THE MOTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

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We confine ourselves, for simplicity, to first-order algebraic differential equations (ADE's), although analogous considerations may be made for higher-order ADE's:

$$P(x, y(x), y'(x)) = 0. \tag{*}$$

A *motion* of (*) is a change of independent variable that takes solutions to solutions, that is, a suitable map φ of the underlying interval I into itself so that if y is a solution of (*) then $y \circ \varphi$ is a solution of (*), i.e.

$$P(x, y(\varphi(x)), y'(\varphi(x))\varphi'(x)) = 0.$$

In this paper we prove that the motions satisfy their own second order ADE

$$Q(x, \varphi(x), \varphi'(x), \varphi''(x)) = 0 \tag{\#}$$

and that, in general, (#) cannot be replaced by a first-order ADE. We find this surprising.

THEOREM 1. *Consider the equation* (*) *on an open interval* $I \subseteq \mathbb{R}$. *There exists an equation* (#) *that is satisfied by every* C^2 *motion* φ *of* (*) *for which there exists a* C^1 *solution* y *for which* $y \circ \varphi$ *is not a constant on any subinterval of* I.

It is clear that *some* restriction on the ADE is needed. Consider, for example the ADE $y' = 0$, where the solutions are the constant functions, and *any* mapping is a motion.

Before proving this, let us briefly review some classical things about resultants (see [**1**]).

For

$$A(Y) = a_0 Y^q + a_1 Y^{q-1} + \ldots + a_q,$$
$$B(Y) = b_0 Y^r + b_1 Y^{r-1} + \ldots + b_r,$$

we define the Y-resultant of $A(Y)$ and $B(Y)$ by the formula

$$\operatorname{Res}_Y(A(Y), B(Y)) = \det \begin{bmatrix} a_0 & a_1 & \ldots & a_q & 0 & \ldots & \ldots & 0 \\ 0 & a_0 & a_1 & \ldots & a_q & 0 & \ldots & 0 \\ \vdots & & & & & & & \\ 0 & \ldots & a_0 & a_1 & \ldots & \ldots & \ldots & a_q \\ b_0 & b_1 & \ldots & b_r & 0 \ldots & \ldots & \ldots & 0 \\ 0 & b_0 & b_1 & \ldots & b_r & 0 & \ldots & 0 \\ \vdots & & & & & & & \\ 0 & \ldots & b_0 & b_1 & \ldots & \ldots & \ldots & b_r \end{bmatrix} \begin{matrix} \left.\right\} r \text{ rows} \\ \left.\right\} q \text{ rows} \end{matrix}$$

This has the following property, where for simplicity we assume $a_0 = 1 = b_0$: $A(Y) = 0$ and $B(Y) = 0$ have a common solution if and only if $\operatorname{Res}_Y(A(Y), B(Y)) = 0$.

Now let us prove Theorem 1. Substituting $\varphi(x)$ for x in (*), we have

$$P(\varphi(x), y(\varphi(x)), y'(\varphi(x))) = 0 \tag{1}$$

and

$$P(x, y(\varphi(x)), y'(\varphi(x))\varphi'(x)) = 0 \tag{2}$$

because φ is a motion. Considering the left hand sides of (1) and (2) as polynomials in $y'(\varphi(x))$ we see, on factoring out the highest powers of $y'(\varphi(x))$ permissible, that

$$\tilde{P}(\varphi(x), y(\varphi(x)), y'(\varphi(x)) = 0, \tag{1'}$$

$$\tilde{P}(x, y(\varphi(x)), y'(\varphi(x))\varphi'(x)) = 0, \tag{2'}$$

where $\tilde{P}(x, y, 0) \neq 0$. (This is because of our hypothesis which implies that there exist solutions of (1) and (2) with $y'(\varphi(x)) \neq 0$.)

Taking resultants of these two polynomials in $y'(\varphi(x))$, we eliminate $y'(\varphi(x))$ to get

$$S(x, \varphi(x), \varphi'(x), y(\varphi(x))) = 0. \tag{3}$$

Now differentiate this expression to get

$$T(x, \varphi(x), \varphi'(x), \varphi''(x), y(\varphi(x), y'(\varphi(x))) = 0. \tag{4}$$

This time, use resultants to eliminate $y'(\varphi(x))$ between (1) and (4) to get

$$U(x, \varphi(x), \varphi'(x), \varphi''(x), y(\varphi(x))) = 0. \tag{5}$$

Now we may divide (3) and (5) by the highest permissible powers of $y(\varphi(x))$ to get

$$\tilde{S}(x, \varphi(x), \varphi'(x), y(\varphi(x))) = 0, \tag{3'}$$

$$\tilde{U}(x, \varphi(x), \varphi'(x), \varphi''(x), y(\varphi(x))) = 0, \tag{5'}$$

where $\tilde{S}(x, y, z, 0) \neq 0$ and where $\tilde{U}(x, y, z, w, 0) \neq 0$. This is possible since (3) and (5) have solutions where $y(\varphi(x))$ is not identically zero on any subinterval of I. Again taking a resultant, we eliminate $y(\varphi(x))$ from (3′) and (5′) to get

$$Q(x, \varphi(x), \varphi'(x), \varphi''(x)) = 0, \tag{\#}$$

which is the desired result.

THEOREM 2. *In the context of the above theorem, one may not generally take φ to satisfy a first-order algebraic differential equation.*

Proof. Let the given ADE be

$$y' + 2xy = 1 \tag{*}$$

whose general solution is

$$y = e^{-x^2}\int_0^x e^{t^2}\,dt + de^{-x^2}, \ d \text{ a constant.}$$

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If φ is a motion of (*) then

$$e^{-\varphi^2}\int_0^{\varphi} e^{t^2}\,dt = e^{-x^2}\int e^{t^2}\,dt + ce^{-x^2},$$

$$c = c(\varphi) = \text{const}.$$

Take the derivative of this last expression

$$e^{x^2-\varphi^2}(2x-2\varphi\varphi')\int_0^{\varphi} e^{t^2}\,dt + e^{x^2-\varphi^2}e^{\varphi^2}\varphi' - e^{x^2} = 0$$

which we write as

$$\varphi' = \frac{1-2x\,\text{err}\,\varphi}{1-2\varphi\,\text{err}\,\varphi} \tag{6}$$

where

$$\text{err}\,\varphi = e^{-\varphi^2}\int_0^{\varphi} e^{t^2}\,dt.$$

Now (6) is a first-order differential equation, but it is not algebraic. Now over a rectangle R in the (x, z) plane, there is a unique solution to (6) with $\varphi(x_0) = z_0$ for $(x_0, z_0) \in R$. Thus for $(x_0, z_0) \in R$, we would have

$$Q(x_0, z_0, \left(\frac{1-zx_0\,\text{err}\,z_0}{1-2z_0\,\text{err}\,z_0}\right)) = 0$$

if we were to have

$$Q(x, \varphi(x), \varphi'(x)) = 0$$

for some polynomial Q in three variables. We will show that this is impossible. Let us rewrite this as

$$Q\left(x, z, \frac{1-2x\,\text{err}\,z}{1-2z\,\text{err}\,z}\right) = 0 \quad \text{for} \quad (x, z) \in R.$$

Hold $x = x_0$ fixed in R. Then for an interval I of values of z

$$Q\left(x_0, z, \frac{1-2x_0\,\text{err}\,z}{1-2z\,\text{err}\,z}\right) = 0. \tag{7}$$

Now (7) implies, unless $Q(x_0, z, w) \equiv 0$, that err z is an algebraic function of z over I. However from [**2**, pp. 48–49] this is not so. The only way to resolve this contradiction is that

$$Q(x_0, z, w) \equiv 0 \quad \text{for } (x_0, z) \text{ in } R,\ w \in \mathbb{C}.$$

This implies $Q(x, y, z) \equiv 0$, and thus we have our conclusion that there is no non-trivial first-order ADE satisfied by all the motions φ.

REFERENCES

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