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TWO APPROACHES TO MÖBIUS INVERSION

I-CHIAU HUANG

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Abstract

The Möbius inversion formula for a locally finite partially ordered set is realized as a Lagrange inversion formula. Schauder bases are introduced to interpret Möbius inversion.

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1. Introduction

In combinatorics, there are various inversion formulas including Lagrange inversion, inverse relations and Möbius inversion. Some theorems are in fact also inversions, for instance: MacMahon's master theorem [12]; Saalschütz's theorem [1, p. 9]; (a terminating form of) Dixon's theorem [1, p. 13]; and Jacobi's formula [10]. An inversion formula is regarded as a phenomenon of change of 'coordinate systems'. It is shown that Lagrange inversion [5], MacMahon's master theorem [4, Example 1], Saalschütz's theorem [6, Identity 11] and Dixon's theorem [6, Identity 12] are phenomena of changes of variables; an inverse relation is a phenomenon of a change of Schauder bases [7]; Jacobi's formula is a phenomenon of a change of parameters [9]. The purpose of this article is to show that Möbius inversion for a locally finite partially ordered set is within the same view.

Throughout the article, κ is a field and X is a partially ordered set, which is *locally finite* (that is, there are only finitely many elements between any $x, y \in X$). Let $\chi^n : X \times X \to \kappa$ be the function such that $\chi^n(x, y)$ is the number of distinct chains $x = x_0 < x_1 < \cdots < x_n = y$. Since X is locally finite, there is an integer n for each pair (x, y) such that $\chi^i(x, y) = 0$ for $i > n$, so we can define the *Möbius function of* X as

$$\mu := \chi^0 - \chi^1 + \chi^2 - \chi^3 + \cdots .$$

For $x \in X$, we call the set $\{y \in X : y \geq x\}$ *the principal filter on* X *generated by* x. Let $f : X \to \kappa$ be a function. If all principal filters on X are finite, we can define a function $g(x) := \sum_{y \geq x} f(y)$; the Möbius inversion formula asserts that

$$f(x) = \sum_{y \geq x} g(y)\mu(x, y). \tag{1.1}$$

Dually, we call the set $\{y \in X : y \leq x\}$ *the principal ideal on X generated by x*. If all principal ideals on X are finite, we can define a function $g(x) := \sum_{y \leq x} f(y)$; the dual version of the Möbius inversion formula asserts that

$$f(x) = \sum_{y \leq x} g(y)\mu(y, x). \tag{1.2}$$

Möbius inversion first occurred in number theory. Its significance in combinatorics is shown by Rota [15] by introducing the incidence algebra of a locally finite partially ordered set. In terms of the incidence algebra, the function χ^n is indeed the nth power of χ^1; the Möbius function μ is the inverse of the zeta function $\chi^0 + \chi^1$. Observed by Stanley [17], a locally finite partially ordered set can be recovered by its incidence algebra over a field. Incidence algebras are a subject of much research. See the textbook [16]. This article provides alternative views on Möbius inversion by enhancing the method of generating functions with differentials. See [8] for a review on such a method.

Our approach to Möbius inversion as well as to other inversion formulas is supported by convenient operational tools: local cohomology residues are available for changes of variables and certain Schauder bases; for changes of parameters, we can employ logarithmic residues. In Section 2, we briefly recall the notion of local cohomology residues, which is used in this article.

For a fixed x, there are only finitely many y involved in the formulas (1.1) and (1.2). Hence we may assume that X is finite for Möbius inversion and its dual version. For the power series over κ with the elements of X as variables, we can construct other sets of variables using the order on X (Lemma 3.1). The Möbius function can be computed by residues (equation (3.2) in the proof of Theorem 3.2). As a type of Lagrange inversion, we recover the Möbius inversion formula in Section 3 as the interplay of the representations of a homogeneous polynomial of degree one by these sets of variables. Regarding Lagrange inversion and incidence algebras, another viewpoint is provided by Haiman and Schmitt [2].

Without assuming that X is finite, the set of functions from X to κ can be uniquely represented by a Schauder basis (Definition 4.1). For X with finite principal filters or with finite principal ideals, we can define other Schauder bases using the order on X (Propositions 4.2 and 4.3). Section 4 concludes that Möbius inversion is in fact a phenomenon of a change of Schauder bases.

In Section 5, certain computational aspects of our approaches to Möbius functions are presented, including a determinant formula of Lindström and Wilf [11, 18] and another of Redheffer and Wilf [14, 19].

2. Operational tool

The concrete features of local cohomology residues have been used in algebraic geometry to realize Grothendieck duality. It is first observed in [4] that these algebraic

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residues are a natural tool for taking coefficients of a power series. With elegant operational rules, local cohomology residues are applied to various combinatorial problems [6]. In this section, we collect basic notions of local cohomology residues used in this article. More details can be found in [4, 6, 8]. See also [3] for a treatment in a more general setting.

A power series ring R over κ can be characterized as a complete regular local ring with κ as a coefficient field. A regular system of parameters $x_1, \ldots, x_n$ of R serves as variables. In other words, R can be described as $\kappa[[x_1, \ldots, x_n]]$. The module $\widetilde{\Omega}_{R/\kappa}$ of finite differentials of R over κ is free of rank n. It comes with a κ-derivation $d : R \to \widetilde{\Omega}_{R/\kappa}$, with which $\widetilde{\Omega}_{R/\kappa} = R\,dx_1 + \cdots + R\,dx_n$. The exterior power $\wedge^n\widetilde{\Omega}_{R/\kappa}$ is free of rank one. Indeed, $\wedge^n\widetilde{\Omega}_{R/\kappa} = R\,dx_1 \wedge \cdots \wedge dx_n$.

The nth local cohomology module of $\wedge^n\widetilde{\Omega}_{R/\kappa}$ supported at the maximal ideal of R consists of generalized fractions

$$\begin{bmatrix} \varphi \\ f_1, \ldots, f_n \end{bmatrix},$$

where the numerator $\varphi \in \wedge^n\widetilde{\Omega}_{R/\kappa}$ and the denominators $f_1, \ldots, f_n$ form a system of parameters of R. Generalized fractions are linear in numerators (linearity law). A change of denominators is compensated by a determinant in the numerator (transformation law). A generalized fraction vanishes if and only if the ideal generated by denominators contains the power series, which represents the numerator in terms of the basis $dx_1 \wedge \cdots \wedge dx_n$ (vanishing law).

Taking residues of a generalized fraction is a κ-linear map determined by

$$\mathbf{res}\begin{bmatrix} dx_1 \wedge \cdots \wedge dx_n \\ x_1^{i_1}, \ldots, x_n^{i_n} \end{bmatrix} = \begin{cases} 1 & \text{if } i_1 = \cdots = i_n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Residue maps are transitive in variables (transitivity law). The most important property of residue maps is that they are invariant under changes of regular systems of parameters (invariance law). We look at an example to see how the invariance law appears explicitly. Consider the power series ring $\kappa[[x]] = \kappa[[y]]$, where $y = x + x^2 + x^3 + \cdots$. Leibniz's rule of derivations gives

$$dy = dx(1 - x)^{-1} = (1 - x)^{-2}\,dx.$$

Hence

$$\Theta := \begin{bmatrix} dy \\ y^n \end{bmatrix} = \begin{bmatrix} (1 - x)^{n-2}\,dx \\ x^n \end{bmatrix}$$

by the transformation law. For $n = 1$, linearity and vanishing laws give

$$\Theta = \begin{bmatrix} (1 + x + x^2 + \cdots)\,dx \\ x \end{bmatrix} = \begin{bmatrix} dx \\ x \end{bmatrix} + \begin{bmatrix} (x + x^2 + \cdots)\,dx \\ x \end{bmatrix} = \begin{bmatrix} dx \\ x \end{bmatrix};$$

for $n \geq 2$, the power series $(1 - x)^{n-2}$ becomes a polynomial of degree $n - 2$. Thus the residue of the generalized fraction Θ computed in terms of the variable y is indeed the same as that obtained in terms of x.

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3. Möbius function by residues

In this section, we work on a power series ring $R = \kappa[[x_1, \ldots, x_n]]$, whose variables $x_1, \ldots, x_n$ form a partially ordered set X. We use the notation

$$u_i := \sum_{x_j \leq x_i} x_j \quad \text{and} \quad v_i := \sum_{x_j \geq x_i} x_j \tag{3.1}$$

for $1 \leq i \leq n$.

LEMMA 3.1. *The elements $u_1, \ldots, u_n$ (respectively $v_1, \ldots, v_n$) form a regular system of parameters of R. In other words,*

$$\kappa[[x_1, \ldots, x_n]] = \kappa[[u_1, \ldots, u_n]] = \kappa[[v_1, \ldots, v_n]].$$

PROOF. We prove the lemma for u_i and leave the case for v_i to the reader. As an application of Nakayama's lemma [13, Theorem 2.3], it suffices to prove that $u_1, \ldots, u_n$ form a basis of the κ-vector space generated by $x_1, \ldots, x_n$. We proceed by induction on the number of variables. The case $n = 1$ is trivial. Assume that the assertion holds if the number of variables is less than n and, without loss of generality, assume also that x_n is a maximal element. Then $u_1, \ldots, u_{n-1}$ form a basis of the vector space generated by $x_1, \ldots, x_{n-1}$. The assertion follows since the subspace generated by $u_1, \ldots, u_n$ is also generated by $u_1, \ldots, u_{n-1}, x_n$. □

The above proof shows that the relation between $u_1, \ldots, u_n$ (respectively $v_1, \ldots, v_n$) and $x_1, \ldots, x_n$ is linear over κ.

THEOREM 3.2. *For $1 \leq j \leq n$,*

$$x_j = \sum_{i=1}^{n} \mu(x_i, x_j) u_i = \sum_{i=1}^{n} \mu(x_j, x_i) v_i.$$

PROOF. We prove the theorem for u_i and leave the case for v_i to the reader. Since x_j is a κ-linear combination of $u_1, \ldots, u_n$, what we need to prove is that

$$\mathbf{res} \begin{bmatrix} x_j \, du_1 \wedge \cdots \wedge du_n \\ u_1, \ldots, u_{i-1}, u_i^2, u_{i+1}, \ldots, u_n \end{bmatrix} = \mu(x_i, x_j). \tag{3.2}$$

By switching the indices of the sequence $x_1, \ldots, x_n$, we only need to prove that

$$\mathbf{res} \begin{bmatrix} x_n \, du_1 \wedge \cdots \wedge du_n \\ u_1^2, u_2, \ldots, u_n \end{bmatrix} = \mu(x_1, x_n), \tag{3.3}$$

which clearly holds if x_n is a minimal element.

Therefore, we assume that x_n is not a minimal element. If X has a minimal element not equal to x_1, say x_2, we let $u_i' = u_i - x_2$ for $x_i > x_2$ and let $u_i' = u_i$ otherwise. Then

$$\begin{bmatrix} x_n \, du_1 \wedge \cdots \wedge du_n \\ u_1^2, u_2, \ldots, u_n \end{bmatrix} = \begin{bmatrix} x_n \, du_1' \wedge dx_2 \wedge du_3' \wedge \cdots \wedge du_n' \\ (u_1')^2, x_2, u_3', \ldots, u_n' \end{bmatrix}.$$

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By the transitivity law of residues, we get

$$\mathbf{res}\begin{bmatrix} x_n\, du_1 \wedge \cdots \wedge du_n \\ u_1^2, u_2, \ldots, u_n \end{bmatrix} = \mathbf{res}\begin{bmatrix} x_n\, du_1' \wedge du_3' \wedge \cdots \wedge du_n' \\ (u_1')^2, u_3', \ldots, u_n' \end{bmatrix}.$$

Since $\mu(x_1, x_n)$ is invariant if we take away a minimal element not equal to x_1 from X, to prove (3.3), we may assume furthermore that $x_1 = \inf X$.

Now we prove the theorem by induction on n. The case $n = 1$ is trivial. Assume the theorem holds for the case that the number of variables is less than n. Consider a minimal element of $X \setminus \{x_1\}$, say x_2. The generalized fraction in (3.3) can be written as

$$\begin{aligned} \begin{bmatrix} x_n\, du_1 \wedge \cdots \wedge du_n \\ u_1^2, u_2, \ldots, u_n \end{bmatrix} &= \begin{bmatrix} x_n\, dx_1 \wedge (dx_1 + dx_2) \wedge du_3 \wedge \cdots \wedge du_n \\ x_1^2, x_1 + x_2, u_3, \ldots, u_n \end{bmatrix} \\ &= \begin{bmatrix} x_n(x_1 - x_2)\, dx_1 \wedge dx_2 \wedge\ du_3 \wedge \cdots \wedge du_n \\ x_1^2, x_1^2 - x_2^2, u_3, \ldots, u_n \end{bmatrix} \\ &= \begin{bmatrix} x_n\, dx_1 \wedge dx_2 \wedge du_3 \wedge \cdots \wedge du_n \\ x_1^2, x_2, u_3, \ldots, u_n \end{bmatrix} \\ &\quad - \begin{bmatrix} x_n\, dx_1 \wedge dx_2 \wedge du_3 \wedge \cdots \wedge du_n \\ x_1, x_2^2, u_3, \ldots, u_n \end{bmatrix}. \end{aligned}$$

Let $u_i' = u_i - x_2$ for $x_2 < x_i$ and let $u_i' = u_i$ otherwise. Let $u_i'' = u_i - x_1$. Then

$$\begin{aligned} \begin{bmatrix} x_n\, du_1 \wedge \cdots \wedge du_n \\ u_1^2, u_2, \ldots, u_n \end{bmatrix} &= \begin{bmatrix} x_n\, dx_1 \wedge dx_2 \wedge du_3' \wedge \cdots \wedge du_n' \\ x_1^2, x_2, u_3', \ldots, u_n' \end{bmatrix} \\ &\quad - \begin{bmatrix} x_n\, dx_1 \wedge dx_2 \wedge du_3'' \wedge \cdots \wedge du_n'' \\ x_1, x_2^2, u_3'', \ldots, u_n'' \end{bmatrix}. \end{aligned}$$

Taking residues, we get

$$\begin{aligned} \mathbf{res}\begin{bmatrix} x_n\, du_1 \wedge \cdots \wedge du_n \\ u_1^2, u_2, \ldots, u_n \end{bmatrix} &= \mathbf{res}\begin{bmatrix} x_n\, dx_1 \wedge du_3' \wedge \cdots \wedge du_n' \\ x_1^2, u_3', \ldots, u_n' \end{bmatrix} \\ &\quad - \mathbf{res}\begin{bmatrix} x_n\, dx_2 \wedge du_3'' \wedge \cdots \wedge du_n'' \\ x_2^2, u_3'', \ldots, u_n'' \end{bmatrix}. \end{aligned}$$

Let μ' be the Möbius function of the partially ordered set $X \setminus \{x_2\}$ and let μ'' be the Möbius function of the partially ordered set $X \setminus \{x_1\}$. By the induction hypothesis,

$$\mathbf{res}\begin{bmatrix} x_n\, du_1 \wedge \cdots \wedge du_n \\ u_1^2, u_2, \ldots, u_n \end{bmatrix} = \mu'(x_1, x_n) - \mu''(x_2, x_n) = \mu(x_1, x_n).$$

This concludes the proof. □

COROLLARY 3.3. *If $x \neq z$, then $\sum_{x \le y \le z} \mu(y, z) = \sum_{x \le y \le z} \mu(x, y) = 0$.*

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PROOF. We may assume that X consists of elements $x_1, \ldots, x_n$ with $x = x_1 = \inf X$ and $z = x_n = \sup X$. Then

$$\sum_{x \le y \le z} \mu(y, z) = \sum_{j=1}^{n} \mathbf{res} \begin{bmatrix} x_j \, dv_1 \wedge \cdots \wedge dv_n \\ v_1, \ldots, v_{n-1}, v_n^2 \end{bmatrix}$$
$$= \mathbf{res} \begin{bmatrix} v_1 \, dv_1 \wedge \cdots \wedge dv_n \\ v_1, \ldots, v_{n-1}, v_n^2 \end{bmatrix} = 0$$

and

$$\sum_{x \le y \le z} \mu(x, y) = \sum_{j=1}^{n} \mathbf{res} \begin{bmatrix} x_j \, du_1 \wedge \cdots \wedge du_n \\ u_1^2, u_2, \ldots, u_n \end{bmatrix}$$
$$= \mathbf{res} \begin{bmatrix} u_n \, du_1 \wedge \cdots \wedge du_n \\ u_1^2, u_2, \ldots, u_n \end{bmatrix} = 0.$$

This concludes the proof. □

Consider the power series ring $\kappa[[x_1, \ldots, x_n]] = \kappa[[y_1, \ldots, y_n]]$. Given a power series f represented as

$$f = \sum a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

with $a_{i_1 \cdots i_n} \in \kappa$, Lagrange inversion seeks formulas of $b_{i_1 \cdots i_n} \in \kappa$ for a new representation

$$f = \sum b_{i_1 \cdots i_n} y_1^{i_1} \cdots y_n^{i_n}$$

in terms of $a_{i_1 \cdots i_n}$ and the coefficients $c^{(j)}_{i_1 \cdots i_n}$ in

$$y_j = \sum c^{(j)}_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}.$$

With the partial order on the variables $x_1, \ldots, x_n$, the Möbius inversion formula is a Lagrange inversion formula: in the power series ring $\kappa[[x_1, \ldots, x_n]] = \kappa[[u_1, \ldots, u_n]]$, consider $f := \sum a_j x_j$. The elements

$$b_i := \mathbf{res} \begin{bmatrix} f \, du_1 \wedge \cdots \wedge du_n \\ u_1, \ldots, u_{i-1}, u_i^2, u_{i+1}, \ldots, u_n \end{bmatrix} = \sum a_j \mu(x_i, x_j)$$

give a new representation $f = \sum b_i u_i$.

4. Schauder bases

The set of functions from X to κ can be identified with the set $\bar{I}(X)$ consisting of formal sums $\sum_{x \in X} a_x x$ with $a_x \in \kappa$. With termwise addition and scalar multiplication, $\bar{I}(X)$ is a vector space over κ. We remark that $\bar{I}(X)$ can also be identified with the incidence algebra of X modulo its Jacobson radical [17]. Given $f = \sum a_x x \in \bar{I}(X)$, the subset

$$\operatorname{supp} f := \{x \in X : a_x \neq 0\}$$

of X is called the *support of* f.

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DEFINITION 4.1. Elements $u_i \in \bar{I}(X)$ form a *Schauder basis of* $\bar{I}(X)$ if the following conditions hold.

- There are only finitely many u_i whose support contains any given $x \in X$. (So we can define $\sum a_i u_i$ for any $a_i \in \kappa$.)
- Any element $\bar{I}(X)$ can be written uniquely as $\sum a_i u_i$ for $a_i \in \kappa$.

If X is finite, the notion of Schauder bases agrees with the notion of bases of a vector space. If X is the set of monomials in the power series ring $\kappa[[x]]$ with the order $x^i < x^j \Leftrightarrow i < j$, then $\bar{I}(X)$ can be identified with $\kappa[[x]]$ as a vector space. In [7, Definition 2.1], a slightly different definition of Schauder bases is given for $\kappa[[x]]$. One can show that a Schauder basis of $\bar{I}(X)$ in the sense of this article is countable and can be listed as a Schauder basis $f_0, f_1, f_2, \ldots$ of $\kappa[[x]]$ in the sense of [7].

PROPOSITION 4.2. *Assume that all principal filters on X are finite. The elements $u_y := \sum_{x \le y} x$, where $y \in X$, form a Schauder basis of $\bar{I}(X)$.*

PROOF. For any $x \in X$, the set consisting of those $y \in X$ such that $x \in \operatorname{supp} u_y$ is exactly the principal filter generated by x; hence is finite.

We need to represent $\sum_z a_z z \in \bar{I}(X)$ in terms of u_y. Recall that Corollary 3.3 asserts that $\sum_{x \le y \le z} \mu(y, z) = 0$ for $x \ne z$. Together with the fact that $\mu(z, z) = 1$, we have $z = \sum_{y \le z} \mu(y, z) u_y$. Therefore,

$$\sum_y \Big(\sum_{y \le z} a_z \mu(y, z) \Big) u_y = \sum_z a_z z.$$

For the uniqueness of representations, it suffices to show that all $b_x \in \kappa$ vanish if $\sum_x b_x u_x = 0$. For $y \in X$, the elements $a_y := \sum_{y \le z} b_z$ have to be zero, since $\sum_y a_y y = \sum_z b_z u_z = 0$. On the other hand,

$$b_x = \sum_{x \le y \le z} \mu(x, y) b_z = \sum_{x \le y} \mu(x, y) a_y = 0.$$

This concludes the proof. □

Similarly we can show the following.

PROPOSITION 4.3. *Assume that all principal ideals on X are finite. The elements $v_y := \sum_{x \ge y} x$, where $y \in X$, form a Schauder basis of $\bar{I}(X)$.*

Recall that an inverse relation is a pair of identities of the form

$$\begin{cases} b_n = \displaystyle\sum_{k=0}^{n} c_{nk} a_k, \\ a_n = \displaystyle\sum_{k=0}^{n} d_{nk} b_k, \end{cases}$$

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where $a_i, b_i, c_{ji}, d_{ji} \in \kappa$. It is shown in [7, Theorem 2.1] that an inverse relation with the orthogonal property comes from representations of a power series by two Schauder bases in the sense of [7]. Assume that all principal filters on X are finite. Möbius inversion consist of the information

$$\begin{cases} b_x = \sum_{y \geq x} \mu(x, y) a_y, \\ a_x = \sum_{y \geq x} b_y, \end{cases}$$

which comes from two representations of an element $\sum a_x x = \sum b_x u_x \in \bar{I}(X)$ by the Schauder bases $\{x\}_{x\in X}$ and $\{u_x\}_{x\in X}$.

5. Computational aspects

In this section, we provide three examples to exhibit how differentials and residues naturally appear in Möbius functions and their applications.

In the first example, X is the set of integers with the usual order. We identify an integer i with a variable x_i. By Theorem 3.2,

$$\mu(n, n+1) = \mathbf{res} \begin{bmatrix} x_{n+1}\, dx_n \wedge d(x_n + x_{n+1}) \\ x_n^2,\, x_n + x_{n+1} \end{bmatrix}.$$

The generalized fraction in the above formula can be simplified as

$$\begin{aligned} \begin{bmatrix} x_{n+1}\, dx_n \wedge d(x_n + x_{n+1}) \\ x_n^2,\, x_n + x_{n+1} \end{bmatrix} &= \begin{bmatrix} x_{n+1}(x_n - x_{n+1})\, dx_n \wedge dx_{n+1} \\ x_n^2,\, x_n^2 - x_{n+1}^2 \end{bmatrix} \\ &= \begin{bmatrix} x_{n+1}(x_n - x_{n+1})\, dx_n \wedge dx_{n+1} \\ x_n^2,\, -x_{n+1}^2 \end{bmatrix} \\ &= \begin{bmatrix} dx_n \wedge dx_{n+1} \\ x_n,\, -x_{n+1} \end{bmatrix}. \end{aligned}$$

By taking the residue,

$$\mu(n, n+1) = \mathbf{res} \begin{bmatrix} dx_n \wedge dx_{n+1} \\ x_n,\, -x_{n+1} \end{bmatrix} = -1.$$

For $m > n + 1$,

$$\begin{aligned} \mu(n, m) &= \mathbf{res} \begin{bmatrix} x_m\, dx_n \wedge d(x_n + x_{n+1}) \wedge \cdots \wedge d(x_n + \cdots + x_m) \\ x_n^2,\, x_n + x_{n+1}, \ldots, x_n + x_{n+1} + \cdots + x_m \end{bmatrix} \\ &= \mathbf{res} \begin{bmatrix} x_m\, dx_n \wedge d(x_n + x_{n+1}) \wedge \cdots \wedge d(x_n + \cdots + x_m) \\ x_n^2,\, x_n + x_{n+1}, \ldots, x_n + x_{n+1} + \cdots + x_{m-1},\, x_m \end{bmatrix} = 0. \end{aligned}$$

For the rest of this section, we consider a partially ordered finite set $X = \{x_1, \ldots, x_n\}$ and use the notation of (3.1) in Section 2.

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In the second example, we revisit a determinant formula obtained independently by Lindström [11] and Wilf [18]. For a sequence $a_1, \dots, a_n \in \kappa$, the differential

$$\omega_i := d\Big(\sum_{x_\ell \geq x_i} a_\ell u_\ell\Big)$$

can be written as

$$\omega_i = \sum_{j=1}^{n} b_{ij}\, dx_j$$

for $b_{ij} \in \kappa$. Indeed, $b_{ij} = \sum_{x_\ell \geq x_i, x_j} a_\ell$. For a maximal element x_m of X,

$$\omega_1 \wedge \cdots \wedge \omega_n = \omega_1' \wedge \cdots \wedge \omega_{m-1}' \wedge (a_m\, dx_m) \wedge \omega_{m-1}' \wedge \cdots \wedge \omega_n',$$

where $\omega_i' := d(\sum_{x_i \leq x_\ell \neq x_m} a_\ell u_\ell)$. By induction on n,

$$\omega_1 \wedge \cdots \wedge \omega_n = (a_1\, dx_1) \wedge \cdots \wedge (a_n\, dx_n).$$

Hence

$$\det(b_{ij}) = \mathbf{res}\begin{bmatrix} \omega_1 \wedge \cdots \wedge \omega_n \\ x_1, \dots, x_n \end{bmatrix} = a_1 \cdots a_n.$$

Let $b_i := b_{ii}$. In terms of the Möbius function, the above determinant formula can also be written as

$$\det(b_{ij}) = \prod_{i=1}^{n} \sum_{j=1}^{n} \mu(x_i, x_j) b_j.$$

For our third example, we recall $u_i = \sum_{x_j \leq x_i} x_j$ and $x_j = \sum_{i=1}^{n} \mu(x_i, x_j) u_i$. The Möbius function also appears in the identity

$$dx_j \wedge du_2 \wedge \cdots \wedge du_n = \mu(x_1, x_j)\, du_1 \wedge du_2 \wedge \cdots \wedge du_n$$

of exterior products of differentials. Let R be the matrix whose (i, j) entry is 1 if $x_i \leq x_j$ or $j = 1$, and otherwise is 0. The determinant considered by Redheffer [14] and Wilf [19] can be written as

$$\det R = \mathbf{res}\begin{bmatrix} d(x_1 + \cdots + x_n) \wedge du_2 \wedge \cdots \wedge du_n \\ x_1, \dots, x_n \end{bmatrix}. \tag{5.1}$$

To compute the residue, we use the relation

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} (\chi^0 + \chi^1)(x_1, x_1) & \cdots & (\chi^0 + \chi^1)(x_n, x_1) \\ \vdots & \ddots & \vdots \\ (\chi^0 + \chi^1)(x_1, x_n) & \cdots & (\chi^0 + \chi^1)(x_n, x_n) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

of matrices to change the denominators of the generalized fraction in (5.1) from $x_1, \dots, x_n$ to $u_1, \dots, u_n$. Note that $\det((\chi^0 + \chi^1)(x_i, x_j))$ is invariant under any

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permutation of the indices of the sequence $x_1, \ldots, x_n$. Since there is a permutation of the indices of $x_1, \ldots, x_n$ such that the matrix $((\chi^0 + \chi^1)(x_i, x_j))$ is upper triangular with ones on the diagonal, we have $\det((\chi^0 + \chi^1)(x_i, x_j)) = 1$. By the transformation law,

$$\begin{bmatrix} d\Big(\sum_{j=1}^{n} x_j\Big) \wedge du_2 \wedge \cdots \wedge du_n \\ x_1, \ldots, x_n \end{bmatrix} = \begin{bmatrix} d\Big(\sum_{j=1}^{n} x_j\Big) \wedge du_2 \wedge \cdots \wedge du_n \\ u_1, \ldots, u_n \end{bmatrix}.$$

We recover the determinant formula of Redheffer [14] as well as its generalization by Wilf [19] as follows:

$$\begin{aligned} \det R &= \mathbf{res} \begin{bmatrix} d\Big(\sum_{j=1}^{n} x_j\Big) \wedge du_2 \wedge \cdots \wedge du_n \\ u_1, \ldots, u_n \end{bmatrix} \\ &= \sum_{j=1}^{n} \mathbf{res} \begin{bmatrix} dx_j \wedge du_2 \wedge \cdots \wedge du_n \\ u_1, \ldots, u_n \end{bmatrix} = \sum_{j=1}^{n} \mu(x_1, x_j). \end{aligned}$$

We remark that the assumption in [19] that x_1 is the smallest element of X is irrelevant in the above proof.

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I-CHIAU HUANG, Institute of Mathematics, Academia Sinica, 6F,
Astronomy-Mathematics Building, No. 1, Sec. 4, Roosevelt Road,
Taipei 10617, Taiwan, ROC
e-mail: ichuang@math.sinica.edu.tw