# SEMI-SIMPLE LOCALLY COMPACT MONOTHETIC SEMI-ALGEBRAS

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(Received 22nd January 1968)

## 1. Introduction

Bonsall and Tomiuk have shown, in (**3**), the connection between the local compactness of a monothetic semi-algebra and the spectral properties of a generating element. This theme was developed, in (**4**), to give a complete characterisation of prime, strict locally compact monothetic semi-algebras in terms of the spectrum of a generator (Theorem $A$). Here we extend this result to the case of a semi-simple locally compact monothetic semi-algebra (Theorem $B$).

In Section **2** we collect the relevant terminology and state the main result. The proofs are given in Section **3**. Theorem 1 in Section **3** was suggested to us by F. F. Bonsall. Its use has considerably simplified our original proof.

## 2. Terminology and main result

Let $B$ be a complex Banach algebra with identity $e$. $\boldsymbol{R}$ will be the set of real numbers, $\boldsymbol{R}^+$ the set of non-negative real numbers and $\boldsymbol{R}^{++}$ the set of strictly positive real numbers. A non-empty subset $A$ of $B$ is called a *semi-algebra* if $x+y$, $xy$ and $\alpha x$ are in $A$ whenever $x$ and $y$ are in $A$ and $\alpha$ is in $\boldsymbol{R}^+$. A semi-algebra $A$ is *strict* if $A\cap(-A) = (0)$; it is *locally compact* if $A$ contains non-zero elements and if, in addition,

$$A\cap\{x:\| x \| \leqq 1\}$$

is a compact subset of $B$.

The semi-algebra $A$ is said to be *monothetic* if $A$ has a single generator, that is, if there exists an element $t\in A$ such that $A$ is the closure in $B$ of the set

$$P(t) = \{\alpha_1 t+\ldots+\alpha_k t^k\colon\ \alpha_i \geqq 0\ (i = 1, \ldots, k);\ k = 1, 2, \ldots\}.$$

In this case we write $A = A(t)$. Obviously, $A(t)$ is commutative.

The commutative semi-algebra $A$ is *semi-simple* if $a^2 \neq 0$ for each non-zero $a\in A$, and it is *prime* if there are no divisors of zero in $A$, in other words, if $a, b\in A$ and $a \neq 0, b \neq 0$ then $ab \neq 0$.

The resolvent set of any element $t\in B$ is denoted by $\rho(t)$. Its complement $\sigma(t)$ in the complex plane $\boldsymbol{C}$ is the spectrum of $t$. The spectral radius of $t$ is denoted by $r(t)$, and the resolvent operator $(ze-t)^{-1}$ by $R(z;\ t)$. A point $\lambda\in\sigma(t)$ is called a *simple pole* of $t$ if it is a pole of the function

$$z\to R(z;\ t)$$

of order one.

https://doi.org/10.1017/S0013091500012712 Published online by Cambridge University Press

The following theorem is essentially Theorem 8 in (**4**).

**Theorem A.** *Let t be a non-zero element of B. A(t) is a prime, strict, locally compact semi-algebra if and only if*

$$0 < r(t) \in \sigma(t)$$

*and*

$$\sigma(t) \cap \{\lambda : |\lambda| = r(t)\}$$

*is a finite set of simple poles of t.*

We now formulate the main result of this paper.

**Theorem B.** *Let t be a non-zero element of B. A(t) is a semi-simple, locally compact semi-algebra if and only if σ(t) decomposes uniquely into two disjoint closed subsets* $\sigma_1$ *and* $\sigma_2$ *such that*

(i) $\sigma_1$ *is a finite (possibly empty) set of simple poles of t and*

$$\sigma_1 \cap R^{++} = \varnothing;$$

(ii) *either* $\sigma_2 = \varnothing$ *or there exists an* $\alpha > 0$ *in* $\sigma_2$ *such that*

$$\sigma_2 = \sigma(t) \cap \{\lambda : |\lambda| \leqq \alpha\}$$

*and*

$$\sigma_2 \cap \{\lambda : |\lambda| = \alpha\}$$

*is a finite set of simple poles of t.*

**Remarks.** (*a*) If we write $t_2$ for the " part " of $t$ associated with the spectral set $\sigma_2$ then, by Theorem *A*, $A(t_2)$ is prime, strict and locally compact.

(*b*) If $\sigma_1 = \varnothing$, $A(t)$ is strict. This is a simple consequence of Theorem *A*. If $A(t)$ is locally compact, strict and semi-simple then by (**2**), Lemma 8, $r(t) > 0$, and hence a slight modification of Theorem 7 in (**4**) shows that $A(t)$ is prime. Thus the converse of (*b*) also follows from Theorem *A*.

## 3. Proofs

**Theorem 1.** *Let t be a non-zero element of B. Then A(t) is locally compact and semi-simple if and only if there exists an idempotent p in A(t) such that*

(i) $A(t) = A(tp) \oplus A(t-tp)$;

(ii) *either* $A(tp) = (0)$ *or* $A(tp)$ *is locally compact, semi-simple and*

$$A(tp) = -A(tp);$$

(iii) *either* $A(t-tp) = (0)$ *or* $A(t-tp)$ *is locally compact, prime and strict.*

**Proof of necessity.** Suppose that $A(t)$ is locally compact and semi-simple. Put $C = A(t) \cap (-A(t))$; then $C$ is a finite dimensional semi-simple algebra over $R$, and so $C$ contains a unit element $p$ ((**1**) p. 37, Theorem 1). Observing that $tp$ and $t-tp$ are in $A(t)$, and writing $t = tp + (t-tp)$, we have

$$A(t) = A(tp) + A(t-tp).$$

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Suppose that $x \in A(tp) \cap A(t-tp)$. Then there exist $y$ and $z$ in $A(t)$ such that $x = yp = z-zp$. Hence $x = 0$. Therefore

$$A(tp) \cap A(t-tp) = (0) \tag{1}$$

and (i) follows.

If $x \in C$, then $x = xp$ and hence $x \in A(tp)$. Conversely, take $x \in A(tp)$. Then $x = yp$ for some $y \in A(t)$. Thus $-x = y(-p) \in A(t)$. But then $x \in C$. Hence $C = A(tp)$. This proves (ii).

Suppose that $x$ and $-x$ are in $A(t-tp)$. Then $x \in C = A(tp)$. By (1), $x = 0$. Therefore $A(t-tp)$ is strict. It is also locally compact and semi-simple as a subsemi-algebra of $A(t)$. A slight modification of Theorem 7 in **(4)** shows that it is therefore prime. This proves (iii).

**Proof of sufficiency.** Let there exist an idempotent $p$ such that conditions (i), (ii) and (iii) hold. Local compactness of $A(tp)$ and $A(t-tp)$ and condition (i) clearly imply that $A(t)$ is locally compact.

Let $a$ in $A(t)$ be such that $a^2 = 0$. Then $(ap)^2 = a^2p = 0$ and $ap \in A(tp)$; hence, by (ii), $ap = 0$. Similarly $(a-ap)^2 = a^2-a^2p = 0$ and $a-ap \in A(t-tp)$; hence, by (iii), $a-ap = 0$. Thus $a = ap+(a-ap) = 0$. Hence $A$ is semi-simple.

**Lemma 2.** *Let $t$ be a non-zero element in $B$. Then $A(t)$ is locally compact, semi-simple and $A(t) = -A(t)$ if and only if $\sigma(t)$ is a finite set of simple poles of $t$ and $\sigma(t) \cap \mathbf{R}^{++}$ is empty.*

**Proof of necessity.** The hypotheses imply that $A(t)$ is a real finite dimensional semi-simple algebra. So we can apply the Corollary in **(1)** p. 40 to show that $A(t)$ is algebraically isomorphic to a direct sum of fields, all of which are finite commutative extensions of $\mathbf{R}$. But the finite commutative extension fields over $\mathbf{R}$ are either copies of $\mathbf{R}$ or of $\mathbf{C}$. Thus $A(t)$ is algebraically isomorphic to $\mathbf{R}^n \times \mathbf{C}^m$ (with coordinate-wise multiplication) for some positive integers $n$ and $m$. We can extend the inverse of this isomorphism to an algebraic homomorphism $\phi$ from $\mathbf{C}^{n+m}$ onto the complex Banach algebra $B(t)$ generated by $t$. Let $N$ be the kernel of $\phi$, then $B(t)$ is algebraically isomorphic to the quotient algebra $\mathbf{C}^{n+m}/N$. Since $N \neq \mathbf{C}^{n+m}$, $\mathbf{C}^{n+m}/N$ is algebraically isomorphic to $\mathbf{C}^r$ for some positive integer $r$. So there exists an algebraic isomorphism $\psi$ from $B(t)$ onto $\mathbf{C}^r$. Clearly this implies that the spectrum of $t$ as an element of $B(t)$ is a finite set of simple poles of $t$. Since $t = tp$, where $p$ is the unit of $B(t)$, the spectrum of $t$ as an element of $B$ is a finite set of simple poles of $t$.

Let $\psi(t) = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Note that

$$\sigma(t) \setminus \{0\} \subset \{\lambda_1, \ldots, \lambda_k\}.$$

Suppose that $\sigma(t) \cap \mathbf{R}^{++} \neq \emptyset$. Then $\lambda_i > 0$ for some $i$. This implies that, for any $a$ in $A(t)$, the $i$th coordinate of $\psi(a)$ is non-negative. In particular, since $-t \in A(t)$, $-\lambda_i \geqq 0$ contradicting $\lambda_i > 0$. This shows that $\sigma(t) \cap \mathbf{R}^{++} = \emptyset$.

**Proof of sufficiency.** Suppose that

$$\sigma(t) = \{\lambda_1, \ldots, \lambda_k\}$$

is a finite set of simple poles of $t$ such that $(t\sigma) \cap \mathbf{R}^{++} = \emptyset$.



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For each $i$, let $p_i$ be the spectral idempotent corresponding to $\lambda_i$. Then

$$t = \lambda_1 p_1 + \ldots + \lambda_k p_k.$$

Hence each $a$ in $A(t)$ is of the form

$$a = \alpha_1 p_1 + \ldots + \alpha_k p_k,$$

where $\alpha_i \in C$ $(i = 1, \ldots, k)$. Therefore $A(t)$ is algebraically isomorphic to a subsemi-algebra of $C^k$ with coordinatewise multiplication. Therefore $A(t)$ is locally compact and semi-simple.

Now we can apply Theorem 1 to show the existence of an idempotent $p$ in $A(t)$ such that

(i) $A(tp) = -A(tp)$,

(ii) $A(t-tp) = (0)$ or $A(t-tp)$ is a prime strict locally compact semi-algebra.

Let $\sigma_1$ be the spectral set associated with $p$. Standard spectral theory shows that

$$\sigma(t-tp) = \begin{cases} \{\sigma(t)\setminus\sigma_1\}\cup\{0\} & \text{if } p \neq 0, \\ \sigma(t) & \text{if } p = 0. \end{cases}$$

Hence $\sigma(t-tp)\cap R^{++} = \varnothing$. If $t-tp \neq 0$, then by (2), Lemma 8, $r(t-tp)>0$. Applying Theorem $A$ we get $r(t-tp) \in \sigma(t-tp)$. This contradicts the fact that $\sigma(t-tp)\cap R^{++} = \varnothing$. Hence $t-tp = 0$, and thus $A(t) = -A(t)$.

**Theorem B. Proof of necessity.** Suppose that $A(t)$ is locally compact and semi-simple. Choose $p$ as in Theorem 1. Since $p$ is a spectral projection, standard spectral theory shows that if $0 \neq \lambda \in \sigma(tp)$ or $0 \neq \lambda \in \sigma(t-tp)$ then $\lambda \in \sigma(t)$, and that, if $\lambda$ is a non-zero simple pole of $tp$ or of $t-tp$, then it is a simple pole of $t$. Also

$$\{\sigma(t)\cap\sigma(tp)\}\cup\{\sigma(t)\cap\sigma(t-tp)\} = \sigma(t).$$

Now suppose that $t-tp \neq 0$. Then, by Theorem 1, $A(t-tp)$ is locally compact prime and strict. Hence, by Theorem $A$,

$$0<r(t-tp) \in \sigma(t-tp)$$

and

$$\sigma(t-tp)\cap\{\lambda:|\lambda| = r(t-tp)\}$$

is a finite set of simple poles of $t-tp$.

Theorem 1 and Lemma 2 show that $\sigma(tp)$ is a finite set of simple poles of $t$ and

$$\sigma(tp)\cap R^{++} = \varnothing.$$

Put $\alpha = r(t-tp)$,

$$\sigma_1 = \sigma(t)\cap\{\lambda:|\lambda|>\alpha\},$$

and

$$\sigma_2 = \sigma(t)\cap\{\lambda:|\lambda| \leqq \alpha\}.$$

Then $\alpha>0$ and, by the spectral theory summarized above, $\sigma_1$ and $\sigma_2$ satisfy the conditions of Theorem $B$.

If $t-tp = 0$, take $\sigma_1 = \sigma(t)$ and $\sigma_2 = \emptyset$; again the conditions of Theorem $B$ are satisfied. Clearly, in both cases the decomposition is unique.

**Proof of sufficiency.** Suppose that $\sigma(t)$ is decomposed into disjoint subsets $\sigma_1$ and $\sigma_2$ satisfying conditions (i) and (ii) of Theorem B. If $\sigma_1 = \emptyset$ then $\sigma_2 = \sigma(t)$. Applying Theorem A we see that $A(t)$ is locally compact and prime, hence also semi-simple.

If $\sigma_1 \neq \emptyset$, let $p$ be the spectral projection associated with the spectral set $\sigma_1$. Then $tp \neq 0$ and $\sigma(tp)$ is a finite set of simple poles of $tp$ such that

$$\sigma(tp) \cap R^{++} = \emptyset.$$

So, applying Lemma 2, we see that $A(tp)$ is locally compact and semi-simple. Now, either $t-tp = 0$, or

$$\sigma(t-tp) \cap \{\lambda : |\lambda| = r(t-tp)\}$$

is a finite set of poles of $t-tp$ which contains the point $\alpha = r(t-tp) > 0$. Hence, either $A(t-tp) = (0)$, or, by Theorem A, $A(t-tp)$ is locally compact, strict and prime. Thus $A(t-tp)$ is also semi-simple. The argument used in the proof of sufficiency of Theorem 1 shows that $A(t)$ is locally compact and semi-simple.

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https://doi.org/10.1017/S0013091500012712 Published online by Cambridge University Press