

## GENERALIZED BLOB ALGEBRAS AND ALCOVE GEOMETRY

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*Abstract*

A sequence of finite-dimensional quotients of affine Hecke algebras is studied. Each element of the sequence is constructed so as to have a weight space labelling scheme for Specht/standard modules. As in the weight space formalism of algebraic Lie theory, there is an action of an affine reflection group on this weight space that fixes the set of labelling weights. A linkage principle is proved in each case. Further, it is shown that the simplest non-trivial example may essentially be identified with the blob algebra (a physically motivated quasihereditary algebra whose representation theory is very well understood by Lie-theory-like methods). An extended role is hence proposed for Soergel's tilting algorithm, away from its algebraic Lie theory underpinning, in determining the simple content of standard modules for these algebras. This role is explicitly verified in the blob algebra case. A tensor space representation of the blob algebra is constructed, as a candidate for a full tilting module (subsequently proven to be so in a paper by Martin and Ryom-Hansen), further evidencing the extended utility of Lie-theoretic methods. Possible generalisations of this representation to other elements of the sequence are discussed.

1. *Introduction*

Affine Hecke algebras are currently the subject of widespread interest in representation theory [1, 31]. It is customary to study them through certain families of quotient algebras, among which the usual choice is the cyclotomic Hecke algebras [2, 10]. These quotients, though finite-dimensional, are still complicated, and complete knowledge of their representation theory remains a significant challenge. Following our consideration of the *physical* context of affine Hecke algebras in the formalism of transfer matrix algebras in statistical mechanics [58], we introduce another class of quotient algebras that are smaller and more accessible. We show that a number of techniques and organisational schemes from algebraic Lie theory work in this context, such as weight spaces and linkage [36] (always) and Soergel's tilting module algorithm [64, 65] (in at least one highly non-trivial case), despite the fact that these algebras lie outside the known regime of applicability of these methods (that is, as underpinned by classical theory).

The blob algebra  $b_n$  is a two-parameter generalisation of the Temperley–Lieb algebra  $T_n(q)$  that may be defined in terms of a basis of decorated Temperley–Lieb diagrams [55, 57] (hence its name). As with  $T_n(q)$ , there is also a definition by the generators

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$\{e_-, U_1, U_2, \dots, U_{n-1}\}$  and relations  $e_-e_- = e_-, U_iU_i = -(q + q^{-1})U_i,$

$$U_iU_{i\pm 1}U_i = U_i, \tag{1}$$

$$U_1e_-U_1 = y_eU_1, \tag{2}$$

and generators commuting otherwise, where  $q$  and  $y_e$  are parameters. (Here,  $T_n(q)$  is the subalgebra generated by the  $U_i$ s.) The representation theory of this algebra is very well understood [15], and it is known to have several beautiful properties evocative of algebraic Lie theory. In this paper we consider  $b_n$  as an affine Hecke quotient, and our sequence of affine Hecke quotients is a sequence of generalisations of  $b_n$ , chosen so as to be likely to be amenable to the same high level of understanding. As indicated above, we show that this choice succeeds in a number of key areas, culminating in a linkage principle [36]. We further explore the blob algebra’s connection to Lie theory, describing in particular an action on a suitable tensor space evocative of the tilting theory and Ringel duality of quantum groups [61, 22, 56]. (Indeed, it has now been shown [53] that this action endows this tensor space with the property of a full tilting module.)

### 1.1. Background/overview of the paper

Soergel has given a beautiful procedure [65, 64] for analysing tilting modules for quantum groups at roots of unity through parabolic Kazhdan–Lusztig polynomials. The procedure itself may be applied formally to an alcove geometry, *without* reference to representation theory. Hence it may be applied, in principle, in cases that are beyond the scope of Soergel’s proof of representation-theoretic interpretation. It is interesting, then, to try to find algebras for which the resultant combinatorial data *has* a representation-theoretic interpretation, even though Soergel’s proof is not applicable. The output of the usual procedure in type- $A$  may be mapped by Ringel duality [23] to the content of projective modules for certain quotients of ordinary Hecke algebras. (There, it may be understood in terms of idempotent decompositions of 1; see [56].) This leads to a determination of decomposition numbers for standard modules of the Hecke algebras themselves. Here we consider generalising the implementation of the procedure on this Ringel dual side. We do this by constructing generalized Hecke algebra quotients that (mildly) generalize the usual role of alcove geometry.

One example where the formal procedure gives the correct decomposition numbers is the blob algebra  $b_n$  (a two-parameter *affine* Hecke algebra quotient); see [55, 57]. We demonstrate the procedure for this example in Section 1.2 below. There is a set of key properties of  $b_n$  (see Section 4.2), which it has in common with the ordinary Hecke algebra quotients (see Section 1.3), which may serve to explain the phenomenon. In this paper we discuss generalisations of  $b_n$  that also possess these properties.

To generalise  $b_n$  suitably, we first place it in the context of affine/cyclotomic Hecke and Ariki–Koike–Levy algebras [2, 10, 52] (although these are *not* themselves the generalisations that we require). The study of these algebras is interesting both abstractly and also since they are useful in studying solutions to the reflection equation in integrable statistical mechanics (see [52] for references). This parallels the role of ordinary Hecke algebras in solving the Yang–Baxter equation. In both cases the ‘physical’ representation theory focuses attention on specific quotients, and implies that decomposition number data should be organized in a certain specific way. In the ordinary Hecke case this is the ‘Soergel’ rather than the ‘LLT’ way [43]. (As complete data sets these are equivalent, but computationally they are not [56].) Thus, while the algebras that we shall construct have representation theory that is accessible in principle by LLT methods [1], this does not remove the need for

a generalized (dual) Soergel approach. In Section 1.3 we discuss quotients of affine Hecke algebras generalizing  $b_n$  which, like  $b_n$ , realize certain key ingredients of this approach – in particular, they possess a weight space.

In Section 3 we imbue this space with an affine reflection group action and an alcove structure, and verify a linkage principle [36] for each such algebra. This states that, given any two standard modules  $\Delta(\mu)$  and  $\Delta(\nu)$ , say, if there is a non-trivial homomorphism between any subquotient of  $\Delta(\mu)$  and any subquotient of  $\Delta(\nu)$ , then the weights  $\mu$  and  $\nu$  lie in the same orbit of the appropriate affine reflection group.

Each generalized algebra is a quotient of a cyclotomic Hecke algebra by certain primitive and central idempotents (independent of  $n$ ). In Section 4 we show how the simplest non-trivial such quotient may effectively be *identified* with the blob algebra. The rest of the paper discusses outstanding technical issues in showing the validity of the generalized Soergel procedure for the generalized algebras (the primitive and central idempotents of the relevant cyclotomic Hecke algebras are computed in a convenient form in Section 5). Such issues include:

- the absence of a diagram calculus (indeed, unlike in the blob case, we provide only a conjecture for a basis for each generalisation – see Section 5.3),
- quasi-heredity (if the algebra *is* quasi-hereditary, for example, the linkage principle implies that weights in different orbits of the appropriate affine reflection group are in different blocks), and
- a global limit (recollement between different  $n$  values [12, 58]).

One property of the ordinary Hecke quotients *not* possessed by these generalisations is the defining representation on ‘tensor space’, realizing Ringel duality [22] with a quantum group quotient (compare [3, 20, 63]). Such a faithful tensor space representation is not manifestly necessary for our purpose, but would be very useful in resolving the issues mentioned above. In Section 6 we address this problem, culminating in the construction of some intriguing new concrete representations of  $b_n$ , which are candidates. (The serendipitous constructions of a number of other interesting new representations of  $b_n$  and its generalisations are outlined in Sections 7 and 8.)

We shall argue that the representation theory of these algebras, while *containing* that of ordinary Lie-theoretic objects, is in a sense more simply described. Given that the representation theory of  $b_n$  is itself known for  $q$  a root of unity in arbitrary characteristic [15], the possibility that open questions in ordinary Lie representation theory may be accessible by this route makes these algebras particularly interesting objects for study.

## 1.2. Alcove geometry and decomposition numbers for $b_n$

There is a ‘good’ parameterisation [57, Section 2] of the blob algebra  $b_n$  by  $q$  and  $m$ , where  $y_e = -[m-1]/[m]$  (see also Section 4.2). The decomposition numbers in the ‘doubly critical’ case ( $q$  a primitive  $l$ th root of 1;  $m$  an integer,  $|m| < l$ ) in characteristic zero are determined in [57, Section 9] by algebraic methods. A formal application of Soergel’s procedure to this case works as follows.

First recall, quite generally, that a Euclidean space with reflection hyperplanes removed has a set  $\mathfrak{A}$ , say) of connected components, called *alcoves*; see [36, Chapter 6] and [9]. For  $s$  a reflection hyperplane and  $B$  an alcove, we denote by  $Bs$  the image of  $B$  in  $s$ . Each nonempty intersection of the closure of an alcove with a hyperplane is called a *wall* of the alcove (here we shall confuse each such wall with the hyperplane containing it). We make

no assumption about the relation of hyperplanes to the origin 0, except that the origin lies in the interior of an alcove called the *fundamental alcove* and denoted  $A^0$ . Define  $|B|$  as the number of hyperplanes between  $B$  and 0 (so  $|A^0| = 0$ ).

In algebraic Lie theory, one starts with a set of ordinary (non-affine) reflections generating the ordinary Weyl group. The  $(q)$ -group weight space is the underlying Euclidean space with its origin  $\rho$ -shifted [36]. In particular, even when an affine reflection is added, the origin is at a fixed position at the base of the dominant region. In our case there is effectively no ordinary Weyl group and no dominant region; that is to say, the placement of *all* hyperplanes is controlled by parameters of the algebra. Thus the weight space for the blob algebra, just as for  $sl_2$ , is the Euclidean space associated to the  $A_1$  Coxeter system – that is, it is effectively  $\mathbb{R}$  (see [57, Section 6]). Now, however, (compare  $sl_2$ ) *all* integral weights are dominant; that is to say, simple modules may be indexed by  $\mathbb{Z}$ . (We shall explain this, in the context of our generalised construction, shortly.) In the  $b_n$  case, the reflection hyperplanes are just points, and those generating the affine Weyl group lie at  $-m$  and  $l - m$ . (An alcove  $B \in \mathfrak{A}$  is a connected component of  $\mathbb{R}$  with the reflection points removed.) A reflection is ‘upward’ if  $|B| < |Bs|$  (compare the usual  $sl_2$  situation).

For each alcove  $A$ , one defines a map

$$n_A : \mathfrak{A} \rightarrow \mathbb{Z}[v],$$

where  $v$  is a formal parameter, as follows. (For simplicity, we ignore features of Soergel’s procedure that do not arise in our case.) Firstly,  $n_A(A) = 1$  and  $n_A(B) \neq 0$  implies that  $|B| \leq |A|$ . Note that  $n_{A^0}$  is determined immediately by this, and proceed inductively on  $>$ . For each alcove  $A$ , there will be a wall  $s$  of  $A$  such that  $|As| > |A|$ . Each alcove  $B$  has one wall that is in the affine Weyl orbit of  $s$ , and we shall write  $B.s$  for the image of  $B$  in that wall (thus  $As = A.s$ ). Then, with  $n_A$  known, we set

$$n'_{As}(B.s) = \begin{cases} n_A(B) + v^{-1}n_A(B.s), & |B.s| > |B|, \\ v^{-1}n_A(B) + n_A(B.s), & |B.s| < |B|, \end{cases}$$

and we define  $n_{As}$  by

$$n_{As}(C) = n'_{As}(C) - \sum_{B : |B| < |As|} n'_{As}(B)|_{v=0} n_B(C).$$

(That this procedure is well defined is not trivial [64].)

Evaluating  $n_A(B)|_{v=1}$ , this construction is (formally) computing the standard module content of tilting modules in a Ringel dual algebra. (Any two weights that are in different affine Weyl orbits [36] are in different blocks [7, Chapter 1]. Thus each block intersects each alcove in at most one weight and, fixing a block, it is the modules with these weights that  $n_A(B)|_{v=1}$  describes.) The corresponding data for  $b_n$  is, in effect, the standard module content of projective modules. By reading by column instead of by row, as it were, we convert this to the simple module content of standard modules [22]. (Truncation to a finite column interval, such as that pictured in the example which follows, corresponds to localisation to some finite  $n$  – see ingredient I2 below.)

As noted, this construct is entirely formal; however, the following proposition holds.

**PROPOSITION 1.1.** *For  $\lambda \in A$  and  $\mu \in B$ , the  $b_n$  standard composition multiplicity*

$$[\Delta(\mu) : L(\lambda)] = \begin{cases} n_A(B)|_{v=1}, & \mu \text{ in the affine Weyl orbit of } \lambda, |\lambda| \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

*Further, the power of the formal parameter  $v$  determines the Loewy layer.*

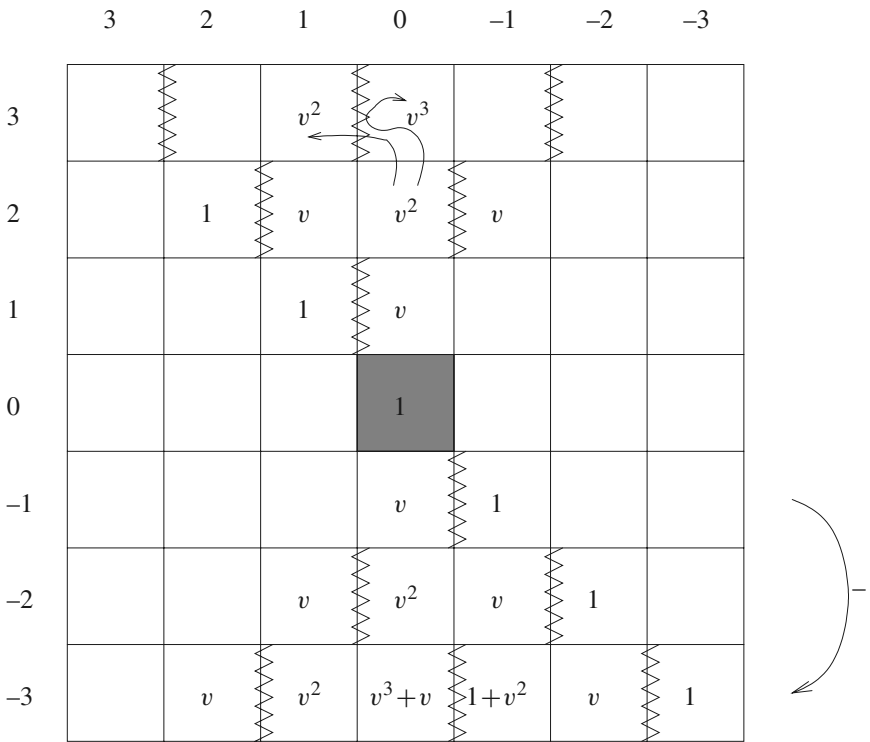


Figure 1: Computation of the polynomials  $n_A(B)$ .

*Proof.* The composition multiplicity data is determined in [57, Section 9]. The computation of the polynomials  $n_A(B)$  is illustrated in Figure 1. The row position in the table gives  $A$ , and the column position  $B$ ; and  $n_{A^0}(A^0)$  is shaded. (The rows and columns, and hence the alcoves, are labelled using  $\mathbb{Z}$ . These labels should not be confused with points in the underlying space  $\mathbb{Z}$  (each alcove contains  $l - 1$  such points), or with weights.) The table is complete for the rows shown, except for the top and bottom rows. For the top row, the arrows within the table illustrate the contributions to the  $n'_{As}$  from a particular  $n_A(B)$  (the shaded lines are the relevant walls for reflection in each case). The arrow outside the table illustrates a required subtraction to obtain  $n_A$  in the bottom row (which is complete except for this subtraction). The pattern is clear, and one sees immediately that the formal procedure reproduces the multiplicity and layer data. □

The case  $b_n(q, m = 1)$  contains the ordinary Ringel dual,  $\text{End}_{U_q \mathfrak{sl}_2}(V_2^{\otimes n})$ , as a quotient. Note that the ordinary Soergel procedure is embedded accordingly in this version (in the ‘dominant’ region of case  $m = 1$ ).

The ‘idempotent splitting’ analysis described in [56] applies in principle in this situation, giving a heuristic explanation of why Soergel’s procedure is relevant here. Following this paradigm, there is a set of natural generalisations for which an analogous method should work.

The ingredients are (in precis; see also [56, 58]) as follows.

- I1 A tower of unital algebras  $A_n \subset A_{n+1}$  over a ring  $R$  with indeterminates, and a multiplicity-free [69] semisimple specialisation [13] (split, and we shall consider only characteristic 0 here).

For a tower as in I1, let  $B_{A_-}$  denote the Bratteli diagram of the semisimple case. Regarded as a set,  $B_{A_-}$  will here mean the vertex set of this graph. Now pass to the tower over a field (a not necessarily semisimple specialisation).

- I2 (i) A quasi-hereditary global limit via an idempotent  $e \in A_m$  (for some  $m$ ) and isomorphisms [29]

$$eA_{n+m}e \cong A_n \tag{3}$$

(and hence a tower of *recollement* [12]).

(Preferably,  $R$  is a unique factorisation domain,  $e$  makes sense over the field of fractions, and we simply exclude consideration of specialisations in which  $e$  is not well defined – see later.)

- (ii) A map  $\mathfrak{P}_A$  from  $B_{A_-}$  to a global index set  $\Lambda$ , which localises at each  $n$  to an index set  $\Lambda(n)$  for standard modules  $\Delta(\mu)$  of  $A_n$  (that is, such that  $\Lambda(n) \hookrightarrow \Lambda(n+m)$  via the full embedding of  $A_n$ -mod in  $A_{n+m}$ -mod consequent on equation (3), while  $\dot{\cup}_n \Lambda(n) \cong B_{A_-}$ ).

Note that here we mean *standard* in the quasi-hereditary sense, but we might also hope that these modules would be ‘nice’ in some Kazhdan–Lusztig sense [25, 41]. Let  $\text{Res}(\mu) \subset \Lambda$  denote the set of weights of standard factors of the restriction  $\text{Res}_{A_n}^{A_{n+1}} \Delta(\mu)$ , and  $\text{Ind}(\mu)$  that of the corresponding induction.

- I3 A space  $V$  (for definiteness, we shall assume that this is a real Euclidean space) and a map  $\Lambda \hookrightarrow V$  with the following properties. The convex hull of  $\text{Res}(\mu)$  intersects  $\Lambda$  in a subset of  $\{\mu\} \cup \text{Ind}(\mu) \cup \text{Res}(\mu)$  (*locality* of induction and restriction); the set  $\rho_\mu$  of reflections in  $V$  that fix  $\{\mu\}$  and  $\text{Res}(\mu)$  fixes  $\Lambda$ ; the group  $W$  generated by  $\cup_\mu \rho_\mu$  is an affine Weyl group [34, Section 4.2]; and  $\Lambda$  is a subset of the set of point facets in the alcove geometry induced by  $W$  on  $V$ .
- I4 Control of bases for the algebras and standard modules – including the means in principle to compute Gram matrices in the case of indeterminate parameters.
- I5 Explicit forms for the simplest primitive idempotents (in particular any primitive *and* central idempotents).
- I6 A linkage principle (see [36]):  $\Delta(\mu)$  and  $\Delta(\nu)$  are in different blocks if there does not exist any  $w \in W_l$  (where  $W_l$  is a suitably rescaled version of  $W$ , depending on the specialisation) such that  $w\mu = \nu$ .

The idea is that the structure of  $A_n$  for small  $n$  would be determined by brute force, and that of subsequent levels would be determined largely by embedding the category of left modules  $A_n$ -mod in  $A_{n+m}$ -mod using equation (3) (see [12, 29]). The simple modules  $L$  of  $A_{n+m}$ -mod such that  $eL = 0$  are added by hand, and their morphisms are determined partly by expedient use of I4 and I5, and then by using Frobenius reciprocity (with respect to the tower of subalgebras) and I6 to provide an emulation of translation functors [15, 36]. The process is best illustrated with reference to the examples of interest in this paper, to which we now turn.

1.3. Role and realization of ingredients 1, 2(ii) and 3

Recall (see [44]; compare [42]) the affine Hecke algebra  $H(n)$  defined by generators  $\{1, X, g_1, \dots, g_{n-1}\}$  and relations

$$g_i g_{i\pm 1} g_i = g_{i\pm 1} g_i g_{i\pm 1}, \quad g_i g_j = g_j g_i, \quad i \neq j \pm 1, \tag{4}$$

$$g_1 X g_1 X = X g_1 X g_1, \quad g_j X = X g_j, \quad j > 1, \tag{5}$$

$$(g_i - q)(g_i + q^{-1}) = 0. \tag{6}$$

The cyclotomic Hecke algebra [10]  $H = H(n, d)$  is the quotient  $\Psi_d$  of  $H(n)$  by

$$\prod_{i=1}^d (X - \lambda_i) = 0. \tag{7}$$

Here  $q, \lambda_1, \dots, \lambda_d, \dots$  are parameters, which we may begin by regarding as indeterminates. Write  $\mathcal{A}$  for  $\mathbb{Z}[q, q^{-1}, \lambda_1, \dots, \lambda_d]$  and  $\mathcal{K}$  for the quotient field. Write  $H^{\mathcal{A}}(n, d)$  for  $H(n, d)$  over  $\mathcal{A}$ . This is a free module over  $\mathcal{A}$  (see [2] and Section 2 below). The case  $d = 2$ ,  $\lambda_1 = -\lambda_2^{-1}$ , is essentially the  $B$ -type Hecke algebra (see [33, 49]). For any  $d' > d$ , let  $\Psi_d : H(n, d') \rightarrow H(n, d)$  also denote the quotient by equation (7). Denote by  $H(-, d)$  the sequence of inclusions

$$H(n, d) \subset H(n + 1, d). \tag{8}$$

Usually, we shall fix an  $\mathcal{A}$ -algebra  $k$  which is a field, and which as a field is  $\mathbb{C}$ , and we shall consider  $H(n, d) = H^{\mathcal{A}}(n, d) \otimes_{\mathcal{A}} k$ . The semisimple generic structure of  $H(n, d)$  over  $\mathbb{C}$  is well known, through that of the specialisation to the group algebra of the group  $Z_d \wr S_n$  (confer [13, 33, 35] as in [2]). We recall it briefly. An integer partition  $\mu$  of degree  $n$  is a list  $(\mu_1, \mu_2, \dots)$  of non-negative integers such that  $\mu_i \geq \mu_{i+1}$  and  $\sum_i \mu_i = n$ . There is a natural correspondence with Young diagrams of degree  $n$ . Denote by  $\Lambda_n = \Lambda_n^d$  the set of ordered lists of  $d$  integer partitions, of summed degree  $n$  (called  $d$ -partitions of degree  $n$ ). For example,

$$\Lambda_2^2 = \{((2), 0), ((1^2), 0), ((1), (1)), (0, (2)), (0, (1^2))\}.$$

The conjugacy classes of  $Z_d \wr S_n$  are readily seen to be indexed by  $\Lambda_n^d$  [45, 58]; thus  $H(n, d)$  has simple modules  $\Delta_\mu$  indexed by  $\mu \in \Lambda_n^d$ . Similarly, the Bratteli diagram  $B = B_{H(-, d)}$  of the natural tower of semisimple algebras  $H(-, d)$  is determined by

$$\text{Res}_{H(n, d)}^{H(n+1, d)} \Delta_\mu = \bigoplus_{i=1}^d \bigoplus_j \Delta_{\mu - e_j^i}, \tag{9}$$

where the sum over  $j$  is over possible subtractions of one box from the  $i$ th Young diagram of  $\mu$ .

For each  $n > 1$ , there are  $2d$  one-dimensional irreducible representations  $R_{\pm l}$  ( $l = 1, 2, \dots, d$ ) of  $H(n, d)$ , given by  $R_{\pm l}(X) = \lambda_l$  and  $R_{\pm l}(g_i) = \pm q^{\pm 1}$ . The representation  $R_l$  corresponds to the module  $\Delta_\mu$  with multipartition  $\mu = (\cdot, \mu_l, \cdot)$ , in which all component integer partitions are empty except the  $l$ th partition, which is either  $(n)$  (for the case  $l > 0$ ) or  $(1^n)$  (case  $l < 0$ ). For each  $n$  we may associate a unique primitive (and central) idempotent to each of these representations, in the algebra over generic  $k$  [32]. We write these idempotents

as  $e_n^{\pm l}$ . For  $l \in \{1, 2, \dots, d\}$ , the element  $e_n^{\pm l}$  of  $H(n, d)$  uniquely obeys

$$\begin{aligned} (g_i \mp q^{\pm 1})e_n^{\pm l} &= 0, & i = 1, 2, \dots, n - 1, \\ (X - \lambda_l)e_n^{\pm l} &= 0, \\ (e_n^{\pm l})^2 &= e_n^{\pm l}. \end{aligned} \tag{10}$$

The inclusion (8) allows us to regard  $e_n^{\pm l}$  as an idempotent in  $H(n + 1, d)$ , albeit neither primitive nor central in general. Indeed, the idempotent will be expressible as a sum of primitive idempotents in reciprocity with the rule (9). (Let  $1 = \sum_{\mu} e^{\mu}$  be the unique decomposition of 1 into primitive *central* idempotents of  $H(n + 1, d)$ . Then  $e_n^{\pm l} = \sum_{\mu} e_n^{\pm l} e^{\mu}$  is this decomposition; that is, it is also unique, even though the decomposition of 1 into primitive idempotents is not.)

In Section 5 of this paper, we give explicit formulae for all  $e_n^{\pm l}$  for all  $d$ . For now, we shall be concerned particularly with  $e^{\pm l} := e_2^{\pm l}$ . The reason for this is the desire for a small but significant generalisation of the set of dominant weights and the weight spaces underlying Soergel’s procedure for the case  $A_{m-1}$  (that is,  $U_qsl_m$ ). Although the induction and restriction rules are straightforward, and satisfy ingredient II, the ‘weight space’ of  $H(-, d)$  (in which the distance  $d(\mu, \lambda)$  is the minimum number of steps on  $\mathbb{B}$  from  $\mu$  to  $\lambda$ ) is somewhat unmanageable geometrically; compare our desired ingredients 2 and 3.

What is wanted is *something like* an analogue for  $H(-, d)$  of the quotients

$$H_n^m \cong \text{End}_{U_qsl_m}(V_m^{\otimes n}) \tag{11}$$

of the ordinary Hecke algebra  $H_n$ . The Bratteli diagram  $\mathbb{B}_{H_-}$  of the ordinary Hecke algebra is the Young ‘graph’, but via (11) the quotients  $H_n^m$  are the natural incarnations of the Ringel duals of  $A_{m-1}$  quantum groups, and hence may be associated to the same weight spaces, and satisfy ingredients 1–6. Let us briefly review this. Let  $e_{m+1}^{\pm}$  denote the two primitive and central idempotents of  $H_{m+1}$  (for simplicity, assume for now that  $[m + 1]! \neq 0$ ). The  $H_n^m$  are such that

$$0 \longrightarrow H_n e_{m+1}^- H_n \longrightarrow H_n \longrightarrow H_n^m \longrightarrow 0 \tag{12}$$

is exact. For  $[m]! \neq 0$  they are quasi-hereditary and satisfy ingredient 2: for example, through

$$e_m^- H_{n+m}^m e_m^- \cong H_n^m. \tag{13}$$

Let  $\Lambda^{1,m}$  denote the set of Young diagrams of less than  $m$  rows, regarded as a subset of  $\mathbb{Z}^{m-1}$ . Let  $v = (1, 1, \dots, 1) \in \mathbb{Z}^{m-1}$ , and let  $\mathbb{Z}^{m-1}/v$  denote the corresponding quotient set. Note that the injective map from  $\mathbb{Z}^{m-2}$  into  $\mathbb{Z}^{m-1}$  given by  $(\mu_1, \mu_2, \dots, \mu_{m-2}) \mapsto (\mu_1, \mu_2, \dots, \mu_{m-2}, 0)$  has image a set of representative elements of  $\mathbb{Z}^{m-1}$  in  $\mathbb{Z}^{m-1}/v$ . Denote by  $\mathfrak{P}^{m-1}$ , the corresponding surjective map from  $\mathbb{Z}^{m-1}$  to  $\mathbb{Z}^{m-2}$  (and also its restriction to  $\Lambda^{1,m}$ , whose image is  $\Lambda^{1,m-1}$ ). Note that  $\mathbb{B}_{H_-^m}$ , the set of weights for  $H_n^m$  for all  $n$ , is  $\Lambda^{1,m+1}$ . In the sense of ingredient 2(ii), the set of isomorphisms (13) collapses  $\mathbb{B}_{H_-^m}$  into  $\Lambda^{1,m}$ , which is the set of dominant weights of  $sl_m$ , via  $\mathfrak{P}^{m+1}$  [56]. That is,  $\mathfrak{P}_{H^m} = \mathfrak{P}^m$ , and  $\Lambda^{1,m}(n)$ , the index set for  $H_n^m$ , is the subset of  $\Lambda^{1,m}$  of diagrams of degree less than or equal to  $n$  and congruent to  $n$  modulo  $m$ .

Delightfully, we find that only the  $n = 2$  idempotents are needed for an analogue of equation (12) for  $H(n, d)$  (see also [58]). Denote the sum of the ideals generated by  $\{e_2^{-l} \mid \text{all } l\}$  by

$$\mathcal{D}_d = \mathcal{D}_d(n) := +_{l=1}^d He_2^{-l} H.$$

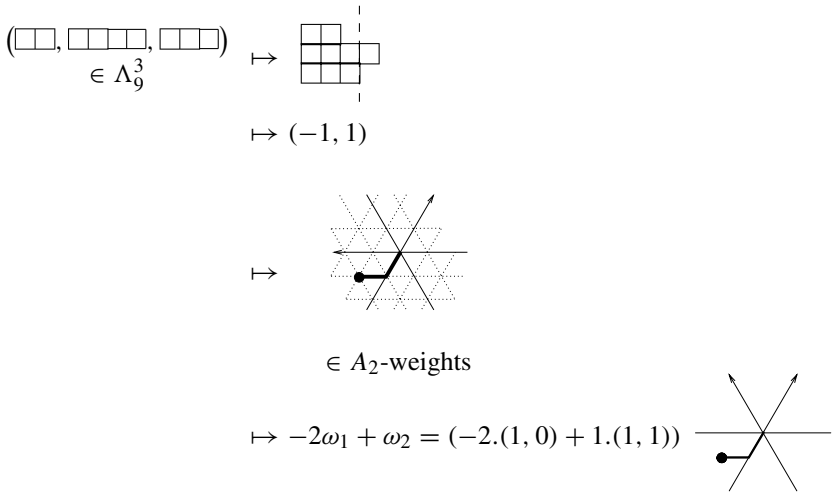


(We shall modify this definition very slightly later.) Define algebra  $H^{\mathcal{D}} = H^{\mathcal{D}}(n, d)$  by  $H^{\mathcal{D}}(1, d) = H(1, d)$  and for  $n > 1$  by exactness of the sequence

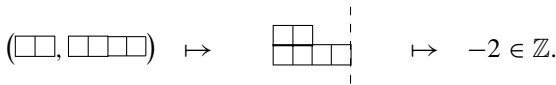
$$0 \longrightarrow \mathcal{D}_d \longrightarrow H \longrightarrow H^{\mathcal{D}} \longrightarrow 0. \tag{14}$$

The idea is to restrict consideration to the subset  $\Lambda^{\mathcal{D}}$  of  $H(-, d)$ -weights in which each integer partition is the trivial partition of that degree. Such an  $H(-, d)$ -weight is characterised by a sequence of  $d$  non-negative integers; that is, the degrees of the component integer partitions (in this way, we have an action of  $\mathfrak{P}^d$  on  $\Lambda^{\mathcal{D}}$ ). This sequence need not be ordered as an integer partition, and hence the set of such weights maps onto the set of all weights of  $A_{d-1}$  (that is, not just the usual dominant weights).

For example, with  $d = 3$ , the weight  $((2), (4), (3))$  becomes  $(2, 4, 3)$ , and  $\mathfrak{P}^3((2, 4, 3)) = (-1, 1)$ , as shown below.



(The final row just shows the same weight in terms of fundamental weights.) Meanwhile, with  $d = 2$ ,



(This last is the weight set used in Section 1.2.) Note also that  $\Delta_\mu \in H\text{-mod}$  is also in  $H^{\mathcal{D}}\text{-mod}$  if and only if  $\mu \in \Lambda^{\mathcal{D}}$  (in which case, as an  $H^{\mathcal{D}}$ -module, we write it  $\Delta(\mathfrak{P}^d(\mu))$ ). To see this note, from (9), that the restriction of  $\Delta_\mu$  to  $n = 2$  contains a copy of one of the excluded one-dimensional modules (not necessarily a *direct* summand in general) if and only if  $\mu$  contains at least one integer partition with a second part (that is, a Young diagram with a second row).

By construction, then, the tower  $H^{\mathcal{D}}(-, d)$  has ingredients 1, 3 (restriction is local via (9) and induction is via Frobenius reciprocity) and, at least formally, the final part of 2. Of course, the construction is most interesting if it can be made to include non-semisimple specialisations (or else the *fundamental alcove* of Section 1.2 is the whole weight space, and the Soergel procedure is trivial). In the remainder of this paper we address ingredients 2, 4, 5 and 6 from this point of view. In particular, we identify the  $d = 2$  case with the blob algebra. This has useful implications for all  $d > 2$ . Note that, fixing  $k$ , there are a number of distinct ways to impose a quotient relation of the form of  $\Psi_d$  on  $H(n, d')$ ,

corresponding to the choice of factors to be removed in the strengthening of the relation (7). For  $\lambda' = (\lambda_1, \lambda_2, \dots, \lambda_{d'})$  and  $\lambda$  any subsequence of this of length  $d' - d$ , let  $\Psi_d^\lambda$  denote the strengthening by omission of the factors  $(X - \lambda_i)$  with  $\lambda_i$  in  $\lambda$ . The quotient  $\Psi_d^{(\lambda_i)}$  commutes with the quotient to  $H^{\mathcal{D}}$ , so the following proposition holds.

**PROPOSITION 1.2.** *Fixing  $k$ , there are  $d + 1$  ways  $(\Psi_d^{(\lambda_i)}, i = 1, 2, \dots, d + 1)$  to quotient in order to pass from  $H^{\mathcal{D}}(n, d + 1)$  to  $H^{\mathcal{D}}(n, d)$ , where  $n > 0$ .*

Returning to the example of  $d = 3$ , we see that as  $n$  varies, the image of  $\Psi_2^{(\lambda_i)}$  sweeps out a  $2\pi/3$  radian arc of weight space, with the union of these arcs over  $i = 1, 2, 3$  giving the complete space (see Figure 4, illustrating this, in Section 5.2). If  $k$  gives a non-semisimple specialisation, then – as we shall see – there is at least one  $i$  such that the corresponding tower of blob algebras (that is,  $d = 2$  algebras) is critical (in the sense of [55, 57], or Section 1.2); that is, it has one or more reflection points. As  $n$  varies, a given  $d = 2$  reflection point sweeps out a straight line in this arc in  $d = 3$  weight space (see Section 5.2 for details). This is then a reflection line of the  $d = 3$  alcove geometry, to which the Soergel procedure may be applied.

Our approach to ingredient 2 is through representation theory, and the last part of the paper addresses this. It includes a ‘walk-through’ review of some earlier work in statistical mechanics, which explains our approach to this problem, and concludes by defining certain representations of  $b_n$  (and  $H^{\mathcal{D}}(n, d)$ ), which are candidates for ‘tensor space’ representations. (That is, they would establish ingredient 2 if faithful; see Section 6. The question of faithfulness is *not* resolved here.)

## 2. Preliminaries

Let the involution  $t : \mathbb{Z}[q, q^{-1}] \rightarrow \mathbb{Z}[q, q^{-1}]$  be given by  $t(q) = -q^{-1}$ , and let  $s$  act on  $\mathbb{Z}[\lambda_1, \dots, \lambda_d]$  by permuting the indices on the  $\{\lambda_i\}$  cyclically. Let

$$[n] = q^{n-1} + q^{n-3} + \dots + q^{1-n}.$$

For  $A$  an algebra, let  $Z(A)$  denote the centre of  $A$ ; and for  $B \subseteq A$ , let  $Z_B(A)$  denote the centralizer of  $B$  in  $A$  (so that  $Z_B(A) \supseteq Z(A)$  and  $Z_A(A) = Z(A)$ ).

For  $a = (a_1, a_2, \dots, a_n)$  an  $n$ -tuple of natural numbers, the *Weyl orbit* of  $a$  in  $\mathbb{N}^n$  is the orbit of the  $S_n$  action permuting indices. Let  $Y = (Y_1, \dots, Y_n)$  be an  $n$ -tuple of variables in some commutative ring  $R$ . Define

$$Y^a = \prod_{i=1}^n Y_i^{a_i}$$

and the *monomial symmetric polynomial* (see, for example, [24])

$$(Y^a)^\Sigma = \sum_{a'} Y^{a'},$$

where the sum is over all the elements in the Weyl orbit of  $a$ .

Let  $R$  be a unique factorisation domain,  $A$  a free  $R$ -module with basis  $\mathbf{A}$ , and  $K$  the field of fractions of  $R$ . Consider any  $e \in A \otimes_R K$ , and let  $a_e \in A$  be such that  $\mathbf{e} = a_e e \in A \subseteq A \otimes_R K$ . If there exists an  $a_e$  such that the coefficient of some  $\mathbf{a} \in \mathbf{A}$  in  $\mathbf{e}$  is 1, then  $a_e$  and  $\mathbf{e}$  are unique with this property, up to a unit. If  $A$  is an  $R$ -algebra, and  $e$  is idempotent, call such an  $\mathbf{e}$  a *preidempotent*, and  $a_e$  the corresponding *normalisation* of  $e$ .

Now let  $R$  as above be  $\mathcal{A}$ , and consider  $A$  as a collection of  $\mathbb{C}$ -algebras by specialisation. We say that a property holds *generically* if it holds on a (Zariski) open subset of parameter space, and that it holds *usually* if the condition for failure to hold may be expressed as a single finite polynomial in a single variable. (For example, [2] is generically and usually invertible;  $(q^2\lambda_1 - \lambda_2)$  is generically but not usually invertible.)

Let  $e = \mathbf{e}/a'_e$  be any explicit expression for an element in  $A$  over  $\mathcal{K}$  as above. For any point  $x$  in parameter space (that is, any  $k$ ), there is an open region with  $x$  in its closure in which the polynomial  $a'_e$  has no root, so  $e$  may be evaluated as a limit at  $k$ . This process does not guarantee a unique finite limit. If  $e$  is a primitive and central idempotent, however, then any two finite limits must be the same, since they will have the properties of a unique primitive and central idempotent in  $A$  over  $k$ . (That is, they will induce the same simple projective module,  $\mathbf{Ae} \otimes k$ .)

Let  $\sigma_i$  denote the elementary transposition  $\sigma_i = (i \ i + 1) \in S_n$ ; so  $\sigma_i(i) = i + 1$ , and so on [32]. Let  $\mathfrak{B}_n$  be a maximal set of inequivalent reduced words in the generators  $\{g_1, \dots, g_{n-1}\}$  of  $H_n$ . For each  $w \in \mathfrak{B}_n$ , note that there is a natural (reduced expression for an) element of  $S_n$  associated to it by substituting  $g_i \rightsquigarrow \sigma_i$ .

Let

$$X_1 := X \quad \text{and} \quad X_j := g_{j-1}X_{j-1}g_{j-1}. \tag{15}$$

So in  $H(n)$ ,

$$[X_j, X_k] = 0, \tag{16}$$

$$[X_j, g_k] = 0, \quad j \neq k, k + 1 \tag{17}$$

$$\begin{aligned} g_k X_{k+1} &= X_k g_k + (q - q^{-1})X_{k+1}, \\ g_k X_k &= X_{k+1} g_k - (q - q^{-1})X_{k+1}, \end{aligned} \tag{18}$$

and so

$$[X_k + X_{k+1}, g_k] = [X_k X_{k+1}, g_k] = 0. \tag{19}$$

Now

$$X_k^j + X_{k+1}^j = (X_k^{j-1} + X_{k+1}^{j-1})(X_k + X_{k+1}) - (X_k^{j-2} + X_{k+1}^{j-2})X_k X_{k+1},$$

so

$$[g_i, (X^a)^\Sigma] = 0$$

for all  $i$  and any  $a$ .

Let  $X^\Sigma$  denote the algebra of symmetric polynomials in the  $X_i$ s. Evidently,  $X^\Sigma \subseteq Z(H(n))$ , and in fact Bernstein has noted that this is an equality (see Appendix A).

It follows from equations (16)–(18) that any product of generators of  $H(n, d)$  can be expressed as a  $\mathbb{Z}[q, q^{-1}, \lambda_1, \dots, \lambda_d]$ -linear combination of words from the set

$$\mathfrak{C}_n^d = \{X^a w \mid a \in \{0, 1, \dots, d - 1\}^n, w \in \mathfrak{B}_n\}.$$

The dimension of this spanning set is clearly  $d^n n!$ , which is also the dimension of  $H(n, d)$ ; thus the following proposition holds.

PROPOSITION 2.1 (SEE [2]). *The set  $\mathfrak{C}_n^d$  is a basis for  $H(n, d)$ .*

We linearly extend  $s$  and  $t$  to act on  $H(n, d)$ , fixing  $\mathfrak{C}_n^d$  pointwise.

PROPOSITION 2.2 (SEE [58]). *Let  $B$  be a basis for  $H(n - 1, d)$ , let  $D$  be a basis for  $\langle X \rangle$ , and let  $G = g_1 g_2 \dots g_{n-1}$ . Then*

$$\{aGb, g_1 aGb, g_2 g_1 aGb, \dots, g_{n-1} \dots g_2 g_1 aGb \mid a \in D, b \in B\}$$

is a basis for  $H(n, d)$ .

Fix  $d$  and set  $p_n^l = \prod_{i \neq l} (q^{2n-2} \lambda_l - \lambda_i)$ . Note that

$$z_n^l = \prod_{k=1}^n \left( \prod_{i \neq l} (X_k - \lambda_i) \right)$$

lies in  $Z(H(n))$  and obeys  $\Psi_d((X - \lambda_l)z_n^l) = 0$ . Comparing this with (10), we thus have

$$e_n^{\pm l} = \frac{z_n^l}{\prod_{k=1}^n p_k^l} e_n^{\pm}.$$

It follows that  $e_n^{\pm l} = (\prod_{k=1}^n ([k] p_k^l)) e_n^{\pm l}$  is a preidempotent for  $e_n^{\pm l}$  (compare with the basis  $\mathfrak{C}_n^d$ ). Now we define

$$\mathcal{D}_d = \mathcal{D}_d(n) := +_{l=1}^d \text{He}_2^{-l} H$$

(a modification of the definition in Section 1.3) and  $H^{\mathcal{D}}$  accordingly. We shall give another expression for  $e_n^{\pm l}$  shortly.

### 3. Standard modules and linkage

The generators and relations in equations (4) (and their inverses) define the ordinary braid group,  $A_n$ -braid. Denote by  $B_n$ -braid the extension by  $g_0 = X$  (and its inverse) obeying equations (5) (confer [33], [50] and the references therein). Thus  $H(n)$  is a quotient of  $\mathbb{C}B_n$ -braid by the quadratic relation in equation (6).

One realization of  $B_n$ -braid is as the group of braids on the cylinder, with  $g_0$  becoming the pure braid in which the first string passes over all the other strings and then around the cylinder. There is a natural ‘Young’ embedding

$$B_n\text{-braid} \times B_m\text{-braid} \hookrightarrow B_{n+m}\text{-braid}.$$

One places the second cylinder concentrically inside the first, and one then allows the two cylinders to converge in such a way that the nodes of  $B_n$ -braid and  $B_m$ -braid respectively remain consecutive on the edges of the cylinder (while of course preserving over/under information). There is a corresponding embedding  $H(n) \times H(m) \hookrightarrow H(n + m)$ . The construction of ‘standard’ modules (in the sense of [11, 62]) follows from this. The quotient  $\Psi_d$  complicates this, in that the spectrum of  $X_{n+1}$  (the image of  $(1, X_1)$ ) is not that of  $X_1$  (the image of  $(X_1, 1)$ ).

DEFINITION 3.1. Let  $\langle X_i \rangle$  denote the commutative subalgebra  $\langle X_i \mid i = 1, \dots, n \rangle \subseteq H(n, d)$ .

Generically, as we shall see, we may determine a *unique* basis of primitive (and of course central) idempotents  $\epsilon_x$  of  $\langle X_i \rangle$  with  $\sum_x \epsilon_x = 1$ . There will be certain specialisations where

this basis will not make sense (certain idempotents will have preidempotents with vanishing normalisation). In any case, any primitive idempotent decomposition of 1 in  $H(n, d)$  will be different, but the unique primitive central idempotent decomposition of 1 in  $H(n, d)$  will be expressible as a crudification of the above (albeit depending on  $k$ ). The generic case will be the least crude (one idempotent per block/isomorphism class of simples), and it will be necessary, formally, to combine certain of these generic idempotents (into non-simple blocks) to make idempotents that make sense over  $k$  in non-semisimple cases. Each  $\epsilon_x$  must obey  $X_i \epsilon_x = x_i \epsilon_x$  with  $x_i$  some scalar. (Thus each induces a left  $H$ -module  $\Pi_x := H\epsilon_x$ . Since the  $H_n$  subalgebra of  $H(n)$  maps isomorphically to its image in the quotient, we have  $H\epsilon_x = H_n \epsilon_x$ , of rank  $n!$ .)

Evidently,  $x_1 \in \{\lambda_i\}$ , and with  $\pi_j := \prod_{i \neq j} (X - \lambda_i)$  we have  $X\pi_j = \lambda_j \pi_j$  and  $\pi_j \epsilon_x \propto \epsilon_x$ . For each such  $\pi_j$  there exists a minimal polynomial  $\pi_{j,-}(X_2) = \prod_k (X_2 - \lambda_{j,k})$  such that  $\pi_j \pi_{j,-} = 0$ . Set  $\pi_{j,k} = \prod_{l \neq k} (X_2 - \lambda_{j,l})$  and  $\epsilon^{(j,k)} = \pi_j \pi_{j,k}$ . Then  $X_2 \epsilon^{(j,k)} = \lambda_{j,k} \epsilon^{(j,k)}$ . For each  $\epsilon^{(j,k)}$  there exists a polynomial  $\pi_{j,k,-}(X_3)$  such that  $\pi_j \pi_{j,k} \pi_{j,k,-} = 0$ , and so on. That is, the roots of such a polynomial are certain of the eigenvalues of the  $X_i$ s.

We can work out these *eigenvalues* of the  $X_i$ s by looking at the properties of generically irreducible representations as given in Section 1.3. Let  $\mu \in \Lambda_n^d$ . A ‘standard’ insertion of  $\underline{n} = \{1, 2, \dots, n\}$  into the boxes of  $\mu$  is one such that deletion of the boxes containing  $\{l, \dots, n\}$  produces a legitimate  $d$ -partition for every  $l$ . Let  $T_\mu$  denote the set of all such standard insertions. For  $i \in \underline{n}$  and  $w \in T_\mu$  there is a  $k \in \{1, 2, \dots, d\}$  such that  $i$  appears in a box in the  $k$ th partition in  $w$ . Define  $w(i) = k$ . It will be evident from the restriction rule (9) that  $T_\mu$  may be used as a basis for  $\Delta_\mu$ , once it has been equipped with a suitable action.

**PROPOSITION 3.2.** *The action of the  $X_i$ s on this basis may be taken to be lower triangular, with diagonal elements of  $X_i$  on  $w \in T_\mu$  given by  $\lambda_{w(i)} q^{2w^i}$ , where  $w^i$  is the distance of  $i$  off the main diagonal in the  $w(i)$ th partition in  $\mu$  (with distances below the diagonal being negative).*

*Proof.* In the case where  $n = 1$  the claim holds, since  $R_i(X) = \lambda_i$ . Suppose that the claim is true at level  $n - 1$ . Then the eigenvalues for  $X_1, \dots, X_{n-1}$  at level  $n$  are given by restriction, using the rule (9). The eigenvalues for  $X_n$  may be determined using equation (10) (in the one-dimensional cases), and then the centrality of  $\sum_i X_i$ , and so on, together with Schur’s lemma. □

Let  $v_i$  denote the  $i$ th elementary vector in  $\mathbb{Z}^d$ . Describe a walk on  $\mathbb{Z}^d$  (or  $\mathbb{Z}^d / (1, 1, \dots, 1)$ ) by a word  $w = w_1 w_2 \dots$  in  $\underline{d}$  such that the vector between the  $i$ th and  $(i + 1)$ th site visited is  $v_{w_i}$ . For a given word  $w$ , define

$$\#_l(i) = \#_l^w(i) = \sum_{j=1}^l \delta_{w_j, i}.$$

A reflection hyperplane  $(i, j; x)$  is characterised by a pair  $v_i, v_j$  ( $i \neq j \in \underline{d}$ ) not parallel to it, and the *signed* distance  $x$  in the direction of  $v_i$  of this hyperplane from 0 (that is, the  $x$  such that  $0 + x v_i$  lies on it). A walk touches this hyperplane at  $l$  if

$$\#_l(i) - \#_l(j) = x. \tag{20}$$

Let  $w$  be a walk that touches hyperplane  $(i, j; x)$  at  $l$ . The walk  $w'$  obtained from  $w$  by applying permutation  $(ij)$  to every  $w_t, t > l$ , is called the (*affine*) reflection of  $w$  in  $(i, j; x)$  at  $l$ . (Note that the touching point  $l$  is not in general uniquely defined by  $w$  and  $(i, j; x)$  or, indeed, existent.) Every point of the reflection after  $l$  is the reflection of this point in  $(i, j; x)$  in the usual alcove geometry sense (see [50, Chapter 7]).

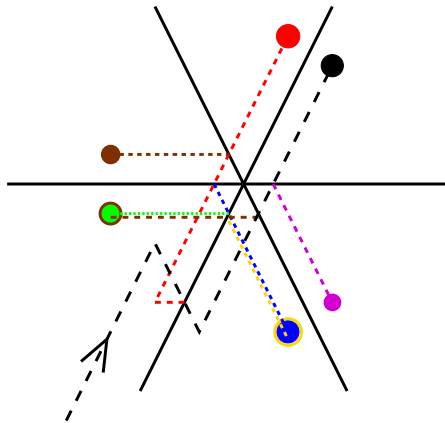


Figure 2: Eight walks in a walk orbit for  $d = 3$  (origin in the bottom left-hand corner, endpoints circled).

If  $w'$  meets another (not necessarily distinct) hyperplane  $(k, m; y)$  at  $l' > l$ , then of course  $w$  meets the image of  $(k, m; y)$  in  $(i, j; x)$  at the same moment. If  $w''$  is the reflection of  $w'$  in  $(k, m; y)$  at  $l'$ , then we say that  $w, w'$  and  $w''$  are in the same *walk orbit* of the reflection group generated by these hyperplanes (the  $i$ th points of these walks are in the same orbit in the usual sense, for each  $i$ ). We may think of folding up the space along the set of hyperplanes in the group – the walk orbits are the sets of walks that are mapped into each other by this folding. For  $G$  a reflection group generated by hyperplanes, we write  $w \sim^G w'$  if walks  $w$  and  $w'$  are in the same walk orbit of  $G$ . Each hyperplane partitions space into two parts (not counting the hyperplane itself). The ‘outside’ of the hyperplane  $(i, j; x), x \neq 0$ , is the part not containing 0.

We shall restrict our attention to hyperplanes *not* touching 0. Note that for each walk  $w$  from 0 that finishes at a point  $\mu$  outside some hyperplane, there is not in general a unique walk in its orbit that finishes at the image point of  $\mu$ , inside it. The orbit structure of walks is more complicated than that of points. For example, the endpoints of elements of the same walk orbit are necessarily in the same point orbit; but the converse does not follow. Further, a walk that stays in the interior of the fundamental alcove is in a singleton orbit; more generally, a walk that touches hyperplanes a total of  $t$  times has  $2^t$  elements in its orbit (see Figures 2 and 3).

For  $G$  a reflection group generated by a set of reflection hyperplanes  $S$ , let  $\bar{S} \subset G$  denote the set of simple reflections. It will be convenient to confuse these reflections with the corresponding hyperplanes, noting that  $\bar{S} \supset S$  in general. In particular, we may now compose hyperplanes by conjugation in  $G$ ; that is,  $(i, j; x) \circ (j, k; y) = (i, k; x + y)$ , and so on. Let  $\mathcal{A}' \subset \mathcal{A}$  consist of those elements of the form  $(\lambda_i - q^{-2x} \lambda_j)$ , for all  $i, j, x$ . Define the *factor set* of  $G, \mathcal{F}(G) \subset \mathcal{A}'$  such that  $(i, j; x) \in \bar{S}$  if and only if  $(\lambda_i - q^{-2x} \lambda_j) \in \mathcal{F}(G)$ . The elements of  $\mathcal{F}(G)$  are called *factors*. Note that there is an equality of ideals

$$\sum_{(i,j;x) \in \bar{S}} \mathcal{A}(\lambda_i - q^{-2x} \lambda_j) = \sum_{(i,j;x) \in \bar{S}} \mathcal{A}(\lambda_i - q^{-2x} \lambda_j) \tag{21}$$

since the composition of hyperplanes above corresponds to the addition (up to a unit) of factors.

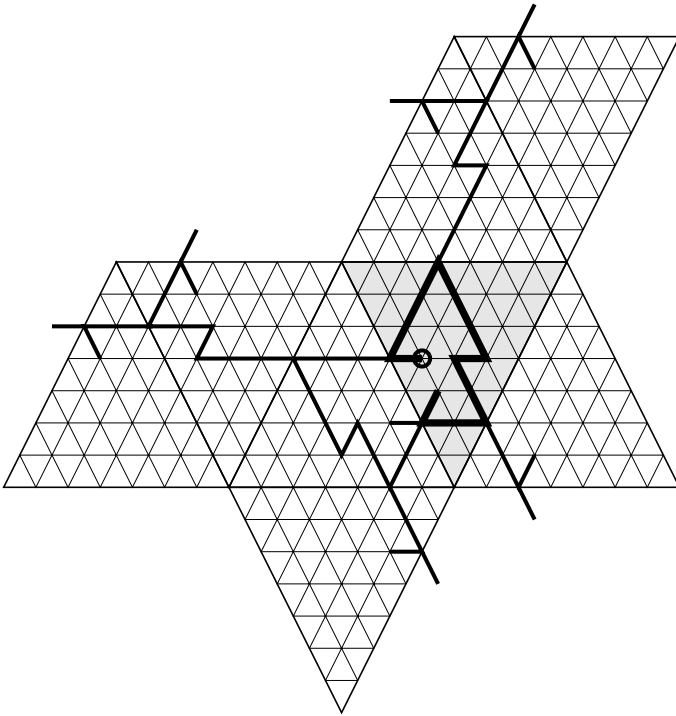


Figure 3: A walk, starting at 0 (circled), in the fundamental alcove (shaded) of a certain  $d = 3$  affine reflection group; and all walks in its walk orbit. Note that each of the two endpoints closest to 0 is reached by two different walks.

For a walk  $w$  of length  $n$ , let  $\mu(w) = (\#_n(1), \#_n(2), \dots)$ , an ordered partition of  $n$ . To each walk  $w$  of length  $n$ , associate an element  $\lambda^w \in \mathcal{A}^n$  as follows:

$$(\lambda^w)_i = \lambda_{w_i} q^{2(\#_i(w_i)-1)}.$$

For example,  $\lambda^{3331312} = (\lambda_3, \lambda_3 q^2, \lambda_3 q^4, \lambda_1, \lambda_3 q^6, \lambda_1 q^2, \lambda_2)$ .

Note that this  $\lambda^-$  gives an injective map from  $\bigcup_{\mu \in \Lambda^{\mathcal{D}(n)}} T_\mu$  into  $\mathcal{A}^n$ .

**PROPOSITION 3.3.** (i) *Let  $w'$  be the reflection of  $w$  in  $(i, j; x)$  at  $l$ ; then every non-zero element of  $\lambda^w - \lambda^{w'}$  is of the form  $\pm q^{2\alpha}(\lambda_i - q^{-2x}\lambda_j)$ , for some  $\alpha \in \mathbb{Z}$ . (Note that there is no mention of  $l$  in the implication.)*

(ii) *Let  $w$  and  $w'$  be two walks. If every element of  $\lambda^w - \lambda^{w'}$  is divisible by  $(\lambda_i - q^{-2x}\lambda_j)$ , then  $w$  and  $w'$  are related by one (or more) reflections in  $(i, j; x)$  for some  $l$  (respectively,  $l < l' < l'' < \dots$ ).*

(iii) *Let  $G$  be the group generated by a set of simple reflections  $\{(i, j; x), (k, l; y), \dots\}$ . Walks  $w$  and  $w'$  are in the same walk orbit of  $G$  (that is, they are related by a sequence of reflections in hyperplanes in  $G$ ) if and only if each non-zero  $(\lambda^w - \lambda^{w'})_i$  is divisible by an element of  $\mathcal{F}(G)$ .*

*Proof.* (i) Consider the first point at which  $w$  and  $w'$  differ, which we may take to be  $w'_{l+1} = (ij)w_{l+1} = j$ .

We have

$$(\lambda^w - \lambda^{w'})_{l+1} = \lambda_i q^{2(\#_{l+1}^w(i)-1)} - \lambda_j q^{2(\#_{l+1}^{w'}(j)-1)}.$$

Now  $\#_{l+1}^w(i) = \#_l^w(i) + 1$  and  $\#_{l+1}^{w'}(j) = \#_l^{w'}(j) + 1$ , so (see equation (20)) the exponents differ by  $2x$ , as required. Since subsequent points  $k$  with  $w_k \neq w'_k$  are *all* those at which  $\{w_k, w'_k\} \cap \{i, j\} \neq \emptyset$ , the *difference* in exponents is preserved.

(ii) This is a special case of (iii).

(iii) ‘*Only if*’ part: Note that, by construction,  $(\lambda^w - \lambda^{w'})_i$  is, up to a unit, an element of  $\mathcal{A}'$ . Each pair  $w \sim^G w'$  may be related by a series of simple reflections:

$$w \sim^G w'' \sim^G \dots \sim^G w',$$

and each intermediate  $(\lambda^{w^i} - \lambda^{w^{i+1}})_i$  is of the required form, by part (i). Evidently,  $(\lambda^w - \lambda^{w'})_i = \sum_i (\lambda^{w^i} - \lambda^{w^{i+1}})_i$ . Now confer equation (21).

‘*If*’ part: Let  $w|_m$  denote the first  $m$  steps in walk  $w$ . Fix  $w$ , and consider  $w'$  such that  $w \not\sim^G w'$ . Suppose that point  $m$  is the first at which  $w|_m \not\sim^G w'|_m$ . Let  $\tilde{w}|_{m-1} = w'|_{m-1}$  and let  $\tilde{w}_m$  be such that  $w|_m \sim^G \tilde{w}|_m$ . Thus  $\tilde{w}|_m$  is a reflection of  $w'|_m$  in some hyperplane  $h$  at  $m - 1$ , with  $h \notin G$ . Thus  $\lambda^{w'} - \lambda^{\tilde{w}} = (0, 0, \dots, 0, (q^{2\alpha}(\lambda_i - q^{-2x}\lambda_j)))$  for some  $\alpha, i, j, x$ , where  $i \neq j$ , since  $w'$  and  $\tilde{w}$  agree at  $m - 1$ . The non-zero factor is not a factor of  $\mathcal{F}(G)$ , since  $h \notin G$ . Now consider  $\lambda^w - \lambda^{w'} = (\lambda^w - \lambda^{\tilde{w}}) + (\lambda^{\tilde{w}} - \lambda^{w'})$ . The  $m$ th element of the first summand on the right is divisible by some  $f \in \mathcal{F}(G)$ , by the *only if* part, while that of the second summand is not, by the above argument. Thus the left-hand side is also not divisible.  $\square$

For example, consider the sequences 333, 331 and 321 with  $\lambda^{333} = (\lambda_3, q^2\lambda_3, q^4\lambda_3)$ ,  $\lambda^{331} = (\lambda_3, q^2\lambda_3, \lambda_1)$  and  $\lambda^{321} = (\lambda_3, \lambda_2, \lambda_1)$ . We have

$$(3, 1; 2)333 = 331;$$

$$(3, 2; 1)331 = 321;$$

$$\lambda^{333} - \lambda^{321} = (0, q^2(\lambda_3 - q^{-2}\lambda_2), q^4(\lambda_3 - q^{-4}\lambda_1)).$$

**PROPOSITION 3.4.** *For each  $w \in \bigcup_{\mu \in \Lambda^{\mathcal{D}(n)}} T_\mu$  there is an element  $\epsilon_w$  of  $\langle X_i \rangle$  as described above, obeying the requirement that*

$$X_i \epsilon_w = (\lambda^w)_i \epsilon_w.$$

*Proof.* For a  $d$ -partition  $\mu$  of form  $((\mu_1), (\mu_2), \dots)$  (that is,  $\mu \in \Lambda^{\mathcal{D}}$ ) and  $w \in T_\mu$  identified with the corresponding word  $w$ , we have  $w_i = w(i)$  and  $\#_i(w_i) = w^i$ . Now see Proposition 3.2.  $\square$

Let  $R_x$  denote the representation of  $\langle X_i \rangle$  corresponding to any  $\epsilon_x$  as above. It follows that the  $\langle X_i \rangle$ -simple character of  $\text{Res}_{\langle X_i \rangle}^H(\Delta_v)$  is

$$[\text{Res}_{\langle X_i \rangle}^H(\Delta_v) : R_w] = \delta_{\mu(w), v}. \tag{22}$$

Note that this is not a *unique* characterisation unless the following corollary holds.

**COROLLARY 3.5.** *If  $k$  is generic (that is,  $\lambda^-$  remains injective on passing from  $\mathcal{A}^n$  to  $(\mathcal{A} \otimes k)^n$ ), then  $R_w \cong R_{w'}$  implies that  $w = w'$ , and  $\langle X_i \rangle^{\mathcal{D}} \subset H^{\mathcal{D}}$  has enough simples to be semisimple.*



DEFINITION 3.6. We define  $\mathcal{G}(k)$ , the reflection group induced by  $k$ , as follows. For each triple  $(i, j; x)$  such that  $(\lambda_i - q^{2x}\lambda_j) \otimes k = 0$ , include the hyperplane  $(i, j; x)$  as a generator.

Note that this is sensible inasmuch as  $(i, j; x) \circ (j, k; y) = (i, k; x + y)$ , while  $\lambda_i - q^{2x}\lambda_j = 0$  and  $\lambda_j - q^{2y}\lambda_k = 0$  imply that  $\lambda_i - q^{2(x+y)}\lambda_k = 0$ . If  $d'$  of the parameters  $\{\lambda_i\}$  are related in this way, then the group generated is  $S_{d'}$ , unless  $q$  is a root of unity – in which case, it is the affine extension. We shall usually refer to the group as the affine Weyl group regardless, to emphasize the fact that even in the finite case, the hyperplanes do not pass through 0. That is, we exclude from consideration any  $k$  in which  $\lambda_i = \lambda_j$  does not imply that  $i = j$ .

PROPOSITION 3.7. Over  $k$ ,  $R_w \cong R_{w'}$  if and only if  $w$  and  $w'$  are in the same walk orbit of the affine Weyl group induced by  $k$  (that is, they are related by some series of reflections).

Proof. There is an isomorphism if and only if  $\lambda^w - \lambda^{w'}$  vanishes over  $k$ . Every term is of the form  $q^\alpha(\lambda_i - q^{2x}\lambda_j)$ . Now apply Proposition 3.3. □

PROPOSITION 3.8 (LINKAGE). If there exists a nontrivial homomorphism  $\Delta_\mu \rightarrow \Delta_\nu$  over  $k$ , then  $\mu$  and  $\nu$  lie in the same orbit of the affine Weyl group induced by  $k$ .

Proof. Consider the following diagram.

$$\begin{array}{ccc} \Delta_\mu & \longrightarrow & \Delta_\nu \\ \downarrow \text{Res} & & \downarrow \text{Res} \\ \sum_{w \in T_\mu} R_w & \longrightarrow & \sum_{w \in T_\nu} R_w \end{array}$$

Now apply Proposition 3.7. □

Note that this strengthens immediately to exclude interorbit maps from any submodule of  $\Delta_\mu$  to any quotient of  $\Delta_\nu$ ; that is,  $\mu \not\sim^{\mathcal{G}(k)} \nu$  implies that  $\Delta_\mu$  and  $\Delta_\nu$  have no composition factors in common. Thus, under the assumption that, as for a quasi-hereditary algebra, every simple occurs in  $\{\text{Head}(\Delta_\mu) \mid \mu \in \Lambda^{\mathcal{D}}\}$ , we have linkage in the form of I6.

PROPOSITION 3.9. Let  $w, w' \in \bigcup_{\mu \in \Lambda^{\mathcal{D}(n)}} T_\mu$  be reflections of each other in  $(i, j; x)$  at any  $l$ . Then over any  $k$  in which  $(\lambda_i - q^{2x}\lambda_j)$  vanishes, we have an isomorphism of left  $H$ -modules

$$H\epsilon_w \cong H\epsilon_{w'}.$$

Proof. By Proposition 3.7,  $\epsilon_w$  and  $\epsilon_{w'}$  induce isomorphic  $\langle X_i \rangle$ -modules. □

Given the injectivity of  $\lambda^-$ , it will be convenient to be able to refer to a walk  $w$  either directly or via its image  $\lambda^w$ . This unifies the labelling schemes for  $\epsilon_w$  following Definition 3.1 and in Proposition 3.4.

Note that  $R_x(\pi_j) = 0$  for all  $R_x$  except those with  $x_1 = \lambda_j$ . Let  $S_j$  be the set of possible values of  $R_x(X_2)$  when  $x_1 = \lambda_j$  (that is,  $S_j = \{\lambda_k \neq \lambda_j, q^2\lambda_j, q^{-2}\lambda_j\}$ ). Then  $R_{(\lambda_j, x_2, \dots)}(\prod_{s \in S_j} (X_2 - s)) = 0$  for any such  $x$ , and  $\pi_j \prod_{s \in S_j} (X_2 - s)$  lies in the radical of  $\langle X_i \rangle$ , and hence some power of it vanishes. This tells us, up to multiplicity, the roots of  $\pi_{j,-}$ , and hence of the preidempotent  $\pi_j \pi_{j,k}$  (obtained by omitting a factor  $(X_2 - \lambda_{j,k})$ ,  $\lambda_{j,k} \in S_j$ , from  $\pi_j \pi_{j,-}$ ). If  $\lambda^-$  is injective, then the radical is  $\{0\}$  and all the roots can be distinguished. Iterating this argument, we have the following proposition.

PROPOSITION 3.10. For  $w$  a walk,

$$\epsilon^{\cdot w} = \prod_{i=1}^{|w|} \prod_{\substack{x \in \cup_{v, |v|=i} T_v \\ \text{such that} \\ x|_{i-1} = w|_{i-1}, \\ x_i \neq w_i}} (X_i - R_x(X_i))$$

where the union is over all multipartitions satisfying the constraints.

For example,  $\epsilon^{\cdot 111} = \epsilon^{\cdot 11}(X_3 - q^{-2}\lambda_1) \prod_{i=2}^d (X_3 - \lambda_i)$ , and

$$\begin{aligned} \epsilon^{\cdot 11122} = \epsilon^{\cdot 111} & \left( (X_4 - q^{-2}\lambda_1)(X_4 - q^6\lambda_1) \prod_{2 < i \leq d} (X_4 - \lambda_i) \right) \\ & \cdot \left( (X_5 - q^{-2}\lambda_1)(X_5 - q^6\lambda_1)(X_5 - q^{-2}\lambda_2) \prod_{2 < i \leq d} (X_5 - \lambda_i) \right). \end{aligned}$$

Note that unless the omitted factor in  $\pi_j$  or  $\pi_{j,k}$  coincides (under  $X_1 \leftrightarrow X_2$ ) with a factor in the other, then the product  $\pi_j \pi_{j,k}$  is automatically in  $Z(H(2, d))$ . (Simply multiply in all the factors apparently required for symmetry, and then replace these with scalars using the eigenvector property – the excluded cases are those where one or more such scalars vanish.)

For example, in the case  $n = 2, d = 2$ , we see that  $\langle X_i \rangle$  has rank 6. We have

$$(X - \lambda_2)(X_2 - q^{-2}\lambda_1)(X_2 - q^2\lambda_1)(X_2 - \lambda_2) = 0;$$

so, for example,

$$\epsilon^{\cdot 11} = \epsilon^{\cdot \lambda_1, q^2\lambda_1} = (X - \lambda_2)(X_2 - q^{-2}\lambda_1)(X_2 - \lambda_2)$$

is a preidempotent with  $X_2 \epsilon^{\cdot \lambda_1, q^2\lambda_1} = q^2\lambda_1 \epsilon^{\cdot \lambda_1, q^2\lambda_1}$ , while

$$\epsilon^{\cdot \lambda_1\lambda_2} = (X - \lambda_2)(X_2 - q^{-2}\lambda_1)(X_2 - q^2\lambda_1)$$

is a preidempotent with  $X_i \epsilon^{\cdot \lambda_1\lambda_2} = \lambda_i \epsilon^{\cdot \lambda_1\lambda_2}$ . The preidempotent  $\epsilon^{\cdot \lambda_1, q^2\lambda_1}$  lies in  $Z(H)$  (provided that  $q^2 \neq 1$ ), since it can be symmetrized:

$$(X_1 - q^{-2}\lambda_1) \epsilon^{\cdot \lambda_1, q^2\lambda_1} = (\lambda_1 - q^{-2}\lambda_1) \epsilon^{\cdot \lambda_1, q^2\lambda_1}.$$

On the other hand,  $\epsilon^{\cdot \lambda_1\lambda_2}$  cannot be rescaled to its symmetrized form because the symmetrizing factor  $X_2 - \lambda_2$  kills it.

For  $H(2, d)$  we have

$$\prod_{j \neq 1} (X - \lambda_j)(X_2 - q^2\lambda_1)(X_2 - q^{-2}\lambda_1) \prod_{j \neq 1} (X_2 - \lambda_j) = 0,$$

so similar considerations apply. Indeed, they do for all  $n$ . In particular,  $\epsilon^{\cdot 11\dots 1} \in Z(H)$  unless  $q^2 = 1$ . More generally, we may proceed as follows.

For  $\mu$  an ordered partition of  $n$ , let  $H_\mu$  denote the corresponding Young subalgebra of  $H_n \subset H(n, d)$ . For  $w$  a walk, let  $e_w \in H_n$  denote the  $q$ -Young symmetrizer (see [14] and [26, Section 9.3]) associated to  $H_{\mu(w)}$ . A walk  $w$  is said to be sorted if it takes the form  $111 \dots 22 \dots$  (more precisely, if  $w_i \leq w_{i+1}$  for all consecutive pairs of steps in  $w$ ). There is a unique sorted walk in each  $T_\mu$ , denoted  $w(\mu)$ . A walk  $w$  is said to be direct if all the steps in a given direction are taken consecutively (thus a sorted walk is direct).

**PROPOSITION 3.11.** For  $w$  sorted, and  $(1 + q^2)$  and each  $\lambda_i$  invertible,  $\epsilon_w$  commutes with  $H_{\mu(w)}$ .

*Proof.* Let  $\mu^t$  denote the  $t$ th interval of  $\{1, 2, \dots, n\}$  in the ordered partition  $\mu(w)$ ; that is, the  $t$ th set of integers fixed under the action of the Young subgroup  $S_{\mu(w)}$  of  $S_n$  on  $\{1, 2, \dots, n\}$ . We require to show, for each  $t$ , that the factors in  $\epsilon_w$  involving  $X_{\mu^t} := \{X_i \mid i \in \mu^t\}$  constitute a symmetric polynomial in these variables, and hence commute with the  $t$ th factor algebra in  $H_{\mu(w)}$ . (The remaining factor algebras commute with these variables by equation (17).) Our walk is of the form  $w = 111 \dots 22 \dots tt \dots$ , and the factors in question are (by Proposition 3.10) those written out explicitly in:

$$\epsilon_w = \epsilon_{111 \dots 22 \dots} \prod_{i=t+1}^d (X_a - \lambda_i) \left( \prod_{i=1}^{t-1} (X_a - q^{-2}\lambda_i)(X_a - q^{2\mu_i(w)}\lambda_i) \right) \prod_{b \in \mu^t \setminus \{a\}} \left( (X_b - q^{-2}\lambda_t) \prod_{i=t+1}^d (X_b - \lambda_i) \left( \prod_{i=1}^{t-1} (X_b - q^{-2}\lambda_i)(X_b - q^{2\mu_i(w)}\lambda_i) \right) \right) \dots$$

(where  $X_a$  is the first  $X_i$  in  $X_{\mu^t}$ ). It will be apparent that this is rendered symmetric by multiplying by  $(X_a - q^{-2}\lambda_t)$ , but  $(X_a - q^{-2}\lambda_t)\epsilon_w = (\lambda_t - q^{-2}\lambda_t)\epsilon_w$  so it is *already* symmetric, provided that  $(1 - q^{-2})\lambda_t$  is invertible.  $\square$

A similar property holds for direct walks.

**PROPOSITION 3.12.** If  $w$  takes the form  $111 \dots 22 \dots$ , then  $\epsilon_w e_w = e_w \epsilon_w$  (and similarly for the preidempotent forms  $\epsilon$  and  $\mathbf{e}$ ). The modules  $\Delta_{\mu(w)} = \mathbf{He}_w \epsilon_w$  are the left standard modules of  $\mathbf{H}$  with these weights.

*Proof.* Note that  $e_w \in H_{\mu(w)}$ , and apply Proposition 3.11.  $\square$

Write  $\mu \geq \nu$  if every change of direction in  $w(\mu)$  occurs at the same step as one in  $w(\nu)$ . (Note that  $w(\nu)$  may have changes at other points as well.) Note also that  $\mu \geq \nu$  implies that  $e_{w(\mu)} e_{w(\nu)} = e_{w(\mu)}$ , and hence  $\mathbf{e}_{w(\mu)} \mathbf{e}_{w(\nu)} = \kappa_\nu \mathbf{e}_{w(\mu)}$  for some  $\kappa_\nu \in \mathcal{A}$ .

**PROPOSITION 3.13.** Suppose that  $w(\mu) \sim^{\mathfrak{g}(k)} w(\nu)$ , and  $\mu \geq \nu$ , and  $\kappa_\nu \otimes k \neq 0$ . Then  $\Delta_\mu \hookrightarrow \Delta_\nu$ .

*Proof.*

$$\Delta_\mu = \mathbf{He}_{w(\mu)} \epsilon_{w(\mu)} = \mathbf{He}_{w(\mu)} \mathbf{e}_{w(\nu)} \epsilon_{w(\mu)} \hookrightarrow \mathbf{He}_{w(\nu)} \epsilon_{w(\mu)} \cong \mathbf{He}_{w(\nu)} \epsilon_{w(\nu)} = \Delta_\nu. \quad \square$$

#### 4. The case $d = 2$ and the blob algebra

##### 4.1. Idempotents in $\mathbf{H}(n, 2)$

The primitive and central idempotents  $e_n^{\pm 1}$  corresponding to the four one-dimensional representations of  $\mathbf{H}(n, 2)$  over  $\mathcal{K}$  may be constructed as follows.

Fixing  $d = 2$ , define

$$P_n = P_n^{+2} = q^{n-1} [n] (q^{2n-2} \lambda_2 - \lambda_1);$$

also,  $P_n^{-2} = t P_n$ ,  $P_n^{+1} = s P_n$  and  $P_n^{-1} = st P_n$ .

PROPOSITION 4.1. *Set*

$$\alpha_j = \frac{-\lambda_1}{P_j}, \quad \beta_j = \frac{qP_{j-1}}{P_j} \quad \text{and} \quad \gamma_j = \frac{q^{2j-2}}{P_j}. \tag{23}$$

Then  $e_0^{+2} = 1$  and

$$e_{j+1}^{+2} = e_j^{+2} (\alpha_{j+1} + \beta_{j+1}g_j + \gamma_{j+1}X_{j+1}) e_j^{+2}. \tag{24}$$

Also,  $e_j^{-2} = te_j^{+2}$ ,  $e_j^{+1} = se_j^{+2}$ , and  $e_j^{-1} = tse_j^{+2}$ .

*Proof.* See Section 5.1 (or simply consider  $g_{j-1}e_j^{+2}$ ). □

*Examples.*

$$\begin{aligned} e_1^{-2} &= e_1^{+2} = \frac{X_1 - \lambda_1}{\lambda_2 - \lambda_1}; \\ e_2^{-2} &= \left( \frac{X - \lambda_1}{\lambda_2 - \lambda_1} \right) \frac{-\lambda_1 + q^{-2}g_1Xg_1 - q^{-1}(\lambda_2 - \lambda_1)g_1}{(1 + q^{-2})(q^{-2}\lambda_2 - \lambda_1)} \left( \frac{X - \lambda_1}{\lambda_2 - \lambda_1} \right). \end{aligned} \tag{25}$$

Define

$$\Sigma = (X_1 + X_2 - (\lambda_1 + \lambda_2)) \quad \text{and} \quad \Pi = (X_1X_2 - \lambda_1\lambda_2).$$

Note that  $\mathfrak{D}_2^2 = \{1, \Sigma, \Pi, \Sigma g_1, \Pi g_1\}$  is a basis of  $Z(H(2, 2))$  (see Appendix A, equation (56)).

In terms of  $\mathfrak{C}_2^2$  and  $\mathfrak{D}_2^2$ , we have

$$\begin{aligned} e_2^{-2} &= \frac{\lambda_1^2 - \lambda_1(X + X_2) + XX_2 + q^{-1}\lambda_1(X + X_2 - (\lambda_1 + \lambda_2))g_1 - q^{-1}(XX_2 - \lambda_1\lambda_2)g_1}{(\lambda_1 - \lambda_2)(1 + q^{-2})(q^{-2}\lambda_2 - \lambda_1)} \\ &= \frac{(-\lambda_1\Sigma + \Pi)(1 - q^{-1}g_1)}{(\lambda_2 - \lambda_1)(q^{-2}\lambda_2 - \lambda_1)(1 + q^{-2})}, \end{aligned} \tag{26}$$

which form manifests the centrality of this idempotent. Note that the preidempotent

$$\mathbf{e}_2^{-2} = (-\lambda_1\Sigma + \Pi)(1 - q^{-1}g_1)$$

coincides with its  $s$ -image (up to a unit in  $\mathcal{A}$ ) if and only if  $\lambda_1 = \lambda_2$  in  $k$ . However,

$$e_2^{-2} + e_2^{-1} = \frac{q^{-2}\Sigma + (1 + q^{-2})\Pi}{(q^{-2}\lambda_2 - \lambda_1)(q^{-2}\lambda_1 - \lambda_2)} \frac{(q - g_1)}{[2]}.$$

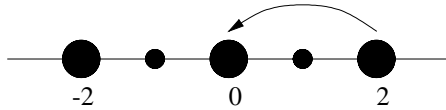
A remark is in order on denominators and idempotent decompositions of 1. The idempotent decomposition of  $1 = e_1^{+2} + e_1^{+1} \in H(1, 2)$  is not defined in the case where  $\lambda_1 = \lambda_2$  over  $k$ , and the radical  $\text{rad}(H(1, 2)) = ke_1^{+1}$ . Obviously,  $R_{\pm 1} = R_{\pm 2}$  for any  $n$  in this case. The decomposition of

$$1 = (e_1^{+2} + e_1^{+1}) + e_{((1),(1))} + (e_1^{-2} + e_1^{-1}) \in H(2, 2)$$

thus has the same limitation: the bracketed sums do not split. Similarly, when  $q = -q^{-1}$ , we have  $R_{+i} = R_{-i}$  (for any  $n$ ). More interesting is the case where  $\lambda_2 = q^2\lambda_1$ . Here, both  $e_2^{-2}$  and  $e_2^{+1}$  are undefined. Clearly,  $R_{+1} \neq R_{-2}$  (unless  $q^2 = -1$ ,  $\lambda_i = 0$ ), so  $e_2^{-2} + e_2^{+1}$  is also undefined, but  $e_2^{+2}$  and  $e_2^{-1}$  are well defined, so

$$e_2^{-2} + e_2^{+1} + e_{((1),(1))}$$

does not split and  $R_{+1}$  and  $R_{-2}$  (that is,  $\Delta_{((2),)}$  and  $\Delta_{(,(1^2))}$ ) must both be composition factors of  $\Delta_{((1),(1))}$ . Note that at first sight this seems problematic for our proposed  $H^{\mathcal{D}}$  weight space; however, over this  $k$  we may identify  $R_{-2}$  with the simple head of  $\Delta_{((1),(1))}$  and label it accordingly (rather than by its  $H(n, d)$  label, which is  $(, (1^2))$ ). This is a good paradigm for the subtleties with labels in realising ingredient I2(ii) (confer [59]). In terms of the  $A_1$  integral weight set  $\mathbb{Z} \subset \mathbb{R}$  (recall that  $\mathfrak{P}^2((, (2))) = -2$ ,  $\mathfrak{P}^2(((1), (1))) = 0$  and  $\mathfrak{P}^2(((2), )) = 2$ ), we depict the standard modules and morphism in the following manner.



It will be convenient to note the equality

$$(X - \lambda_2)(-\lambda_1 \Sigma + \Pi) = 0 \tag{27}$$

and its  $s$  image; and hence that

$$(-\lambda_1 \Sigma + \Pi)e_2^{-2} = (-\lambda_1 \Sigma + \Pi) \frac{g_1 - q}{-q^{-1} - q} \tag{28}$$

and its  $s$  image.

Now consider the algebras  $H^{\mathcal{D}}(n, 2)$  obtained by quotienting by  $\mathbf{e}_2^{-1} = 0$  and  $\mathbf{e}_2^{-2} = 0$ .

First consider  $n = 2$ . A basis of the ideal  $\mathcal{D}_2(2)$  is simply  $\{\mathbf{e}_2^{-1}, \mathbf{e}_2^{-2}\}$ , for  $\lambda_1 \neq \lambda_2$ . The basis  $\mathcal{E}_2^{\mathcal{D}}$  of  $H(2, 2)$  is

$$\{1, X_1, X_2, X_1 X_2, g_1, X_1 g_1, X_2 g_1, X_1 X_2 g_1\}.$$

We can think of using the relation  $e_2^{-2} = 0$  to eliminate  $X_1 X_2 g_1$ , and then using its  $s$ -partner to eliminate  $g_1$ , to obtain a basis for  $H^{\mathcal{D}}(2, 2)$  (see also Section 5.3). Specifically, it is convenient to represent the quotient relations as

$$\begin{aligned} (X_1 + X_2 - (\lambda_1 + \lambda_2))(g_1 - q) &= 0; \\ (X_1 X_2 - \lambda_1 \lambda_2)(g_1 - q) &= 0. \end{aligned} \tag{29}$$

Note that these two relations generate the *same* ideal.

Alternatively, rewriting  $g_i - q =: u_i$  (so  $u_i^2 = -(q + q^{-1})u_i$ ) and  $X - \lambda_i =: v$ , we have

$$v u_1 v u_1 = (\lambda_1 q^{-1} - \lambda_2 q) v u_1 \quad \text{and} \quad u_1 v u_1 v = (\lambda_1 q^{-1} - \lambda_2 q) u_1 v, \tag{30}$$

and similarly for the image under  $s$ . Combining (or by applying  $(g_1 - q)$  to the first equation in (29) from the left), we obtain

$$u_1 v u_1 = (\lambda_1 q^{-1} - \lambda_2 q) u_1 \tag{31}$$

(and its  $s$  image).

#### 4.2. $H^{\mathcal{D}}(n, 2)$ and the blob algebra

The blob algebra  $b_n$  [55] may be defined as in Section 1. (That the presentation in Section 1 is equivalent to the usual definition in terms of a basis of decorated Temperley–Lieb diagrams may be verified by a straightforward (if tedious) generalisation of the corresponding exercise for the ordinary Temperley–Lieb algebra as in [50].) The properties I1–I6 are established for the blob algebra in [57].

In the parametrisation of  $b_n$  by  $q$  and  $m$ , where  $y_e = -[m - 1]/[m]$ , it is convenient to replace  $e_-$  by the rescaled generator  $U_0 = -[m]e_-$ , or rather to replace  $e_-$  with  $U_0$ , with relations  $U_0^2 = -[m]U_0$  and

$$U_1U_0U_1 = [m - 1]U_1. \tag{32}$$

The variant with these relations is isomorphic to the original, except (obviously) when  $[m] = 0$ .

**PROPOSITION 4.2.** *Let  $m, q, \lambda_1$  and  $\lambda_2$  be such that*

$$[m - 1](\lambda_1 - \lambda_2) = (\lambda_1q^{-1} - \lambda_2q)[m].$$

(Note that  $\lambda_1 \neq \lambda_2$  unless  $q^2 = 1$  and  $m \rightarrow \infty$ , or  $[m] = 0$ ; otherwise, putting  $\lambda_1/\lambda_2 = q^{-2r}$  and  $m = r$ , we may take  $\lambda_1 = q^m/(q - q^{-1})$ , and then  $\lambda_1 - \lambda_2 = [m]$ ). Then there is an isomorphism

$$\phi : H^{\mathcal{D}}(n, 2) \rightarrow b_n$$

given by  $\phi(u_i) = U_i$  and  $\phi(v) = U_0$ . There is a homomorphism  $\psi : b_n \rightarrow H^{\mathcal{D}}(n, 2)$ , given by  $\psi(U_i) = u_i, \psi(U_0) = v$ .

*Proof.* It is straightforward to verify the cyclotomic Hecke relations under  $\phi$ . A direct calculation also shows that the image under  $\phi$  of the numerator of  $e_2^{-2}$ , as in equation (25), vanishes identically if we use the form of  $\lambda_1$  given in the proposition. The image of  $e_2^{-1}$  vanishes similarly. Thus  $\phi$  is a surjective homomorphism (except possibly when  $q^2 = 1$ ). A dimension count reveals that this surjective map is generically an isomorphism.

For  $\psi$ , equation (31) verifies relation (32). It is now enough to check that  $u_1u_2u_1 = u_1$  (that is, in  $H(3, 2)$ , that  $u_1u_2u_1 - u_1 = u_1(u_2u_1 - 1) = [3]!e_3^- \in \mathcal{D}_2$ ). Write  $f = u_1u_2u_1 - u_1$  and consider  $f\mathcal{D}_2f$ . (If  $f \in \mathcal{D}_2$ , then  $f \in f\mathcal{D}_2f$ , at least if  $[3]!$  is invertible, and this is a much smaller and simpler object to work with.) This (pre)idempotent subalgebra includes

$$\begin{aligned} f(X_1 + X_2 - (\lambda_1 + \lambda_2))f &= f((1 + q^{-2})X - (\lambda_1 + \lambda_2))f, \\ f(X_1X_2 - \lambda_1\lambda_2)f &= f(-q^{-1}Xg_1X - \lambda_1\lambda_2)f \end{aligned}$$

and

$$\begin{aligned} fXg_1g_2(X_1 + X_2 - (\lambda_1 + \lambda_2))f &= f(Xg_1g_2X + Xg_1g_2g_1Xg_1 - Xg_1g_2(\lambda_1 + \lambda_2))f \\ &= f(-q^{-1}Xg_1X + Xg_2g_1g_2Xg_1 - q^{-2}(\lambda_1 + \lambda_2)X)f \\ &= f(-(q^{-1} + q^{-3})Xg_1X - q^{-2}(\lambda_1 + \lambda_2)X)f. \end{aligned}$$

A linear combination of these is

$$[3]!(1 + q^{-4})\lambda_1\lambda_2 - q^{-2}(\lambda_1^2 + \lambda_2^2))f,$$

so  $f \in f\mathcal{D}_2f$ , at least in an open subset of the parameter space. The coefficient may be rewritten as  $[3]!q^{-2}\lambda_1\lambda_2(q - q^{-1})^2[m + 1][m - 1]$ , so this covers most of the interesting cases. The remainder require a more tedious calculation.  $\square$

**COROLLARY 4.3.** *The finite characteristic  $b_n$  decomposition matrices in [15] are a subset of type-B Hecke decomposition matrices, via the correspondence given in Section 1.3.*

Recall that  $m = \pm 1$  implies the existence of a quotient algebra isomorphic to the Temperley–Lieb algebra (in addition to the noted Temperley–Lieb subalgebra). The quotient identifies  $X = 1$  (or  $q^2$ ).

5. On general  $d$

5.1. Idempotents in  $H(n, d)$

The primitive and central idempotents corresponding to the  $2d$  one-dimensional representations of  $H(n, d)$  over  $\mathcal{K}$  may be constructed as follows. Fixing  $d$ , define

$$P_n = P_n^{+1} = q^{n-1}[n] \prod_{i \neq 1} \left( q^{2n-2} \lambda_1 - \lambda_i \right). \tag{33}$$

$P_n^{-1} = t P_n$ ,  $P_n^{+2} = s P_n$ ,  $P_n^{+3} = s^2 P_n$ , and so on. Define  $\lambda^{(0)} = 1$  and

$$\lambda^{(i)} = \sum_{d \geq j_1 > j_2 > \dots > j_i > 1} \left( \prod_{l=1}^i -\lambda_{j_l} \right)$$

(a sum over descending positive integer sequences  $(j_1, j_2, \dots, j_i)$ , with  $j_1 \leq d$ ).

PROPOSITION 5.1. Set

$$\beta_{j+1} = \frac{q P_j}{P_{j+1}}; \tag{34}$$

$$\alpha_j^0 = \frac{\prod_{i \neq 1} (-\lambda_i)}{P_j}; \tag{35}$$

$$\alpha_{j+1}^{d-i} = \frac{q^{2j} \lambda^{(i-1)} + \sum_{l=2}^i q^{(i-1)(2j-2)} (q^{2j} - 1) \lambda_1^{l-1} \lambda^{(i-l)}}{P_{j+1}} \quad (1 \leq i < d).$$

Then  $e_0^{+1} = 1$ ,

$$e_{j+1}^{+1} = e_j^{+1} \left( \beta_{j+1} g_j + \sum_{i=0}^{d-1} \alpha_{j+1}^i X_{j+1}^i \right) e_j^{+1}, \tag{36}$$

$e_j^{-1} = t e_j^{+1}$ , and so on.

For example,  $\beta_1 = 0$ ,  $\alpha_1^i = \lambda^{(d-1-i)}$ , and

$$e_1^{+1} = \frac{\prod_{i \neq 1} (X_1 - \lambda_i)}{P_1}.$$

(There are more examples in Section 5.2.)

*Proof of Proposition 5.1.* We work by induction on  $j$ , with the example above as base. Firstly, note that

$$e_{j+1}^{+1} e_j^{+1} = e_j^{+1} e_{j+1}^{+1} = e_{j+1}^{+1};$$

so, from Proposition 2.2, equation (36) gives a correct form for the  $R_{+1}$  idempotent up to coefficients. Then note that in this form it is sufficient (see equation (10)) to check the identity  $(g_j - q) e_{j+1}^{+1} = 0$ , and the normalisation (that is, the idempotency). The former is a direct calculation, and the latter may be checked by evaluating in  $R_{+1}$ ; that is, by substituting  $q$  for  $g_i$  and  $\lambda_1$  for  $X$  (in which case  $e_{j+1}^{+1}$  must evaluate to 1). To begin, we rewrite the claimed

expression for  $(g_j - q)e_{j+1}^{+1}$  as

$$(g_j - q) \left( e_j^{+1} (\beta_{j+1} g_j) e_j^{+1} + \left( \sum_{i=0}^{d-1} \alpha_{j+1}^i X_{j+1}^i \right) e_j^{+1} \right)$$

using commutation properties. By the inductive assumption, this is equal to

$$\begin{aligned} & (g_j - q) \left( e_{j-1}^{+1} \left( \beta_j g_{j-1} + \sum_{i=0}^{d-1} \alpha_j^i X_j^i \right) (\beta_{j+1} g_j) e_j^{+1} + \left( \sum_{i=0}^{d-1} \alpha_{j+1}^i X_{j+1}^i \right) e_j^{+1} \right) \\ &= e_{j-1}^{+1} \beta_{j+1} \left( \beta_j (g_j - q) g_{j-1} g_j + (g_j - q) \left( \sum_{i=0}^{d-1} \alpha_j^i X_j^i \right) g_j \right) e_j^{+1} \\ & \quad + (g_j - q) \left( \sum_{i=0}^{d-1} \alpha_{j+1}^i X_{j+1}^i \right) e_j^{+1} \\ &= e_{j-1}^{+1} \beta_{j+1} \left( (g_j - q) \left( \sum_{i=0}^{d-1} \alpha_j^i X_j^i \right) g_j \right) e_j^{+1} + (g_j - q) \left( \sum_{i=0}^{d-1} \alpha_{j+1}^i X_{j+1}^i \right) e_j^{+1}. \end{aligned}$$

Now, using the commutation rules on the first summand to bring the factor  $g_j$  forward through the  $X_j^i$ 's, we find that the expression equals

$$\begin{aligned} & (g_j - q) \left( \left( g_j \beta_{j+1} \alpha_j^0 + \alpha_{j+1}^0 \right) \right. \\ & \quad \left. + \sum_{i=1}^{d-1} \left( \beta_{j+1} \left( -q \alpha_j^i - (q - q^{-1}) \left( \sum_{l=i+1}^{d-1} \alpha_j^l (X_j)^{l-i} \right) \right) + \alpha_{j+1}^i \right) X_{j+1}^i \right) e_j^{+1}. \end{aligned}$$

But  $X_j e_j^{+1} = q^{2j-2} \lambda_1 e_j^{+1}$ , so we see that this is equal to

$$\begin{aligned} & (g_j - q) \left( \left( -q^{-1} \beta_{j+1} \alpha_j^0 + \alpha_{j+1}^0 \right) \right. \\ & \quad \left. + \sum_{i=1}^{d-1} \left( \beta_{j+1} \left( -q \alpha_j^i - (q - q^{-1}) \left( \sum_{l=i+1}^{d-1} \alpha_j^l (q^{2j-2} \lambda_1)^{l-i} \right) \right) + \alpha_{j+1}^i \right) X_{j+1}^i \right) e_j^{+1}. \end{aligned}$$

Then, equating coefficients to zero, we get

$$\begin{aligned} \alpha_{j+1}^0 &= q^{-1} \alpha_j^0 \beta_{j+1}, \\ \alpha_{j+1}^{d-1} &= q \alpha_j^{d-1} \beta_{j+1}, \\ \alpha_{j+1}^{d-2} &= \left( q \alpha_j^{d-2} + (q - q^{-1}) \alpha_j^{d-1} (q^{2j-2} \lambda_1) \right) \beta_{j+1}, \end{aligned}$$

and

$$\alpha_{j+1}^i = \left( q \alpha_j^i + (q - q^{-1}) \sum_{l=i+1}^{d-1} \alpha_j^l (q^{2j-2} \lambda_1)^{l-i} \right) \beta_{j+1} \quad (0 < i < d).$$



Without regard for normalization, any one coefficient may be chosen arbitrarily, so (without loss of generality) try  $\beta_{j+1} = qP_j/P_{j+1}$ . Then  $\alpha_{j+1}^{d-1} = \rho_{j+1}^{d-1}/P_{j+1}$ , where

$$\rho_{j+1}^{d-1} = q^{2j}, \quad \alpha_{j+1}^0 = \frac{\lambda^{(d-1)}}{P_{j+1}}, \quad \text{and} \quad \alpha_{j+1}^{d-2} = \frac{\rho_{j+1}^{d-2}\lambda_1 + \rho_{j+1}^{d-1}\lambda^{(1)}}{P_{j+1}},$$

for some  $\rho_{j+1}^{d-2}$ . Then

$$\rho_{j+1}^{d-2} = q^2\rho_j^{d-2} + (q^2 - 1)q^{2j-2}q^{2j-2} = q^{2j-2}(q^{2j} - 1).$$

Similarly,

$$\alpha_{j+1}^{d-3} = \frac{\rho_{j+1}^{d-3}\lambda_1^2 + \rho_{j+1}^{d-2}\lambda^{(1)}\lambda_1 + \rho_{j+1}^{d-1}\lambda^{(2)}}{P_{j+1}},$$

and, for  $1 \leq i < d$ ,

$$\alpha_{j+1}^{d-i} = \frac{\sum_{l=1}^i \rho_{j+1}^{d-l}\lambda_1^{l-1}\lambda^{(i-l)}}{P_{j+1}}$$

for some  $\rho_{j+1}^-$ . Then

$$\begin{aligned} \rho_{j+1}^{d-3} &= q^2\rho_j^{d-3} + (q^2 - 1)(q^{2j-2}q^{2j-4}(q^{2j-2} - 1) + (q^{2j-2})^3) = q^{4j-4}(q^{2j} - 1) \\ \rho_{j+1}^{d-4} &= q^2\rho_j^{d-4} \\ &\quad + (q^2 - 1)((q^{2j-2})q^{4j-8}(q^{2j-2} - 1) + (q^{2j-2})^2q^{2j-4}(q^{2j-2} - 1) + (q^{2j-2})^4) \\ &= q^{6j-6}(q^{2j} - 1), \end{aligned}$$

and

$$\rho_{j+1}^{d-i} = q^{(i-1)(2j-2)}(q^{2j} - 1) \quad (1 < i < d).$$

Finally, the normalization condition,

$$\begin{aligned} \lambda^{(d-1)} + \sum_{i=1}^{d-1} \left( \left( q^{2j}\lambda^{(i-1)} + \sum_{l=2}^i q^{(l-1)(2j-2)}(q^{2j} - 1)\lambda_1^{l-1}\lambda^{(i-l)} \right) (q^{2j}\lambda_1)^{d-i} \right) \\ = P_{j+1} - q^2P_j \end{aligned}$$

is verified by direct computation. □

By inspection of these idempotents, we see that the algebras will not be generic in specialisations in which  $\lambda_i/\lambda_j = q^{2r}$  for some  $i, j, r \in \mathbb{N}$ , nor in certain specialisations in which  $q$  is a root of unity (this also follows immediately from [1, 59]). Let us disregard, for the moment, the cases in which  $X$  is degenerate or non-invertible. Then, noting that rescaling all the  $\lambda_i$ s by the same factor produces an isomorphic algebra, we can fix  $\lambda_1 = 1$  (say) and adopt as parameters  $q$  and  $\lambda_i/\lambda_1$ ; see [59]. It is illuminating to proceed by example (and confer [58, Section 5]).

### 5.2. The case $d = 3$

The primitive and central idempotents corresponding to the six one-dimensional representations of  $H(n, 3)$  may be constructed as follows. Fixing  $d = 3$ , define  $P_n = P_n^{+1}$  and

$P_n^{\pm i}$  as in equation (33). For example,

$$P_1 = P_1^{\pm 1} = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3);$$

$$P_2 = q[2](q^2\lambda_1 - \lambda_2)(q^2\lambda_1 - \lambda_3).$$

As before,  $\beta_j = qP_{j-1}/P_j$ , and here

$$\alpha_j^0 = \frac{\prod_{i \neq 1}(-\lambda_i)}{P_j};$$

$$\alpha_j^1 = \frac{q^{2j-4}(q^{2j-2} - 1)\lambda_1 - q^{2j-2}(\lambda_2 + \lambda_3)}{P_j}; \tag{37}$$

$$\alpha_j^2 = \frac{q^{2j-2}}{P_j}.$$

Compute  $e_j^{\pm l}$  by  $e_0^{+1} = 1$ , and then

$$e_{j+1}^{+1} = e_j^{+1} \left( \beta_{j+1}g_j + \alpha_{j+1}^0 + \alpha_{j+1}^1 X_{j+1} + \alpha_{j+1}^2 X_{j+1}^2 \right) e_j^{+1}, \tag{38}$$

$e_j^{-1} = te_j^{+1}$ , and so on. For example

$$e_1^{+1} = \frac{X_1^2 - (\lambda_2 + \lambda_3)X_1 + \lambda_2\lambda_3}{P_1}$$

and

$$e_2^{+1} = e_1^{+1} \left( \frac{qP_1g_1 + \lambda_2\lambda_3 + ((q^2 - 1)\lambda_1 - q^2(\lambda_2 + \lambda_3))X_2 + q^2X_2^2}{P_2} \right) e_1^{+1}$$

$$= q \frac{(X_1^2 - (\lambda_2 + \lambda_3)X_1 + \lambda_2\lambda_3)}{P_1} \frac{(X_2^2 - (\lambda_2 + \lambda_3)X_2 + \lambda_2\lambda_3)(g_1 + q^{-1})}{P_2}.$$

Given these results, let us consider the generalised Soergel procedure for  $d = 3$  corresponding to that in Section 1.2 for  $d = 2$ . For illustration, we consider a field  $k$  in which the equations  $\lambda_i/\lambda_1 = q^{-n_i}$  ( $i = 2, 3$ ) and  $\lambda_1/\lambda_3 = q^{-n_1}$  are solved for positive integer  $n_i$  only in the case  $n_2 = 2, n_3 = 4$  (in particular, in this instance  $q$  is not a root of unity). Figure 4 illustrates the location of the corresponding reflection hyperplanes (shown as thick lines) in the  $H^{\mathcal{D}}$  (that is,  $A_2$ ) weight space. The Kazhdan–Lusztig polynomials for this geometry are given in the figure on [58, p. 1289]. The claim, then, is that if  $\mu$  is a weight in the fundamental alcove  $A^0$  (such as 0), then  $\Delta(\mu)$  has the following simple content (and Loewy structure).

$$L_{\mu}$$

$$L_{\mu.s} \qquad L_{\mu.t}$$

$$L_{\mu.st} \qquad L_{\mu.ts}$$

$$L_{\mu.sts}$$

(Here  $s$  and  $t$  are the walls of  $A^0$  – we are abusing the notation  $A.s$  of Section 1.2 to apply to weights in the obvious way.) Of course, localising at small  $n$ , some of these simple modules will vanish. Let us consider  $n = 0, 1, 2, 3$ .

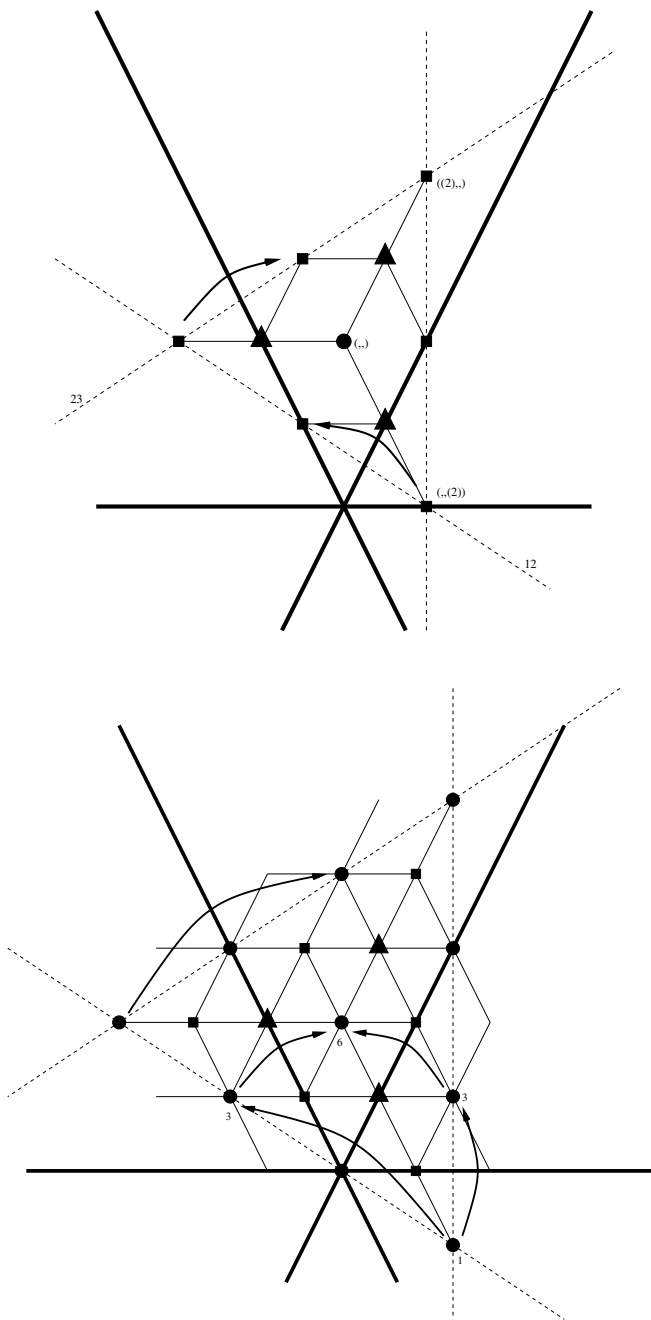


Figure 4: Weights of  $H^{\mathcal{D}}(n, 3)$  in weight space formalism: (a) for  $n = 0, 1, 2$ ; and (b) for  $n = 0, 1, 2, 3$ . Weight set  $\Lambda(0)$  consists of the empty weight (shown as a black circle marked  $(,)$  in (a));  $\Lambda(1)$  consists of the three weights marked with triangles, and so on (see the main text).

(For any  $k$ ) we have  $H^{\mathcal{D}}(0, 3) \cong k$ , with basis  $\{1\}$ . The weight for the corresponding simple module is the innermost dot in the figures (marked  $(, , )$  in Figure 4(a)). For  $H^{\mathcal{D}}(1, 3)$ , the idempotents  $e_1^{+i}$  are all well defined, and  $1 = \sum_{i=1}^3 e_1^{+i}$  is an idempotent decomposition into primitive idempotents. (The corresponding simples are marked as triangles.)

The set of weights for  $H^{\mathcal{D}}(2, 3)$  are marked with squares. (The weights corresponding to the multipartitions  $((2), , )$  and  $(, , (2))$  have been explicitly labelled, to fix the coordinate system.) Note that  $e_2^{+i}$  is divergent in the case  $i = 2, 3$ . The easiest way to see what is happening at  $n = 2$  is to recall the  $d = 3$  version of Proposition 1.2.

**PROPOSITION 5.2.** *Fixing  $k$ , there are three ways to quotient to pass from  $H^{\mathcal{D}}(n, 3)$  to  $H^{\mathcal{D}}(n, 2)$ , where  $n > 0$ .*

The corresponding subsets of the set of weights for  $H^{\mathcal{D}}(2, 3)$  lie in three straight lines: the dashed lines shown in Figure 4(a). Since we have shown that the  $d = 2$  algebra is isomorphic to a blob algebra in each of these cases, we can give a complete description. The lines marked 12 and 23 correspond to singly critical blob algebras, with reflection points the intersection points of these lines with the thick lines shown. Accordingly, there are injective standard module morphisms (‘reflection’ morphisms) as indicated by arrows. The other blob line is a semisimple quotient (from the parameters, it is nominally a singly critical case, but the reflection point lies at the outside edge of localisation of weight space to  $n = 2$ , so there is no image of any  $n = 2$  weight in it). It is straightforward to show that there are no other non-trivial morphisms. Since the  $d = 3$  alcove structure is determined by the three  $d = 2$  quotients, and since the morphisms indicated are all also blob module morphisms, their location is necessarily consistent with the  $d = 3$  Soergel procedure.

For  $H^{\mathcal{D}}(3, 3)$ , the three blob quotients correspond to the three dashed lines shown in Figure 4(b). The reflection morphisms shown within these lines are again simply blob morphisms. (Note that the reflection point on the third line now lies within the localisation of weight space.) The only question, then, concerns the morphisms into the module with  $A_2$ -weight 0. This weight is marked in the figure by the rank of the module,  $\text{Rank}(\Delta(0)|_{n=3}) = 6$ . (This weight coincides with the weight for the unique simple of  $H^{\mathcal{D}}(0, 3)$  in our scheme, since that module is a localisation of  $\Delta(0)|_{n=3}$ .) A straightforward Frobenius reciprocity argument (using the morphisms at level  $n = 2$  and the generic restriction rules) shows that there are at least two homomorphisms into this module; but it does not uniquely determine the domain in either case. To confirm the indicated morphisms, consider the walk basis of  $\Delta(0)|_{n=3}$  and the walk orbits of these walks – see Figure 5. Note from the last of these pictures that  $321 \sim_{\mathcal{G}^{(k)}} 333$ . It follows that there is a homomorphism between the corresponding standard modules, by Proposition 3.13. (Note that walk 321 is not sorted, but that the weaker condition of all steps in a given direction being taken consecutively is sufficient for this construction.) The other claimed morphisms follow similarly.

It is worth noting that  $\Delta(0)|_{n=3}$  may generically be defined as  $\Delta(0)|_{n=3} = H^{\mathcal{D}}(3, 3)e_{(1^3)}$ , where  $e_{(1^3)} = e_3^-$  is the usual  $q$ -antisymmetriser [58] (that is,  $\Delta(0)|_{n=3}$  is the globalisation of the unique simple of  $H^{\mathcal{D}}(0, 3)$ ). This is an illuminating construct to consider in any case. The module  $H(3, 3)e_{(1^3)}$  is, of course, much larger. By Proposition 2.1 it has basis  $\{X^a e_{(1^3)} \mid a \in \{0, 1, 2\}^3\}$ . To determine a basis in our case, one must use the vanishing of  $e_2^{-i}$  (which generates a significant part of  $\text{He}_{(1^3)}$ ). This problem is dealt with elsewhere; see [16].

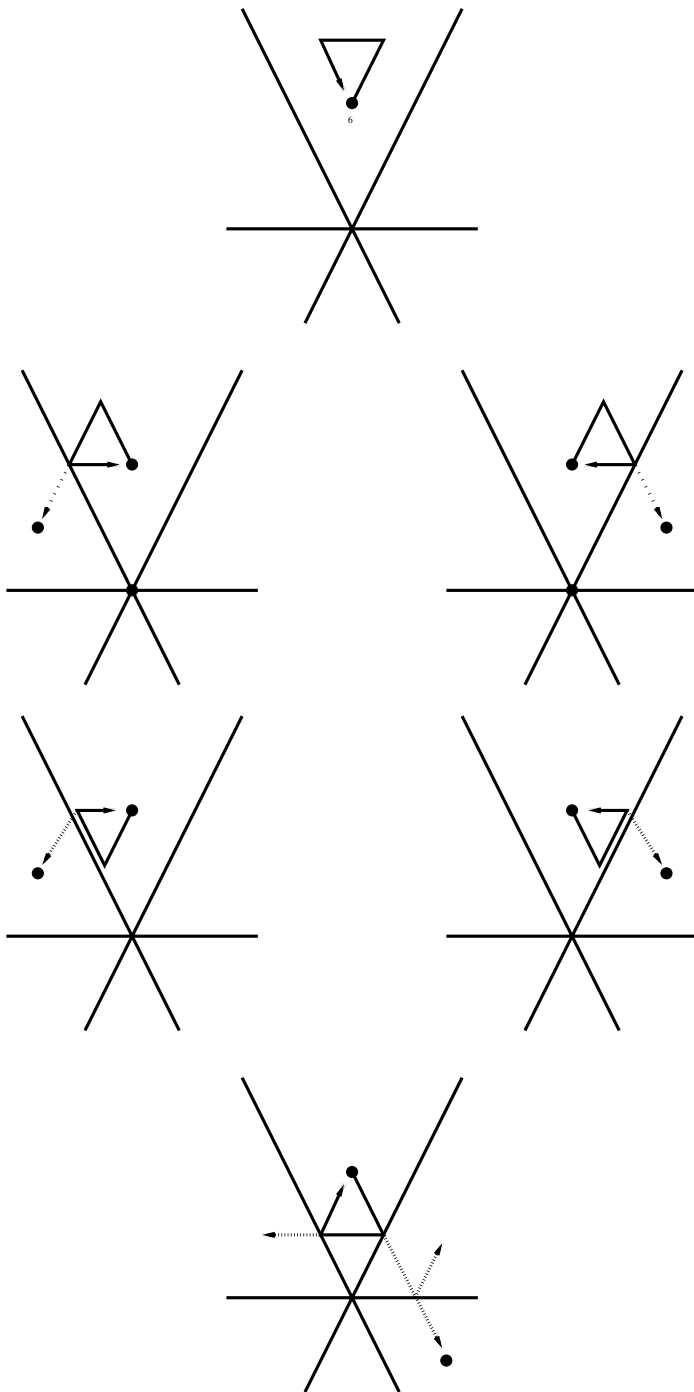


Figure 5: The six walks in  $T_0(3)$ , and their walk orbits.

5.3. Conjectured basis for  $H^{\mathcal{D}}(n, d)$

Consider  $w \in S_n$ , a permutation, and let  $S(w)$  be a corresponding reduced word in the Coxeter generators  $\{\sigma_i\}$ . The set map from  $S(S_n)$  to  $\mathfrak{B}_n$  given by  $\sigma_i \rightarrow g_i$  is an isomorphism [34]. Thus from  $w \in \mathfrak{B}_n$  we may read off a permutation (which we shall also call  $w$ ). Given such a permutation, define a symmetric relation  $(w) \subset \underline{n} \times \underline{n}$  by:  $i(w)j$  if  $i > j$  implies that  $w(i) < w(j)$  (and symmetry). That is,  $i(w)j$  if the lines  $i$  and  $j$  cross in the diagram of  $w$ .

CLAIM 5.3.  $\{X^{(a_1, a_2, \dots)}w \in \mathfrak{C}_n \mid i(w)j \text{ implies that } a_i \neq a_j\}$  is a basis for  $H^{\mathcal{D}}(n, d)$ .

Idea of the proof. Note that the dimensions are right (consider the Robinson–Schensted correspondence in the form given in [67], or, for example, in [66]). Thus it is enough to show spanning or linear independence. For the latter, it is convenient to have a representation  $\mathcal{R}$  of  $H^{\mathcal{D}}$  (if it is linear independent in  $\mathcal{R}$ , then the claim is proven and  $\mathcal{R}$  is faithful).

The remainder of the paper is concerned with representations of  $H^{\mathcal{D}}$ .

6. On representations of  $b_n$  and  $H^{\mathcal{D}}(n, d)$

The ordinary Temperley–Lieb algebra has a powerful diagram calculus (see [50] for a review). One motivation for the introduction of the blob algebra [55] was to bring the utility of such a calculus to the representation theory of the periodic/affine algebras studied in [54] (see also [50, Chapter 8], as well as [60, 38, 27, 30] and the references therein). In fact, all the finite irreducible representations of these infinite-dimensional algebras may be constructed using [55]. (Although completeness is not shown there, see [28, 57].) From the point of view of lattice statistical mechanics,  $b_n$  also renders the ‘seam’ boundary conditions (as in [5]; see also [47], for example) of the ice-type model [4] into the algebraic formalism of Temperley and Lieb [68]. Indeed, there are a number of mathematical and physical reasons (in addition to the pursuit of our ingredient I2 for  $H^{\mathcal{D}}(-, d)$ ) why a faithful ‘tensor space’ representation of the blob would be useful (confer [55, Section 4], as well as [60, 6, 22]).

By a tensor space representation of  $b_n$  (or  $H^{\mathcal{D}}(n, d)$ ), we mean a representation for each  $n$  with underlying module of form  $V(n) = V_{\text{aux}} \otimes V^{\otimes n}$  as a vector space, with  $V_{\text{aux}}$  and  $V$  finite-dimensional vector spaces (confer [40]), on which the generators act ‘locally’ (in particular,  $b_{n-1} \subset b_n$  acts trivially on the last factor  $V$ , so the restriction of  $V(n)$  is a manifest direct sum of  $\dim V$  copies of  $V(n-1)$ ); and a representation that is well defined in arbitrary specialisations.

Let  $e \in H^{\mathcal{D}}(n+d, d)$  be idempotent. A module  $V(n+d)$  for a tensor space representation over  $K$  is globalisable by  $e$  if  $e$  projects the last  $d$  tensor factors  $V^{\otimes d} \rightarrow K$ , and acts trivially on other tensor space factors.

To establish ingredient I2 generically for  $H^{\mathcal{D}}(-, d)$  (that is, to establish that there exists idempotent  $e_d \in H^{\mathcal{D}}(n+d, d)$  such that

$$e_d H^{\mathcal{D}}(n+d, d) e_d \cong H^{\mathcal{D}}(n, d); \tag{39}$$

confer [12, 21]), certain generalised braid diagrams may be used [58]. However, the following proposition holds.

PROPOSITION 6.1. Suppose that  $V(n+d)$  is the module for a faithful tensor space representation of  $H^{\mathcal{D}}(n+d, d)$  over  $\mathbb{C}$ , and is globalisable by  $e_d$ . Then equation (39) holds.



It is desirable not only to have representations of  $b_n$  that act on tensor space, but also that they preserve some version of this charge conservation – that is, they are a direct sum of analogues of permutation representations. (Note that the tensor space representation in [55, Section 4] is neither full tilting [22] in general, nor charge-conserving.)

One way to proceed is to search for maps from  $b_n$  to  $T_{n'}$  (for some  $n'$ ) (or, respectively, from  $H^{\mathcal{D}}(n, d)$  to  $H_{n'}^d$ ), and hence to obtain  $b_n$ -modules by restriction of  $M_2^q$ . Another possibility is to enrich suitable  $T_{n'}$ -modules with the property of a  $b_n$ -module by determining an action of the blob operator on them. We begin by investigating the latter.

### 6.1. Generalised bialgebra construction

Let  $(M, \circ, e)$  be a finite monoid, and let  $A$  be a  $K$ -algebra with basis  $M$  and multiplication defined on this basis by  $m_1 m_2 = k_{m_1, m_2}(\mathbf{q}) m_1 \circ m_2$ , where  $\mathbf{q}$  is some set of parameters and  $k_{12}(\mathbf{q}) \in K$ . (The possibilities for the coefficients will in general be constrained by  $M$  as follows:

$$\begin{aligned} m_1(m_2 m_3) &= m_1 k_{m_2, m_3}(\mathbf{q})(m_2 \circ m_3) = k_{m_2, m_3}(\mathbf{q}) k_{m_1, m_2 \circ m_3}(\mathbf{q}) m_1 \circ (m_2 \circ m_3) \\ &= k_{m_1, m_2}(\mathbf{q})(m_1 \circ m_2) m_3 = k_{m_1, m_2}(\mathbf{q}) k_{m_1 \circ m_2, m_3}(\mathbf{q})(m_1 \circ m_2) \circ m_3. \end{aligned}$$

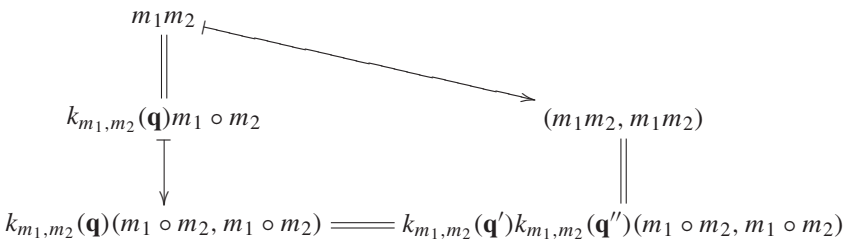
So  $k_{m_2, m_3}(\mathbf{q}) k_{m_1, m_2 \circ m_3}(\mathbf{q}) = k_{m_1, m_2}(\mathbf{q}) k_{m_1 \circ m_2, m_3}(\mathbf{q})$ , but there are plentiful solutions – for example, any finite group algebra.) Suppose that there is a triple of points in parameter space for which

$$k_{m_1, m_2}(\mathbf{q}') k_{m_1, m_2}(\mathbf{q}'') = k_{m_1, m_2}(\mathbf{q}) \tag{42}$$

for all  $m_1, m_2$ . Then  $A$  has a kind of generalised coproduct:

$$\begin{aligned} A &\hookrightarrow A' \times A'', \\ m &\mapsto (m, m), \end{aligned}$$

making it a kind of generalised bialgebra, since



(confer [39, Section 1.1.3(iv)], for example). In particular, the coproduct is an algebra morphism, and if a submanifold  $S$  of parameter space can be found for which each pair  $\mathbf{q}', \mathbf{q}'' \in S$  has a  $\mathbf{q} \in S$  satisfying equation (42), then the sum over all  $\mathbf{q} \in S$  of categories of (left) modules is closed under tensor product. (The example of group algebras is the usual bialgebra and tensor product.)

It is easy to show, using the diagram calculus (or via a mild generalisation of the Temperley–Lieb diagram variant of cabling [48, Section A(iii)]), that  $T_n(q)$  is an algebra of this type, with  $\mathbf{q} = \{q\}$ . The diagram in Figure 6 illustrates the coproduct on  $U_1 U_2 \in T_4(q)$ , using lines of different thickness for different  $q$ . The cabling-like visualisation of the two factors, in which they are embedded in a single pseudodiagram, is possible because the thin and thick lines are arranged into subdiagrams that never meet in any composition.



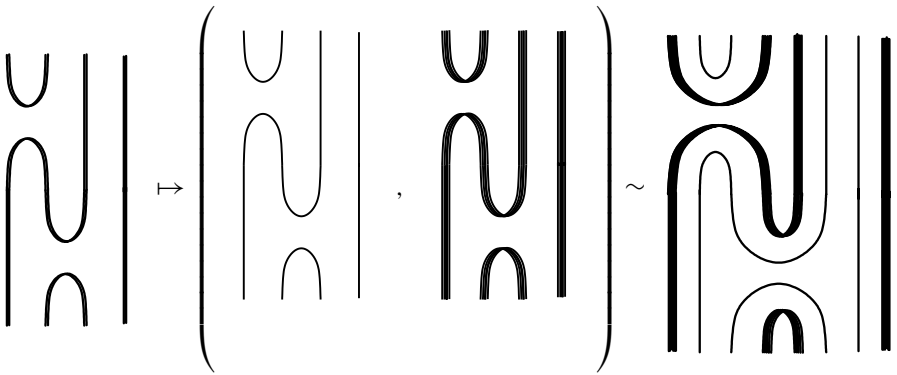


Figure 6: Coproduct on  $U_1U_2 \in T_4(q)$ .

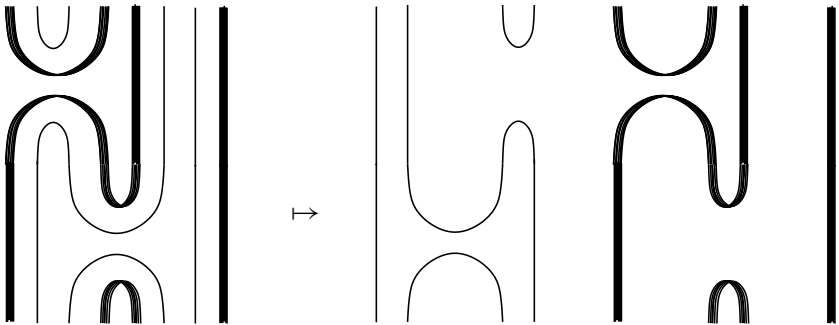


Figure 7: Reflected visualisation.

It will be evident that the set of conditions (42) include  $-[2]_q = [2]_{q'}[2]_{q''}$  in this case (consider a composition in which a closed loop arises, such as  $U_1U_1$ ); and that this is the only non-trivial condition. Through the cabling picture we may pass to another visualisation, in which the thinner lines have been reflected in a vertical line at the left edge of the diagram (confer [19]), as illustrated in Figure 7. There is no significant difference between these two visualisations, except that it is perhaps slightly easier to describe the construction of certain tensor product representations *explicitly* using the reflected form, as we shall see. Again, in the reflected form we may view the picture as a single pseudodiagram (but again there is no sense in which the right- and left-hand sides ever touch).

For example, we may tensor together two tensor space representations in the form:

$$\begin{aligned} \rho &: T_n(q) \longrightarrow \text{End}(V_2^{2n}); \\ \rho &: U_i \mapsto M_2^s(U_{n-i})M_2^t(U_{n+i}). \end{aligned}$$

Here, the set of conditions (42) reduces to  $-[2]_q = [2]_s[2]_t$  via, for example,

$$\begin{array}{ccc} \rho(U_i U_i) & \longleftarrow & \rho(U_i) \rho(U_i) \\ \parallel & & \parallel \\ -[2]_q \rho(U_i) & \longleftarrow & [2]_s [2]_t \rho(U_i). \end{array}$$

Set

$$\mathcal{U}^q(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

and  $\mathcal{U}^q = \mathcal{U}^q(0)$  (confer [18]). Just as  $M_2^r(U_i)$  is a matrix acting trivially on every tensor factor except the  $i$ th and  $(i + 1)$ th, where it acts as  $-\mathcal{U}^r$ , so let  $M_2^{r,x}(U_i)$  denote a matrix differing from this only in acting like  $-\mathcal{U}^r(x)$  in that position.

Note that

$$(\mathcal{U}^s \otimes \mathcal{U}^t)(1 \otimes \mathcal{U}^r(x) \otimes 1)(\mathcal{U}^s \otimes \mathcal{U}^t) = \left( \frac{r}{st} + \frac{st}{r} + x \frac{t}{s} \right) (\mathcal{U}^s \otimes \mathcal{U}^t) \quad (43)$$

for any  $r, s, t, x$  (an explicit calculation).

**PROPOSITION 6.3.** Fix  $q$  and  $m$ , put  $q = e^{i\mu_q}$  and  $u_0 = [m]_q e_-$ , and choose  $r, s$  and  $t$  such that

$$-\cos(\mu_q) = 2 \cos(\mu_s) \cos(\mu_t); \quad (44)$$

$$-\frac{\sin((m-1)\mu_q)}{\sin(m\mu_q)} = \frac{\cos(\mu_s + \mu_t - \mu_r)}{\cos(\mu_r)}. \quad (45)$$

(Note: exclude  $m = 0$ , and beware of  $q = 1$ . A convenient realisation is  $r = i(-q)^m$ ,  $s = -i\sqrt{iq}$ ,  $t = -\sqrt{iq}$ ; that is,  $\mu_r = m(\mu_q + \pi) + \pi/2$ ,  $\mu_s = (\mu_q + \pi)/2 - 3\pi/4$ ,  $\mu_t = (\mu_q + \pi)/2 + 3\pi/4$  – so rational  $\mu_q/\pi$  and  $m$  give rational  $\mu_r/\pi$ ,  $\mu_s/\pi$ ,  $\mu_t/\pi$ .) Then there is an algebra homomorphism

$$\rho_0 : b_n(q, m) \longrightarrow \text{End}(V_2^{2n})$$

given by

$$\rho_0 : e_- \mapsto \frac{1}{[2]_r} M_2^r(U_n); \quad (46)$$

$$\rho_0 : U_i \mapsto M_2^s(U_{n-i}) M_2^t(U_{n+i}).$$

*Proof.* We may readily verify that  $\rho_0(e_- e_- = e_-)$ .

The relations for  $T_n(q) \subset b_n$  may be checked in  $\rho$ ; that is, as above. (Note that  $[m]_q = \sin(m\mu_q)/\sin(\mu_q)$ .) There remains  $U_1 e_- U_1 \propto U_1$  (relation (2)). This is validated by the explicit calculation in equation (43):

$$U_1 e_- U_1 = \frac{[m-1]_q}{[m]_q} U_1 \mapsto -\left( \frac{r}{st} + \frac{st}{r} \right) \rho_0(U_1). \quad \square$$

Another way to see this is to note that, in tensor space, equation (43) allows us to make sense of an extension of the pseudodiagrams in Figure 7, in which the left- and right-hand sides meet as in Figure 8. (Specifically, this figure may be replaced by a scalar multiple of one in which the loop composed of mixed line segments is omitted.)

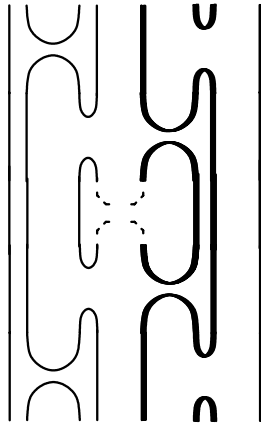


Figure 8: Extension of the reflected visualisation.

There is a similar homomorphism  $\rho_s$ , which simply replaces equation (46) with

$$\rho_s : e_- \mapsto \frac{1}{[2]_r} M_2^{r,[2]_r}(U_n) \tag{47}$$

and equation (45) with

$$-\frac{\sin((m-1)\mu_q)}{\sin(m\mu_q)} = \frac{\cos(\mu_s + \mu_t - \mu_r)}{\cos(\mu_r)} + e^{i(\mu_t - \mu_s)}. \tag{48}$$

This mild complication has the merit that the blob/box symmetry of the algebra [55] maps this representation to one of the same type.

The isomorphism of  $b_n$  to its opposite (note the symmetry of the relations under writing back to front) provides an automorphism that is fixed by  $\rho_0$  (the representations of the generators are symmetric matrices); thus  $\rho_0$  is contravariant self-dual. The same is true of  $\rho_s$ .

These very exciting representations merit further study. Martin has recently shown [51] that  $\rho_0$  is faithful, and Martin and Ryom-Hansen have shown [53] that it is full tilting in the quasihereditary cases. The intriguing question now is: ‘‘How do they generalize to higher  $d$ ?’’ This question is not trivial. We have made considerable use of the Temperley–Lieb diagram calculus here, and there is no such powerful tool in evidence for higher  $d$ , short of the braid group itself. The remainder of the paper is essentially concerned with addressing this question.

### 6.2. Other constructions

The blob algebra  $b_n(q, m)$  is (at least) *singly critical* when  $m \in \mathbb{Z}^+$ , see Section 1.2: if  $q$  is not a root of unity, then the procedure there still works, but with  $l$  set unboundedly large. This means that the usual Pascal triangle of standard modules [55] is complicated by at least one wall of ‘reflection’ homomorphisms from outside to inside across the alcove wall at  $m$ , as exemplified in Figure 9. There is a suggestive combinatorial coincidence with  $T_n$  manifest in the dimension of heads of the blob standard modules in certain singly critical (that is,  $m \in \mathbb{Z}$ ) cases. This starts with the  $m = \pm 1$  cases, where there is a  $b_n$  quotient

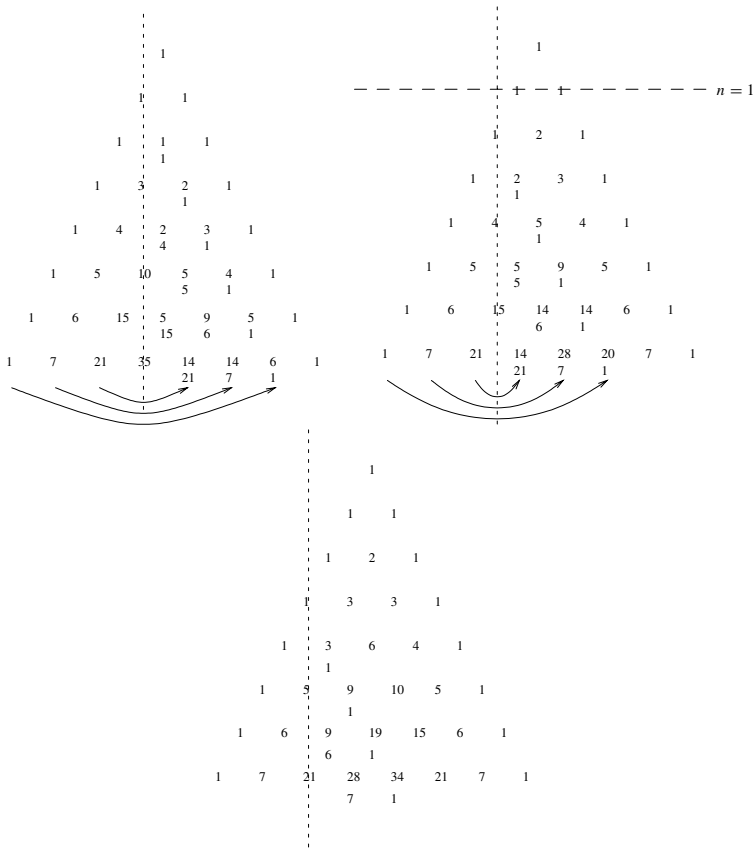


Figure 9: The Pascal triangle of  $b_n$  standard modules (layers  $n = 0, 1, \dots, 7$  are shown) complicated at  $m \in \mathbb{Z}$  by reflection homomorphisms. Here we show, by their dimensions, the composition factors of each standard in cases  $m = -1, -2, -3$ .

given by  $e_- \mapsto 0$  (or, respectively,  $e_- \mapsto 1$ ). It follows immediately from the relations that this quotient is isomorphic to  $T_n$  (and so the coincidence is explained). Figure 9 exhibits a similar phenomenon at  $m = -2$ . These may be regarded as special cases of a more general ‘braid construction’.

In the next sections we describe this braid construction, and review aspects of the connection between the blob algebra and periodic (and affine) systems that lead to other useful maps from  $b_n$  (and other  $H(n, d)$  quotients) into ordinary Hecke quotients, and hence into tensor space.

### 7. Braids and the blob approach to periodic systems

In this paper, the ‘A-type braid group’ A-braid is the group of braidings of a row of initially vertical strings numbered from the top left:  $1, 2, \dots$ , which braidings are trivial on all but finitely many strings. The subgroup  $A_n$ -braid acts trivially on all but the first  $n$  strings (so  $A_0$ -braid =  $A_1$ -braid  $\subset$   $A_2$ -braid  $\subset \dots$ ). Thus  $H_n$  is a quotient of  $\mathbb{C}A_n$ -braid, and  $g_i$  is the braid in which string  $i$  crosses over string  $i + 1$ . Evidently, such elements, and their

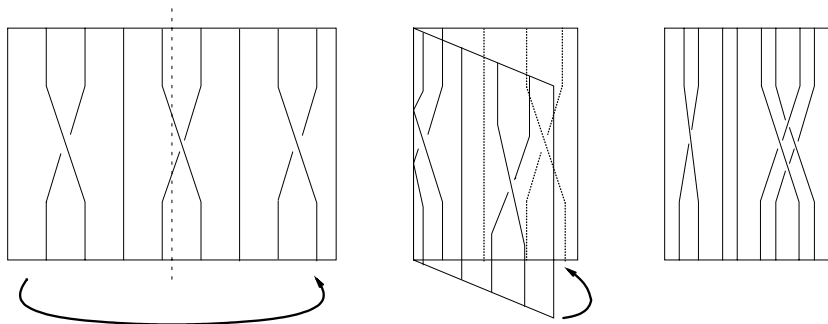


Figure 10: Illustration of the folding injection of  $A_n$ -braid into itself.

inverses, generate the group [46, 8]. Let  $(-): A_n\text{-braid} \rightarrow A_n\text{-braid}$  be the automorphism given by  $g_i \mapsto g_{n-i}$ ; see [50, Section 5.7.2]; confer [19].

Occasionally, we shall need to refer to the group of braidings of precisely  $n$  strings (as, for example, in the Young subgroup construction – see [50, Section 13.1, p. 323]). This group is obviously isomorphic to  $A_n$ -braid, and we shall distinguish them only by context. Let  $1^m$  denote the identity element on precisely  $m$  strings, and let

$$1_C^m : A_n\text{-braid} \rightarrow A_{mn}\text{-braid}$$

be the corresponding *cabling* morphism (replace each string with  $m$  parallel strings).

Let  $\Delta : A_n\text{-braid} \rightarrow A_n\text{-braid} \times A_n\text{-braid}$  be the group comultiplication. Let  $Y : A_n\text{-braid} \times A_n\text{-braid} \hookrightarrow A_{2n}\text{-braid}$  be the natural ‘Young’ embedding, extended to (a version of) the full braid group by extending the numbering of strings to  $\mathbb{Z} \setminus \{0\}$  – that is, essentially the full line, not the half line – and placing the second copy of  $A_n$ -braid on the ‘minus’ side. Let  $F$  be the map back to the full braid group proper, obtained by *folding* the left-hand side of the plane over on to the right-hand side at a point slightly shifted from the origin (so that each negatively numbered string starting point lies just to the left of its positively numbered version, and the system is bounded on the left again), and then renumbering – see Figure 10. Let  $S$  be the map back to  $A$ -braid, obtained by renumbering  $i \mapsto i + n + 1$  ( $i < 0$ ) and  $i \mapsto i + n$  (for  $i > 0$ ) and discarding all strings numbered less than 1. Now define a map  $1_F^2$  from  $A_n\text{-braid} \rightarrow A_{2n}\text{-braid}$  by commutativity of:

$$\begin{array}{ccc}
 A_n\text{-braid} & \xrightarrow{1_F^2} & A_{2n}\text{-braid} \\
 \downarrow \Delta & & \uparrow F \\
 A_n\text{-braid} \times A_n\text{-braid} & \xrightarrow{1 \times (-)} A_n\text{-braid} \times A_n\text{-braid} \xrightarrow{Y} & A_{2n}\text{-braid}
 \end{array}$$

and similarly for  $1_S^2$ . The map  $1_F^2$  is similar also to the  $m = 2$  cabling morphism, in that each string now has a partner running parallel to it, but the over/under information is not the same.

We now recall certain constructions from [50, Section 5.7] (some changes of notation will be necessary). Let  $\hat{C}A_n\text{-braid}$  denote the algebra associated to affine graph  $\hat{A}_n$  there (strictly speaking, only every other such graph has a pregraph, but this need not concern us), let  $\hat{A}_n\text{-braid}$  denote the underlying group (that is, the affinization of the ordinary braid

group), and let  $g$  denote the ‘extra’ generator associated to the affinizing vertex in  $\hat{A}_n$  compared to  $A_{n-1}$ . Define

$$G = g_1 g_2 \dots g_{n-1}$$

in  $A_n$ -braid (the element in which string 1 crosses over strings 2 to  $n$ ), and note that  $Gg_iG^{-1} = g_{i+1}$  ( $i < n - 1$ ).

**PROPOSITION 7.1** (see [50]). *There is a homomorphism  $\phi_0 : \hat{A}_n$ -braid  $\mapsto A_n$ -braid given by identification on the  $A_n$ -braid subalgebra and  $g \mapsto Gg_{n-1}G^{-1}$ .*

As noted for example in [48, Section 3], there are actually a number of closely related ways of building representations of the affine (or periodic) case, corresponding to the choice of periodic boundary conditions (the ‘cohomological seam’) in a physical system. This was systematized, in [55, Section 3], by the introduction of the idempotent blob generator  $e_-$ . Using this, we can build an invertible generator  $g_- = 1 + ae_-$  (for  $a$  a suitable constant) obeying

$$g_- g_1 g_- g_1 = g_1 g_- g_1 g_- \tag{49}$$

(see also Proposition 4.2), and we define  $G_- = g_- G$ . Then

$$g \mapsto G_- g_{n-1} (G_-)^{-1} \tag{50}$$

defines a generalisation  $\phi_a$  of  $\phi_0$  for each suitable choice of  $a$ ,  $y_e$  (see [55, equation (25)]).

Note that neither equation (6) nor the blob construction for  $g_-$  are needed to verify the map in equation (50); only equation (49) is necessary. Thus the map generalises to  $H(n)$ , and even to the level of braids.

The connection between the  $B$ -type and periodic systems now follows, inasmuch as  $B_n$ -braid may be realized as the group of braids on the cylinder, whereupon  $\pi = g_{n-1} g_{n-2} \dots g_1 g_0$  is the braid obtained from the identity braid by turning the bottom edge of the cylinder through one vertex clockwise (that is, so as to take vertex 1 into vertex 2, and so on);  $\pi'$ , the corresponding generalisation of  $G_-$  (that is, with  $g_-$  replaced by  $g_0$ ), is the anticlockwise turn. Thus  $B_n$ -braid may be thought of as having affine  $\hat{A}_n$ -braid as a subgroup, with  $g \cdot = \pi g_1 \pi^{-1} = \pi' g_{n-1} (\pi')^{-1}$ ; *confer* [55, Section 3].

### 7.1. Homomorphisms of $B_n$ -braid to $A_n$ -braid

Let  $A = A_n$ -braid, and let  $\underline{n} = \{1, 2, \dots, n\}$ . For  $b \in A$  and  $i \in \underline{n}$ , define  $b(i)$  to be the final position of string  $i$  in  $b$ .

Let  $p$  be any partition of (or, equivalently, any equivalence relation on)  $\underline{n}$ . Then for each such  $p$ , there is a subset of  $A = A_n$ -braid such that  $b \in A$  implies that  $b(i) \sim i$ . This subset is a subgroup: call it  $p$ - $A_n$ -braid, or just  $p$ -braid. For example, if  $p$  is the ‘trivial’ relation ( $\{\underline{n}\}$ ), then  $p$ -braid =  $A$ ; if  $p$  is the identity relation, then  $p$ -braid is the pure braid group (the normal subgroup whose quotient is the permutation action on  $\underline{n}$ ,  $b(i)$ , described above).

For convenience when dealing with general  $n$ , we shall describe a partition for which each  $i > m$ , for some  $m$ , is in the same part by only giving the other parts. Thus  $\{\}$ -braid =  $\{\underline{n}\}$ -braid =  $A$ ; and  $\{\{1\}\}$ -braid =  $A'$  is the group that is pure on string 1.

Let  $L_i \in A_n$ -braid be the pure braid that takes string  $i$  behind all earlier strings, and then back in front of them (that is,  $L_1 = 1$  and  $L_{i+1} = g_i L_i g_i$ ); see Figure 11.

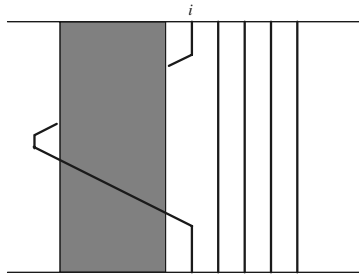


Figure 11:  $L_i$ ; here, the shaded area represents the identity braid on the first  $i - 1$  strings.

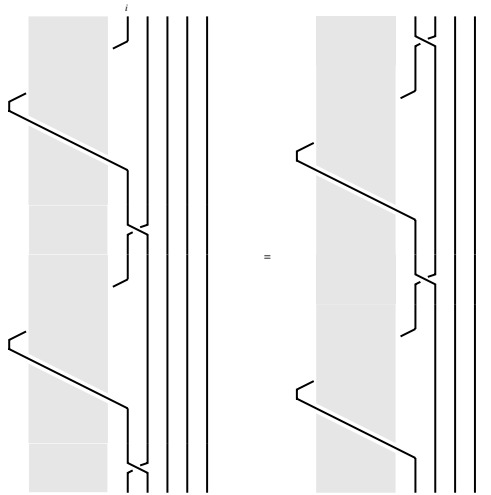


Figure 12: Schematic for generalisations of  $L_i g_i L_i g_i = L_i L_{i+1} = L_{i+1} L_i = g_i L_i g_i L_i$  as in  $f_b$  ( $b$  lives in the shaded region).

Note that in the Hecke algebra quotient,  $L_i$  is the image of  $X_i \in H(n, 1)_{|\lambda_1=1}$ , that  $L_i L_j = L_j L_i$ , and that  $C_n = \prod_{i=1}^n L_i$  is the ('clockwise' or  $g_i$ -built) pure twist element of  $A_n$ -braid, denoted  $M^2$  in [50, Section 5.7.2].

**PROPOSITION 7.2.** *Let  $b$  be any element of  $A_{m-1}$ -braid. Then there is an injective group homomorphism*

$$\begin{aligned}
 f_b^m : B_n\text{-braid} &\longrightarrow \{m-1\} - A_{n+m-1}\text{-braid}; \\
 g_0 &\mapsto C_m b; \\
 g_i &\mapsto g_{i+m-1}.
 \end{aligned}$$

*Proof.* Consider Figure 12. This checks the key relation explicitly. □

For example,  $f_1^2$  is  $g_0 \mapsto g_1 g_1$ ,  $g_1 \mapsto g_2$ , and so on. Likewise,  $f_1^1$  is  $g_0 \mapsto 1$ ,  $g_i \mapsto g_i$  ( $i > 0$ ).

There is a group homomorphism extending the  $1_C^2$  cabling morphism (confer [50, Chapter 13]; note also [17]):

$$\begin{aligned}
 1_C^2 : B_n\text{-braid} &\longrightarrow A_{2n}\text{-braid}; \\
 g_0 &\mapsto g_1; \\
 g_i &\mapsto 1_C^2(g_i).
 \end{aligned}$$

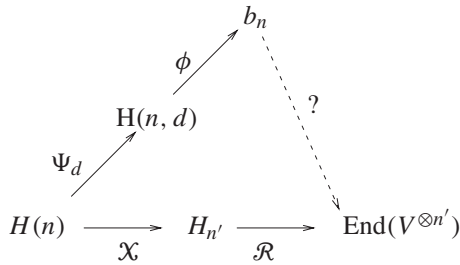
There is a similar homomorphism extending the  $1_S^2$  morphism:

$$\begin{aligned}
 1_S^2 : B_n\text{-braid} &\longrightarrow A_{2n}\text{-braid}; \\
 g_0 &\mapsto g_n; \\
 g_i &\mapsto g_{n-i}g_{n+i}
 \end{aligned}$$

(confer Figure 10), and another extending  $1_F^2$ .

### 8. On tensor space representations of $H(n, d)$

The constructions above allow us to build representations of  $B_n$ -braid from type- $A$  representations (and if these are charge-conserving tensor space representations, then these properties will be preserved, in some sense, as will globalisability). Our strategy now in searching for maps from  $b_n$  to  $T_n$  (and generalisations to  $d > 2$ ) may be summarized by the following picture.



The Northeast-pointing maps are the canonical quotients from Sections 1 and 4 (we shall call the combined map  $\eta$ ), the map  $\mathcal{R}$  is the ordinary representation on tensor space, and the dotted line is the desired map occurring if  $\mathcal{R} \circ \mathcal{X}$  factors through  $b_n$ . The exercise is to find among the maps  $\mathcal{X}$  those for which  $\mathcal{R} \circ \mathcal{X}$  factors thus. (As we shall see, the candidates that we noted in Section 6.2 for maps from  $b_n$  to  $T_n$  are just special cases of the simple representations  $f_b^m$  of  $B_n$ -braid above.)

By Definition 6.2, each  $B$ -braid quotient of form

$$B_n\text{-braid} \xrightarrow{f_b^m} A_{m+n}\text{-braid} \xrightarrow{M_N} V_N^{m+n} \tag{51}$$

(‘auxiliary space’ construction) or

$$B_n\text{-braid} \xrightarrow{1^2} A_{2n}\text{-braid} \xrightarrow{M_N} V_N^{2n}$$

(‘cabling-related’ construction) factors through a partial specialisation of  $H(n, d)$  (some  $d$  in which the  $g_0$  (that is,  $X$ ) eigenvalues are determined in terms of  $q$ , but  $q$  is indeterminate.



Obviously, the  $M_2 \circ f_1^2$  quotient obeys a quadratic  $f(g_0) = 0$ , so  $d = 2$ , and so it factors through some specialisation of  $\text{Hecke}_B(n)$ ; but  $f(g_0)$  has coefficients in  $\mathbb{Z}[q, q^{-1}]$ , so this is still not generic ( $\text{Hecke}_B$  has two parameters,  $q$  and  $Q$ , say).

Since all our maps  $M_2 \circ \mathcal{X}$  map into the  $T_n$  action on  $V_2^n$  (for some  $n$ ), their image breaks up at least as far as in equation (41).

Possible next steps here are: (i) to investigate the generic irreducible content of the  $f_b$  representations (of whatever specialisation of  $H(n, d)$  they might provide); and (ii) to investigate what portion of the parameter space is actually accessible by this construction (that is, what eigenvalues of  $g_0$  we can realise by varying  $b$ ). Fixing  $q$ , this portion is discrete (that is, of measure zero), but then so is the (at least singly critical) portion that we are most interested in, so it is not necessarily too restricted.

Note that if we want to access a dense subset of the parameter space, this *cannot* be via  $M_N \circ f_b^m$ , since this depends continuously only on  $q$ . Instead, we could look at maps ending on, say,  $\otimes_{i=1}^d V_N^n|_{q=q_i}$ .

### 8.1. On cabling-related maps

Underlying the map  $1_F^2$  is the full range of direct product representations of  $A_n$ -braid. In particular, we can regard  $\otimes_i M_N^{q_i}$  as a representation of  $A_n$ -braid. In general, we may not assume that these representations will factor through any particular Hecke quotient, but if one does, then it could provide a generalisation of the extension of  $1_F^2$  to  $B_n$ -braid. We form

$$R^\otimes(g_i) = M_N^q(g_{n-i}) \otimes M_N^r(g_{n+i})$$

and try, say,  $R^\otimes(g_0) = 1 \otimes M_N^s(g_{n-i}) \otimes 1$ , and we compute  $R^\otimes(g_0 g_1 g_0 g_1 - g_1 g_0 g_1 g_0)$ . In this particular case, with  $N = 3$ , the image vanishes only when  $q = r = s$  (a brute-force calculation). An analogous deformation of the extended  $1_C^2$  map at  $N = 3$  fails in the same way.

The  $1_C^2$  cabling map with  $N = 2$  does not in general factor through  $b_n$  either. Further investigations are hindered by the magnitude of the computations required, but these negative results serve well to illustrate the extraordinary nature of the  $\rho$ -representations in Proposition 6.3.

### 8.2. On $N = 2$ auxiliary spaces, $b_n$ , and ‘the coincidence’

In (51) we require that  $M_N \circ f_b^m(x)$  be also a representation ( $R$ , say) of  $\eta(x)$  for all  $x$ . Firstly,

$$\eta : g_i \mapsto q + U_i,$$

so equation (40) verifies equation (1). We also require that  $R(U_1 e_- U_1 - k_- U_1) = 0$  for some  $k_-$  (the relation (5) is not sufficient to ensure this). Whenever we *find* a map, the other question is: “Is it faithful?”

Since

$$\eta : g_0 \mapsto \alpha 1 + \beta e_-,$$

the spectrum of  $M_2(f_b^m(g_0))$  must be quadratic if the map is to factor through the blob as it stands. By equation (41), this spectrum may be determined from the action on the zero charge sector  $\mathcal{P}_0$  of  $V_2^n$ . The following lists are the eigenvalues with multiplicities in this

sector, arranged by standard  $T_n$ -module factor with the ‘spine’ module on the left, and so on.

- $M_2(L_1): \{1\};$
- $M_2(L_2 = C_2): \{q^2, q^{-2}\};$
- $M_2(L_3): \{q^2, q^{-2}, q^{-4}\};$
- $M_2(C_3): \{1, 1, q^{-6}\};$
- $M_2(L_4): \{1, 1, q^2, q^{-4}, q^{-4}, q^{-6}\};$
- $M_2(C_4): \{1, 1, q^{-4}, q^{-4}, q^{-4}, q^{-12}\};$
- $M_2(C_5): \{q^{-4}, q^{-4}, q^{-4}, q^{-4}, q^{-4}, q^{-10}, q^{-10}, q^{-10}, q^{-10}, q^{-20}\};$
- $M_2(C_6): \{q^{-8}, \dots, q^{-8}, q^{-12}, \dots, q^{-12}, q^{-20}, \dots, q^{-20}, q^{-32}\}.$

The pattern for  $C_i$  will be obvious.

Note, therefore, that  $C_1, C_2$  and  $C_3$  are the only possibilities here (unless we further specialize to  $q$  a root of unity). ‘Null’ twist  $C_1$  corresponds to the  $m = \pm 1$  case already discussed. For the other cases, it remains to check that the candidates for images of the generators obey  $U_1 e_- U_1 = k_- U_1$  for some scalar  $k_-$ .

*Example:*  $M_2 \circ f_1^2$ . (Let  $u_1$  and  $u_0$  denote the canonical preimages along  $\eta$  of  $U_1$  and  $e_-$ .) An elementary calculation finds a value of  $k_-$  for which

$$M_2(f_1^2(u_1 u_0 u_1 - k_- u_1)) = 0.$$

This value then determines that the blob parameter  $m = -2$  here.

Since  $\text{span}(1, g_1) = \text{span}(1, g_1 g_1)$  here (and  $V_2^n$  is a faithful  $T_n$ -module), the image of  $B_n$ -braid here is the whole of  $T_{n+1}$ . Thus equation (41) determines the structure of  $V_2^{n+1}$  as a  $B_n$ -braid-module. For  $n = 2$ , it is

$$V_2^3 = 1 + (2 + 1) + (2 + 1) + 1 \tag{52}$$

(representing summands by their dimensions), which is the structure as a  $T_3$ -module. Since we hit the whole of  $T_3$ , equation (52) is the irreducible decomposition with  $q$  generic. Let us call the two inequivalent modules here  $M^1$  and  $M^2$ . The generic simples of  $H(2, 2)$ , as indexed by their 2-partitions (see Section 1), are:

2-partition		((2), )		((1), (1))		(, (2))		((1 <sup>2</sup> ), )		(, (1 <sup>2</sup> ))
dimension		1		2		1		1		1

all but the last two of which survive the quotient to  $b_2$ . Note, therefore, that if the  $T_3$  standard  $M^2$  breaks up no further (that is,  $q$ -generically) it is a blob representation for  $m = -2$ , but that  $M_2 \circ f_1^2$  cannot be a faithful  $b_n$ -module.

For  $B_3$ -braid, we have  $V_2^4 = 1 + (3 + 1) + (2 + 3 + 1) + (3 + 1) + 1$  (as a  $T_4$ -module) as follows; *confer* Hecke<sub>B</sub>.

2-partition		((3), )		((2, 1), )		((1 <sup>3</sup> ), )		((2), (1))		
dimension		1*		2		1		3*		
2-partition		((1 <sup>2</sup> ), (1))		((1), (2))		((1), (1 <sup>2</sup> ))		(, (3))		...
dimension		3		3*		3		1*		...

(Blob representations are indicated with a \*.)

Recall that at  $m = -2$ , blob standards break up as shown in Figure 9. In particular, at  $n = 3$ ,

$$\begin{array}{ccc} 1 & 2 & 3 & 1 \\ \longrightarrow & 1 & & \end{array} .$$

Note that here, and for all  $n$ , the heads of the  $b_n$  standards to the right of the  $m = -2$  line may indeed be identified with the (generic) irreducible  $T_{n+1}$ -modules. This is neat, but it follows that none of these representations is faithful.

To summarize the last two sections, we have not been successful in generalising the  $\rho$ -representations. The search for full tilting modules for general  $d$  continues, and we report these negative results partly in order to avoid unnecessary duplication later. More positively, the representations that we have found are interesting from the point of view of Yang–Baxter equations in physics [4], but we shall discuss these applications elsewhere.

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We commend the following recent papers on related topics to the reader: [63, 31].

#### Appendix A. On the Bernstein centre $Z(H(n))$ and $Z(H(n, d))$

Some of the manipulations of ideals in Section 4 and thereafter are not trivial. The following mechanical exposition of the Bernstein centre and its image in  $Z(H(n, d))$  may help the reader to see where they come from.

Following on from equation (19), we define  $\hat{g}_i = [X_i, g_i]$  (thus  $\hat{g}_i = (X_i - X_{i+1})g_i + (q - q^{-1})X_{i+1}$ ). Then

$$\hat{g}_i X_j = X_{\sigma_i(j)} \hat{g}_i \tag{53}$$

and

$$\begin{aligned} \hat{g}_i \hat{g}_{i+1} \hat{g}_i &= X_i g_i (X_{i+1} g_{i+1} - g_{i+1} X_{i+1}) X_i g_i + \dots \\ &= X_i g_i X_i (X_{i+1} g_{i+1}) g_i - X_i g_i (g_{i+1} g_i X_i g_i) X_i g_i + \dots \\ &\vdots \\ &= X_{i+1} g_{i+1} X_i g_i^{-1} X_{i+1} g_{i+1} - g_{i+1} X_{i+1} X_i g_i^{-1} g_{i+1} X_{i+1} + \dots \\ &= X_{i+1} g_{i+1} X_i (g_i - (q - q^{-1})) X_{i+1} g_{i+1} \\ &\quad - g_{i+1} X_{i+1} X_i (g_i - (q - q^{-1})) g_{i+1} X_{i+1} + \dots \\ &= X_{i+1} g_{i+1} X_i g_i X_{i+1} g_{i+1} - g_{i+1} X_{i+1} X_i g_i g_{i+1} X_{i+1} + \dots \\ &= \hat{g}_{i+1} \hat{g}_i \hat{g}_{i+1}. \end{aligned} \tag{54}$$

Let  $K$  be our ground ring (an integral domain), let  $K[X_-]$  be the ring of polynomials in the  $X_i$ s, and let  $K(X_-)$  be the field of fractions. Note that we can write

$$\alpha_i \hat{g}_i = g_i + \beta_i,$$

where both  $\alpha_i$  and  $\beta_i$  lie in  $K(X_-)$ . (Note that in our quotient  $\Psi_d$ , the image of  $K[X_-]$

itself generically, but not always, contains  $\alpha_i$  and  $\beta_i$ . For example, when  $d = 1$  and  $q^2 = 1$ , then  $X_i - X_{i+1}$  is not invertible.) It then follows from equation (53) that any element of the extension of  $H(n)$  by  $K(X_-)$  can be expressed in the form

$$h = \sum_{w \in \mathfrak{B}_n} c_w \hat{g}_w,$$

where  $c_w \in K(X_-)$  and  $\hat{g}_w$  is obtained by putting hats on the generators in  $w$ . Indeed, the extension may be decomposed as

$$\bigoplus_{w \in \mathfrak{B}_n} K(X_-) \hat{g}_w \tag{55}$$

(an induction on the usual *length* function on  $\mathfrak{B}_n$ ).

Suppose that  $h$  has at least one  $w \neq 1$ , with  $c_w \neq 0$  (that is,  $h \notin K(X_-)$  subalgebra). Then there is at least one  $i$  such that  $w(i) \neq i$  (under the obvious generalisation of the  $\sigma_i$  action in equation (53)), and

$$hX_i = \sum_w c_w \hat{g}_w X_i = \sum_w c_w X_{w(i)} \hat{g}_w,$$

so the  $\hat{g}_w$  component of  $hX_i - X_i h$  is  $c_w(X_{w(i)} - X_i) \neq 0$ . Thus

$$Z_{K[X_-]}(H(n)) = K[X_-] \supseteq Z(H(n)).$$

But with  $c \in K[X_-]$ , then  $\hat{g}_w c - c \hat{g}_w = (c^w - c) \hat{g}_w$ , so  $Z(H(n)) = X^\Sigma$ , as Bernstein says.

Naturally,  $Z(H(n, d)) \supseteq \Psi_d(X^\Sigma)$ , depending in principle on the ground ring. The argument above mostly works in this case (to show equality), although the possible specialisations of the ground ring become more restricted ( $\lambda_i \neq 0$ , plus the restrictions already mentioned, for example). Note also that  $\Psi_d(K[X_-])$  is finite-dimensional. This makes it interesting to study  $\Psi_d(X^\Sigma)$  – an algebra that is, in a sense, more complicated than  $X^\Sigma$  itself. For example,  $H(2, 1) \cong H_n$  and  $\dim(Z(H_2) = H_2) = 2$ , so  $\dim(\Psi_1(X^\Sigma)) = 2$ . Here a basis for  $\Psi_1(X^\Sigma)$  is

$$\{1, X + X_2 = \lambda_1(1 + g_1^2)\}.$$

(Note that this example illustrates the problem with  $q^2 = 1$ .)

For another example, recall that  $\dim(Z(H(2, 2))) = 5$  generically, so  $\dim(\Psi_1(X^\Sigma)) = 5$ . Here a basis for  $\Psi_2(X^\Sigma)$  is

$$\{1, X + X_2, X X_2, X^2 + X_2^2, (X + X_2) X X_2\}$$

(this is not supposed to be obvious!). Another basis, convenient for comparison with the basis of  $H(2, 2)$ , is

$$\{1, X + X_2, X X_2, (X + X_2 - (\lambda_1 + \lambda_2))g_1, (X X_2 - \lambda_1 \lambda_2)g_1\}. \tag{56}$$

The set of monomial symmetric polynomials in two variables  $X_1$  and  $X_2$  (a basis for  $Z(H(2))$ ) may be indexed by the set  $\Lambda^2$  of Young diagrams of not more than two rows. (Write  $m^a = (X^a)^\Sigma$ ; then  $m^{(0)} = 1$  and  $m^{(1)} = X_1 + X_2$ , and so on.) Note that the set of such polynomials in which the degree of no individual variable exceeds  $d - 1$ , is *not* a basis for  $\Psi_d(X^\Sigma)$  in general, as we see from the examples above.

## References

1. S. ARIKI, ‘On the decomposition numbers of the Hecke algebra of  $G(m, 1, n)$ ’, *J. Math. Kyoto Univ.* 36 (1996) 789–808. 249, 250, 273
2. S. ARIKI and K. KOIKE, ‘A Hecke algebra of  $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$  and construction of its irreducible representations’, *Adv. Math.* (1994) 216–243. 249, 250, 255, 260
3. S. ARIKI, T. TERASOMA and H. YAMADA, ‘Schur–Weyl reciprocity for the Hecke algebra of  $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$ ’, *J. Algebra* 178 (1995) 374–390. 251
4. R. J. BAXTER, *Exactly solved models in statistical mechanics* (Academic Press, New York, 1982). 278, 291
5. R. J. BAXTER, S. B. KELLAND and F. Y. WU, ‘Equivalence of the Potts model or Whitney polynomial with an ice-type model’, *J. Phys. A* 9 (1976) 397–406. 278
6. R. BEHREND and P. A. PEARCE, ‘Boundary weights for Temperley–Lieb and dilute Temperley–Lieb models’, *Int. J. Mod. Phys. B* 11 (1997) 2833–2847. 278
7. D. J. BENSON, *Representations and cohomology I* (Cambridge University Press, 1995). 252
8. J. S. BIRMAN, *Braids, links and mapping class groups*, Ann. Math. Stud. 82 (Princeton University Press, Princeton, NJ, 1975). 285
9. N. BOURBAKI, *Groupes et algèbres de Lie*, vols 4–6 (Masson, 1981). 251
10. M. BROUÉ and G. MALLE, ‘Zyklotomische heckealgebren’, *Astérisque* 212 (1993) 119–189. 249, 250, 255
11. I. CHEREDNIK, ‘A unification of Knizhnik–Zamolodchikov and Dunkl operators via affine Hecke algebras’, *Invent. Math.* 106 (1991) 411–431. 260
12. E. CLINE, B. PARSHALL and L. SCOTT, ‘Finite dimensional algebras and highest weight categories’, *J. Reine Angew. Math.* 391 (1988) 85–99. 251, 254, 278
13. E. CLINE, B. PARSHALL, and L. SCOTT, ‘Generic and  $q$ -rational representation theory’, *Publ. Res. Inst. Math. Sci.* 35 (1999) 31–90. 254, 255
14. P. M. COHN, *Algebra*, vol. 2 (Wiley, New York, 1982). 266
15. A. G. COX, J. J. GRAHAM and P. P. MARTIN, ‘The blob algebra in positive characteristic’, City University Preprint, 2001; *J. Algebra* 266 (2003) 584–635. 250, 251, 254, 270
16. A. G. COX, P. P. MARTIN and S. RYOM-HANSEN, ‘Virtual algebraic Lie theory II’, in preparation. 276
17. E. DATE, M. JIMBO, A. KUNIBA, T. MIWA and M. OKADO, ‘Exactly solvable SOS models II: proof of the star-triangle relation and combinatorial identities’, *Adv. Stud. Pure Math.* 16 (1988) 17. 288
18. T. DEGUCHI, ‘Braid group representations and link polynomials derived from generalized  $SU(n)$  vertex models’, *J. Phys. Soc. Japan* 58 (1989) 3441. 282
19. T. TOM DIECK, ‘Symmetrische brücken und knotentheorie zu den Dynkin–diagrammen vom type B’, *J. Reine Angew. Math.* 451 (1994) 71–88. 281, 285
20. R. DIPPER, G. JAMES and A. MATHAS, ‘The  $(Q, q)$  Schur algebra’, *Proc. Lond. Math. Soc.* 77 (1998) 327–361. 251
21. V. DLAB and C. M. RINGEL, ‘A construction for quasi-hereditary algebras’, *Compositio Math.* 70 (1989) 155–175. 278

22. S. DONKIN, 'On tilting modules for algebraic groups', *Math. Z.* 212 (1993) 39–60. [250](#), [251](#), [252](#), [278](#), [280](#)
23. S. DONKIN, *The  $q$ -Schur algebra*, London Math. Soc. Lecture Note Ser. 253 (Cambridge University Press, 1998). [250](#)
24. W. FULTON and J. HARRIS, *Representation theory* (Springer, 1991). [258](#)
25. A. M. GARSIA and T. J. McLARNAN, 'Relations between Young's natural and the Kazhdan–Lusztig representations of  $s_n$ ', *Adv. Math.* 69 (1988) 32–92. [254](#)
26. R. GOODMAN and N. R. WALLACH, *Representations and invariants of the classical groups* (Cambridge University Press, 1998). [266](#)
27. J. J. GRAHAM and G. I. LEHRER, 'The representation theory of affine Temperley–Lieb algebras', *Enseign. Math.* 44 (1998) 173–218. [278](#)
28. J. J. GRAHAM and G. I. LEHRER, 'Diagram algebras, Hecke algebras and decomposition numbers at roots of unity', Preprint, 2003. [278](#)
29. J. A. GREEN, *Polynomial representations of  $GL_n$*  (Springer, Berlin, 1980). [254](#)
30. R. M. GREEN, 'Tabular algebras and their asymptotic versions', *J. Algebra* (2001), to appear. [278](#)
31. I. GROJNOWSKI, 'Affine  $\hat{sl}_p$  controls the modular representation theory of the symmetric group and related Hecke algebras', math.RT/9907129, 1999. [249](#), [291](#)
32. M. HAMERMESH, *Group theory* (Pergamon, Oxford, 1962). [255](#), [259](#)
33. P. N. HOEFSMIT, 'Representations of Hecke algebras of finite groups with BN pairs of classical type', Ph.D. thesis, University of British Columbia, 1974. [255](#), [260](#)
34. J. E. HUMPHREYS, *Reflection groups and Coxeter groups* (Cambridge University Press, 1990). [254](#), [278](#)
35. G. D. JAMES and A. KERBER, *The representation theory of the symmetric group* (Addison-Wesley, London, 1981). [255](#)
36. J. C. JANTZEN, *Representations of algebraic groups* (Academic Press, 1987). [249](#), [250](#), [251](#), [252](#), [254](#)
37. M. JIMBO, 'A  $q$ -difference analogue of  $U(\mathfrak{g})$  and the Yang–Baxter equation', *Lett. Math. Phys.* 10 (1985) 63–69. [279](#)
38. V. F. R. JONES, 'A quotient of the affine Hecke algebra in the Brauer algebra', *Enseign. Math.* 40 (1994) 313–344. [278](#)
39. A. JOSEPH, *Quantum groups and their primitive ideals* (Springer, 1995). [280](#)
40. M. KASHIWARA, T. MIWA and E. STERN, 'Decomposition of  $q$ -deformed Fock spaces', q-alg/9508006, 1995. [278](#)
41. D. KAZHDAN and G. LUSZTIG, 'Representations of Coxeter groups and Hecke algebras', *Invent. Math.* 53 (1979) 165–184. [254](#)
42. S. S. F. LAMBROPOULOU, 'Solid torus links and Hecke algebras of B type', *Proceedings of the Quantum Topology* (World Scientific, Singapore, 1994) 225. [255](#)
43. A. LASCOUX, B. LECLERC and J.-Y. THIBON, 'Hecke algebras at roots of unity and crystal bases of quantum affine algebras', *Commun. Math. Phys.* 181 (1996) 205–263. [250](#)
44. G. LUSZTIG, 'Affine Hecke algebras and their graded version', *J. Amer. Math. Soc.* 2 (1989) 599–685. [255](#)

45. I. MACDONALD, *Symmetric functions and Hall polynomials* (Oxford University Press, 1979). 255
46. W. MAGNUS, A. KARRAS and S. SOLITAR, *Combinatorial group theory* (Wiley, 1966). 285
47. P. P. MARTIN, ‘Analytic properties of the partition function for statistical mechanical models’, *J. Phys. A* 19 (1986) 3267–3277. 278
48. P. P. MARTIN, ‘Block spin transformations in the operator formulation of two-dimensional Potts models’, *J. Phys. A* 22 (1989) 3991–4005. 280, 286
49. P. P. MARTIN, ‘String-like lattice models and Hecke algebras’, *J. Phys. A* 22 (1989) 3103–3112. 255
50. P. P. MARTIN, *Potts models and related problems in statistical mechanics* (World Scientific, Singapore, 1991). 260, 261, 269, 278, 279, 285, 286, 287, 288
51. P. P. MARTIN, ‘A faithful tensor space representation for the blob algebra’, Preprint, 2002. 283
52. P. P. MARTIN and D. LEVY, ‘Hecke algebra solutions to the reflection equations’, *J. Phys. A* 27 (1994) L521–L526. 250
53. P. P. MARTIN and S. RYOM-HANSEN, ‘Virtual algebraic Lie theory: tilting modules and Ringel duals for blob algebras’, *Proc. London Math. Soc.*, to appear; Math. RT/0210063. 250, 283
54. P. P. MARTIN and H. SALEUR, ‘On an algebraic approach to higher dimensional statistical mechanics’, *Comm. Math. Phys.* 158 (1993) 155–190. 278
55. P. P. MARTIN and H. SALEUR, ‘The blob algebra and the periodic Temperley–Lieb algebra’, *Lett. Math. Phys.* 30 (1994) 189–206. 249, 250, 258, 269, 278, 280, 283, 286
56. P. P. MARTIN and D. WOODCOCK, ‘On quantum spin-chain spectra and the structure of Hecke algebras and  $q$ -groups at roots of unity’, *J. Phys. A* 31 (1998) 10131–10154. 250, 253, 256
57. P. P. MARTIN and D. WOODCOCK, ‘On the structure of the blob algebra’, *J. Algebra* 225 (2000) 957–988. 249, 250, 251, 252, 253, 258, 269, 278
58. P. P. MARTIN, D. WOODCOCK and D. LEVY, ‘A diagrammatic approach to Hecke algebras of the reflection equation’, *J. Phys. A* 33 (2000) 1265–1296. 249, 251, 253, 255, 256, 260, 273, 274, 276, 278
59. A. MATHAS, ‘Canonical bases and the decomposition matrices of Ariki–Koike algebras’, Preprint, 1996. 269, 273
60. V. PASQUIER and H. SALEUR, ‘Common structures between finite systems and conformal field theories through quantum groups’, *Nucl. Phys. B* 330 (1990) 523. 278
61. C. M. RINGEL, ‘The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences’, *Math. Zeit.* 208 (1991) 209–225. 250
62. J. D. ROGAWSKI, ‘On modules over the Hecke algebra of a  $p$ -adic group’, *Invent. Math.* 79 (1985) 443–465. 260
63. S. SAKAMOTO and T. SHOJI, ‘Schur–Weyl reciprocity for Ariki–Koike algebras’, *J. Algebra* 221 (1999) 293–314. 251, 291
64. W. SOERGEL, ‘Kazhdan–Lusztig polynomials and a combinatoric for tilting modules’, *Representation Theory* 1 (1997) 83–114. 249, 250, 252

65. W. SOERGEL, 'Charakterformeln für Kipp-Moduln über Kac-Moody-Algebren', *Representation Theory* 1 (1997) 115–132. 249, 250
66. R. P. STANLEY, *Enumerative combinatorics* (Cambridge University Press, 1997). 278
67. D. STANTON and D. WHITE, *Constructive combinatorics*, Undergrad. Texts in Math. (Springer, New York, 1986). 278
68. H. N. V. TEMPERLEY and E. H. LIEB, 'Relations between percolation and colouring problems and other graph theoretical problems associated with regular planar lattices: some exact results for the percolation problem', *Proc. Royal Soc. A* 322 (1971) 251–280. 278
69. A. M. VERSHIK and A. Y. OKUNKOV, 'An inductive method of expounding the representation theory of symmetric groups', *Russian Math. Surveys* 51 (1996) 1237–1239. 254

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