mathsf{PC}$, $L = Th(\mathbb{F}^L) \subseteq Th((\mathfrak{F}^L)^{\clubsuit}) = L$. By Lemma 5.2, (2) holds.

Assume $L \notin \mathsf{PC}$. For a contradiction, suppose that both (1) and (2) hold. By the assumption, there exists a formula $\varphi \notin L$ such that $L' = L \oplus \varphi$ is consistent. By $\varphi \notin L$ and (2), $\varphi^s \notin L$ for some $s \in \mathsf{CS}$. Then $\neg\neg\varphi^s \notin L$. By (1), $\Delta_n\neg\varphi^s \in L$ for some $n \in \omega$. Then $\Delta_n\neg\varphi^s \in L'$. By $\varphi^s \in L'$, we have $\nabla_n\varphi^s \in L'$ which contradicts $\Delta_n\neg\varphi^s \in L'$. $\square$

COROLLARY 5.4. *If $\mathfrak{F} = (W, R)$ is a finite point-generated frame, then $Th(\mathfrak{F}^{\clubsuit})$ is Post-complete.*

Proof. Let $\mathfrak{F} = (W, R)$ be finite point-generated. It suffices to show that the conditions (1) and (2) in Theorem 5.3 hold. For (1), assume $\psi \in \mathcal{L}_t^0$ and $\neg\psi \notin Th(\mathfrak{F}^{\clubsuit})$. Then there exists $w \in W$ with $\mathfrak{F}^{\clubsuit}, w \not\models \neg\psi$. Then $\mathfrak{F}^{\clubsuit}, w \models \psi$. Let $n = |W|$. Then $\mathfrak{F}^{\clubsuit} \models \Delta_n\psi$. Moreover, (2) follows from Lemma 5.2. $\square$

COROLLARY 5.5. *Let $\mathfrak{F} = (W, R)$ be a point-generated frame. If $W = S^n(\mathfrak{F}, w)$ for some $w \in W$ and $n \in \omega$, then $Th(\mathfrak{F}^{\clubsuit})$ is Post-complete.*

Proof. Assume $W = S^n(\mathfrak{F}, w)$ with $w \in W$ and $n \in \omega$. It suffices to show that the conditions (1) and (2) in Theorem 5.3 hold. For (1), assume $\psi \in \mathcal{L}_t^0$ and $\neg\psi \notin Th(\mathfrak{F}^{\clubsuit})$. Then $\mathfrak{F}^{\clubsuit}, u \models \psi$ for some $u \in W$. By the assumption, $w \in S^m(\mathfrak{F}, u)$ for some $m \leq n$. Then $S^{m+n}(\mathfrak{F}, u) = W$ which yields $\Delta_{n+m}\psi \in Th(\mathfrak{F}^{\clubsuit})$. Note that (2) follows from Lemma 5.2. $\square$

REMARK 5.6. *Kracht* [7, *Corollary* 16] *claims that every extension of* K_t4 *of finite codimension is complete and of finite alternativity. This claim is incorrect since Thomason* [16] *gives a tense logic of codimension* 1 *in* $\Lambda(\mathsf{K}_t4)$ *which is Kripke incomplete (cf., e.g.,* [8, *Theorem* 7.9.1]). *Here we give another counterexample. Consider the general frame $\mathfrak{N}^{\clubsuit}$ in Proposition* 5.1. *Actually $Th(\mathfrak{N}^{\clubsuit}) \in \Lambda(\mathsf{K}_t4)$ is of codimension* 1. *Since $\mathfrak{N}$ is transitive and has no infinite descending chain, we have $\mathfrak{N}^{\clubsuit} \models \blacksquare(\blacksquare p \to p) \to \blacksquare p$. For every $\varphi \in \mathcal{L}_t^0$, $V(\varphi) \in A^{\clubsuit}$ is either finite or cofinite. Then $\mathfrak{N}^{\clubsuit} \models \neg(\Diamond p \wedge \Box(p \to \Diamond(\neg p \wedge \Diamond p)))$.*

Table 1. *Some tense logics.*

$\mathsf{K}_t\mathsf{D}^+ = \mathsf{K}_t \oplus \Diamond\top$	$\mathsf{K}_t\mathsf{D}^- = \mathsf{K}_t \oplus \blacklozenge\top$
$\mathsf{K}_t\mathsf{D} = \mathsf{K}_t \oplus \Diamond\top \wedge \blacklozenge\top$	$\mathsf{K}_t\mathsf{B} = \mathsf{K}_t \oplus p \to \Box\Diamond p$
$\mathsf{K}_t\mathsf{T} = \mathsf{K}_t \oplus p \to \Diamond p$	$\mathsf{K}_t4 = \mathsf{K}_t \oplus \Diamond\Diamond p \to \Diamond p$
$\mathsf{K}_t\mathsf{D}^+4 = \mathsf{K}_t\mathsf{D}^+ \oplus \Diamond\Diamond p \to \Diamond p$	$\mathsf{K}_t\mathsf{D}^-4 = \mathsf{K}_t\mathsf{D}^- \oplus \Diamond\Diamond p \to \Diamond p$

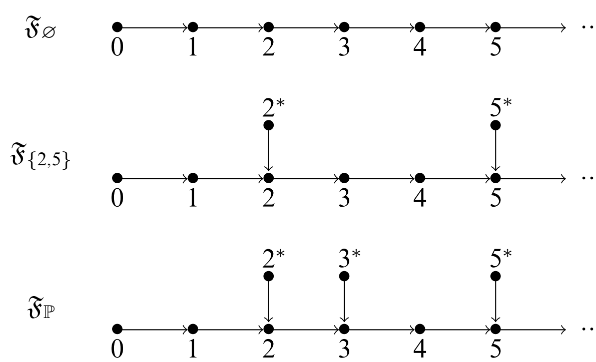

Fig. 3. Frames $\mathfrak{F}_\varnothing$, $\mathfrak{F}_{\{1,5\}}$ and $\mathfrak{F}_\mathbb{P}$.

Every transitive frame validating $\neg(\Diamond p \wedge \Box(p \to \Diamond(\neg p \wedge \Diamond p)))$ does not contain infinite ascending chains. Every frame validating $\blacksquare(\blacksquare p \to p) \to \blacksquare p$ is irreflexive. Then for every $\mathfrak{F} \in \mathsf{Fr}(Th(\mathfrak{N}^\clubsuit))$, $\mathfrak{F} \models \Diamond\top \to \Diamond\Box\bot$. Clearly $\mathfrak{N}^\clubsuit \not\models \Diamond\top \to \Diamond\Box\bot$. Hence $Th(\mathfrak{N}^\clubsuit)$ is Kripke incomplete and so it is a counterexample for Kracht's claim. Furthermore, by checking Kracht's proof, we find that it is based on the following claims:

(K1) *If $L \in \Lambda(\mathsf{K}_t4)$ is of finite codimension, then L_+ is tabular.*
(K2) *For all $S \in \Lambda(\mathsf{K})$ and $L \in \Lambda(\mathsf{K}_t)$, $S \subseteq L_+$ if and only if $S^+ \subseteq L$.*

Here the operations $(.)_+$ and $(.)^+$ are explained in the section of introduction. Note that $(K2)$ holds. This is shown as follows. If $S \subseteq L_+ \subseteq L$, then $S \subseteq L$ and so $S^+ \subseteq L$. Assume $S^+ \subseteq L$. Let $\varphi \in S$. Then $\varphi \in S^+ \subseteq L$ and so $\varphi \in L \cap \mathcal{L}_\Box = L_+$. Hence $S \subseteq L_+$. However $(K1)$ is incorrect. For a contradiction, suppose $(K1)$ holds. Consider again the Post-complete logic $L = Th(\mathfrak{N}^\clubsuit)$. Obviously $Alt_n^F \notin L$ for every $n \in \omega$. Then $L \oplus \mathrm{Alt}_n^F = \mathcal{L}_t$ for all $n \in \omega$. By $(K1)$, L_+ is tabular. Then $\mathsf{K} \oplus \mathsf{alt}_m \subseteq L_+$ for some $m \in \omega$. By $(K2)$, $\mathsf{K}_t \oplus \mathrm{Alt}_m^F = (\mathsf{K} \oplus \mathsf{alt}_m)^+ \subseteq L$ which is impossible. Hence $(K1)$ does not hold.

Using the third Post-completeness theorem and its consequences, we can show some results on the Post number of some tense logics. Recall that the Post number $\mathsf{PN}(L)$ of a tense logic L is the number of Post-complete extensions of L. For tense logics L_1 and L_2, if $L_1 \subseteq L_2$, then $\mathsf{PN}(L_2) \leq \mathsf{PN}(L_1)$ and so $\mathsf{PN}(L_1) = 2^{\aleph_0}$ implies $\mathsf{PN}(L_2) = 2^{\aleph_0}$. We consider tense logics in Table 1.

Proposition 5.7. $\mathsf{PN}(\mathsf{K}_t\mathsf{D}^+4) = \mathsf{PN}(\mathsf{K}_t\mathsf{D}^-4) = 2^{\aleph_0}$ *and hence* $\mathsf{PN}(\mathsf{K}_t\mathsf{D}^+) = \mathsf{PN}(\mathsf{K}_t\mathsf{D}^-) = \mathsf{PN}(\mathsf{K}_t4) = 2^{\aleph_0}$.

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Proof. (1) $\mathsf{PN}(\mathsf{K}_t\mathsf{D}^+4) = 2^{\aleph_0}$. For every subset $I \subseteq \omega \setminus \{0, 1\}$, let $I^* = \{i^* : i \in I\}$. Let $\mathfrak{F}_I = (W_I, R_I)$ be the frame where $W_I = \omega \cup I^*$ and $R_I = \{(n, m) \in \omega \times \omega : n < m\} \cup \{(n^*, m) \in I^* \times \omega : n \leq m\}$. Clearly $S^2(\mathfrak{F}_I, 0) = W_I$ for every $I \subseteq \omega$. (Examples of frames $\mathfrak{F}_\varnothing$, $\mathfrak{F}_{\{2,5\}}$ and $\mathfrak{F}_\mathbb{P}$ where $\mathbb{P}$ is the set of prime numbers are presented in Figure 3.) For every $I \subseteq \omega$, let $L_I = Th(\mathfrak{F}_I^\clubsuit)$. Note that, for every frame $\mathfrak{F} = (W, R)$, $\mathfrak{F} \models \mathsf{K}_t\mathsf{D}^+4$ if and only if $R(w) \neq \varnothing$ and $R(R(w)) \subseteq R(w)$ for all $w \in W$. Obviously $\mathfrak{F}_I^\clubsuit \models \mathsf{K}_t\mathsf{D}^+4$. Then $L_I \in \Lambda(\mathsf{K}_t\mathsf{D}^+4)$ for every $I \subseteq \omega$. By Corollary 5.5, L_I is Post-complete. It suffices to show that $L_I \neq L_J$ when $I \neq J$. Assume $I \neq J$. Let $i \in I \setminus J$ without loss of generality. By $\mathfrak{F}_I^\clubsuit, i \models \blacksquare^{i+1}\bot$ and $i^* R_I i$, we have $\mathfrak{F}_I^\clubsuit, i^* \models \Diamond\blacksquare^{i+1}\bot$. Clearly $R_I(i^*) = \{k \in \omega : k \geq i\}$. Then $\mathfrak{F}_I^\clubsuit, k \models \blacklozenge^i\top$ for each $k \in R_I(i^*)$. Note that $\breve{R}_I(i^*) = \varnothing$. Then $\mathfrak{F}_I^\clubsuit, i^* \models \blacksquare\bot \wedge \Diamond\blacksquare^{i+1}\bot \wedge \Box\blacklozenge^i\top$. By $S^2(\mathfrak{F}_I, i^*) = W_I$, we have $\Delta_2(\blacksquare\bot \wedge \Diamond\blacksquare^{i+1}\bot \wedge \Box\blacklozenge^i\top) \in L_I$. Suppose that there exists $v \in W_J$ with $\mathfrak{F}_J^\clubsuit, v \models \blacksquare\bot \wedge \Diamond\blacksquare^{i+1}\bot \wedge \Box\blacklozenge^i\top$. Then $\breve{R}_J(v) = \varnothing$ which yields $v \in J^* \cup \{0\}$. Assume $v = j^*$ for some $j > i$. Then $R_J(v) \subseteq \{l \in \omega : l > i\}$ and so $\mathfrak{F}_J^\clubsuit, v \not\models \Diamond\blacksquare^{i+1}\bot$. Thus $v \in \{l^* : l < i\} \cup \{0\}$. Note that $\mathfrak{F}_J^\clubsuit, i - 1 \models \blacksquare^i\bot$ and $uR_J(i-1)$ for all $u \in \{l^* : l < i\} \cup \{0\}$. Then $\mathfrak{F}_J^\clubsuit, v \not\models \Box\blacklozenge^i\top$ which contradicts $\mathfrak{F}_J^\clubsuit, v \models \Box\blacklozenge^i\top$. Hence $\mathfrak{F}_J^\clubsuit \models \neg(\blacksquare\bot \wedge \Diamond\blacksquare^{i+1}\bot \wedge \Box\blacklozenge^i\top)$. Then $\nabla_2\neg(\blacksquare\bot \wedge \Diamond\blacksquare^{i+1}\bot \wedge \Box\blacklozenge^i\top) \in L_J$ and $\Delta_2(\blacksquare\bot \wedge \Diamond\blacksquare^{i+1}\bot \wedge \Box\blacklozenge^i\top) \in L_I \setminus L_J$. Hence $L_I \neq L_J$.

(2) $\mathsf{PN}(\mathsf{K}_t\mathsf{D}^-4) = 2^{\aleph_0}$. The proof is similar to (1). It suffices to observe that $\mathfrak{F}_I = (W_I, \breve{R}_I)$ is a frame for $\mathsf{K}_t\mathsf{D}^-4$. □

PROPOSITION 5.8. *For every* $0 < \kappa \leq \aleph_0$, *there exists a consistent tense logic* $L \in \Lambda(\mathsf{K}_t)$ *such that* $\mathsf{PN}(L) = \kappa$.

Proof. For every $0 < j < \aleph_0$, let $\mathfrak{C}_j = (\{0, \dots, j-1\}, \{\langle i, i+1\rangle : i < j\})$ be the chain of j elements. By [9, Theorem 4.8], $Th(\mathfrak{C}_j) \in \mathsf{PC}$. For every $0 < n < \aleph_0$, let $\mathcal{C}_n = \{\mathfrak{C}_i : 1 \leq i \leq n\}$ and $L_n = Th(\mathcal{C}_n)$. By [9, Theorem 4.22], $\Lambda(L_n) \cap \mathsf{PC} = \{Th(\mathfrak{C}_i) : 1 \leq i \leq n\}$. By [9, Corollary 4.5], $Th(\mathfrak{C}_i) \neq Th(\mathfrak{C}_j)$ for $1 \leq i \neq j \leq n$. Hence $\mathsf{PN}(L_n) = n$. By [9, Theorem 4.24], $\mathsf{PN}(T_{1,1}) = \aleph_0$. □

REMARK 5.9. *In the proof of Proposition 5.8, all frames* $\mathfrak{C}_j$ *with* $2 < j < \aleph_0$ *are intransitive and hence not finite frames for the tense logic of linear time* Lin_t *containing the transitivity axioms* $\Box p \to \Box\Box p$ *and linearity axioms* .3 *formulated for* $\Box$ *and* $\blacksquare$. *Wolter [21] gives a description of the lattice* $\Lambda(\mathsf{Lin}_t)$ *and proves that all logics in this lattice are independently axiomatizable. One could prove results on the Post numbers of tense logics in* $\Lambda(\mathsf{Lin}_t)$.

PROPOSITION 5.10. $\mathsf{PN}(\mathsf{K}_t\mathsf{D}) = \mathsf{PN}(\mathsf{K}_t\mathsf{T}) = 1$ *and* $\mathsf{PN}(\mathsf{K}_t\mathsf{B}) = 2$.

Proof. (1) Let $L \in \mathsf{PC} \cap \Lambda(\mathsf{K}_t\mathsf{D})$. Let $\mathbb{F} = (W, R, A)$ be a general frame with $\mathbb{F} \models L$. Then $\mathbb{F} \models \Diamond\top$. Hence $R(w) \neq \varnothing$ for every $w \in W$. Let $\mathbb{F}_\circ = (\{\circ\}, \{(\circ, \circ)\}, \{\varnothing, \{\circ\}\})$. Let $f : W \to \{\circ\}$ be the function with $f(w) = \circ$ for all $w \in W$. Clearly f is a bounded morphism from $\mathbb{F}$ to $\mathbb{F}_\circ$. Then $\mathbb{F}_\circ \models L$. Hence $L = Th(\mathbb{F}_\circ) = \mathsf{K}_t \oplus p \leftrightarrow \Box p$. Then $\mathsf{PN}(\mathsf{K}_t\mathsf{D}) = 1$. By $\mathsf{K}_t\mathsf{D} \subseteq \mathsf{K}_t\mathsf{T}$, $\mathsf{PN}(\mathsf{K}_t\mathsf{T}) = 1$.

(2) Let $L \in \mathsf{PC} \cap \Lambda(\mathsf{K}_t\mathsf{B})$. Let $\mathbb{F} = (W, R, A)$ be a point-generated general frame with $\mathbb{F} \models L$. Then $L = Th(\mathbb{F})$. Suppose $R = \varnothing$. Then $W = \{\bullet\}$ and $A = \{\varnothing, \{\bullet\}\}$. Hence $L = \mathsf{K}_t \oplus \Box\bot$. Suppose $R \neq \varnothing$. Then $R(w) \cup \breve{R}(w) \neq \varnothing$ for every $w \in W$. Clearly $\Diamond\top \leftrightarrow \blacklozenge\top \in \mathsf{K}_t\mathsf{B} \subseteq L$. It follows that $R(w) \neq \varnothing$ and $\breve{R}(w) \neq \varnothing$ for every $w \in W$. Hence $\Diamond\top, \blacklozenge\top \in L$, i.e., $\mathsf{K}_t\mathsf{D} \subseteq L$. By the proof of (1), $L = \mathsf{K}_t \oplus p \leftrightarrow \Box p$. It follows that $\mathsf{PN}(\mathsf{K}_t\mathsf{B}) = 2$. □

§6. Concluding remarks. The present work contributes a series of results on tabularity and Post-completeness in tense logic. A new characterization of tabularity, two characterization theorems on Post-completeness in tabular tense logics, and a characterization of Post-completeness in the lattice of all tense logics are established. The Post numbers of some tense logics are determined. There are many problems which need to be explored. We list here some of them.

The first problem concerns the pretabularity in tense logic. It is well-known that there are exactly five pretabular normal modal logics over $\mathsf{S4}$ [12]; and there are $2^{\aleph_0}$ pretabular normal modal logics over $\mathsf{K4}$ [2]. By [9], there are infinitely many pretabular tense logics in $\Lambda(T_{n,m})$ with $nm \geq 2$. The general question is to find the number of pretabular logics in $\Lambda(L)$ for a tense logic L. An additional problem concerns if there exists a polynomial time algorithm for deciding the Post-completeness of tabular tense logics (cf. Corollary 4.10).

The second problem concerns Post numbers of tense logics. In Section 5, we give Post numbers of some tense logics. An interesting problem is to characterize the set of tense logics $\{L \in \Lambda(\mathsf{K}_t) : \mathsf{PN}(L) = \kappa\}$ for a given cardinal $\kappa \leq 2^{\aleph_0}$. Moreover, the decidability of PC is not known.

The third problem concerns the classification of tense logics $\Lambda(\mathsf{K}_t)$. Makinson's classification theorem in [10] states that, for every consistent normal modal logic L, if $\Diamond\top \in L$, then $L \subseteq \mathsf{K} \oplus \Box p \leftrightarrow p$; and if $\Diamond\top \notin L$, then $L \subseteq \mathsf{K} \oplus \Box\bot$. Here the formula $\Diamond\top$ is called the critical formula for $\Lambda(\mathsf{K})$. The possibility of finding critical formulas for lattices of tense logics needs to be explored.

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