PERIOD DOUBLING WITH HYSTERESIS IN BL HER-TYPE MODELS

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1. Hydrodynamics

We have performed recently a survey of the nonlinear hydrodynamical models of the BL Her-type variables (Buchler & Moskalik 1992). Within this project we have studied several sequences of models, *i.e.*, families in which *only* T_{eff} has been varied from model to model, while all other stellar parameters have been kept constant. The fundamental mode pulsations of each model have been converged to strict periodicity with the relaxation code (Stellingwerf 1974). Such approach speeds up the calculations and simultaneously yields information about the stability properties of the resulting limit cycles. In all studied sequences except one, we have found a narrow range of T_{eff} (typically 100–150K), in which regular solution becomes unstable towards a period doubling bifurcation. The instability has its origin in a half-integer resonance, namely the 3:2 coupling between the fundamental mode and the first overtone (*cf.* Moskalik & Buchler 1990; hereafter MB90). This is the same resonance, which also causes period doubling in the models of classical Cepheids (Moskalik & Buchler 1991). The bifurcation leads to stable *period-two* oscillations, characterized by an RV Tau-like, albeit *strictly periodic* behavior of all variables. In other words, the pulsation light curves and velocity curves will exhibit two alternating minima (as well as maxima) of different values.

In Fig. 1 we show the bifurcation for one of our sequences, by plotting minimum pulsational velocities V_{min} versus T_{eff}. Filled (open) circles correspond to stable (unstable) period-one limit cycles, while asterisks represent differing minima of the period-two solutions. The latter are always stable. The alternations are very pronounced and can reach up to 23.4 km/s. Their size gradually decreases as the edges of the period doubling window are approached. On the low temperature edge of the instability domain, though, the alternations clearly *do not vanish*. This curious behavior prompted us to perform a detailed study of this temperature range.

In Fig. 2 we display the behavior of the models with T_{eff} around 5800K.

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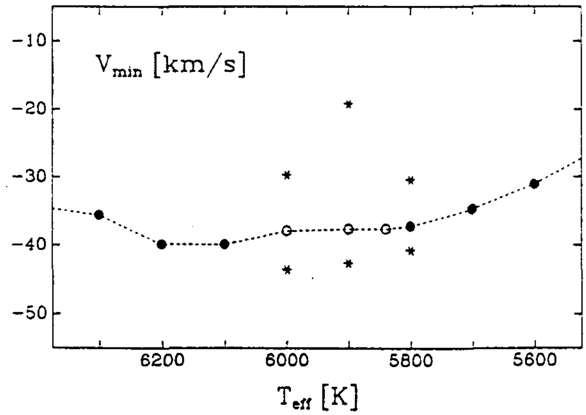

Fig. 1. Minimum pulsational velocities (corrected for the limb darkening) *vs.* $T_{\rm eff}$ for sequence of models with $M = 0.55 M_\odot$, $L = 125 L_\odot$, $X = 0.7$ and $Z = 0.001$. Circles correspond to period-one limit cycles. Solutions, which are unstable towards period doubling are marked with open symbols. Asterisks represent (alternating) minima of the stable period-two cycles.

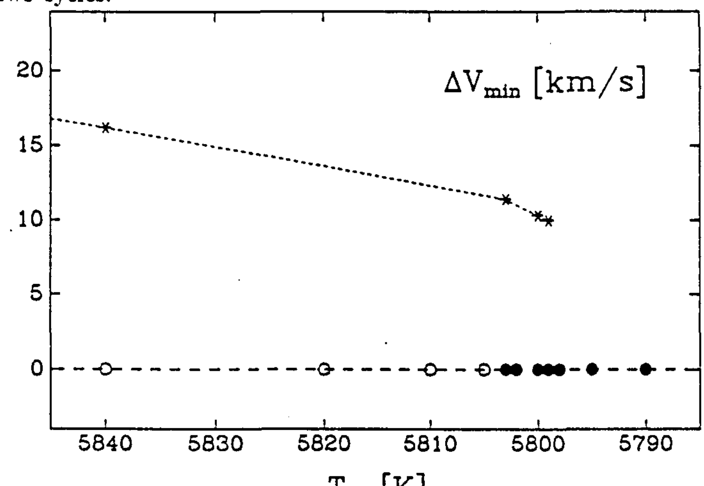

Fig. 2. Differences between velocity minima (corrected for the limb darkening) *vs.* $T_{\rm eff}$ for sequence of Fig. 1. All symbols have the same meaning as in Fig. 1. The period-one limit cycles change their stability at $T_{\rm eff} \simeq 5805$K. For $T_{\rm eff} \in (5799\text{K}–5805\text{K})$ a hysteresis occurs.

For better visualization we plot here the *differences* between consecutive velocity minima, $\Delta V_{\rm min}$, again *versus* $T_{\rm eff}$ of the model. The period-one pulsations (circles), correspond now to $\Delta V_{\rm min} = 0$, since their minima are all equal. The period-two limit cycles (asterisks), on the other hand, correspond to $\Delta V_{\rm min} > 0$. We see that at the point of the stability exchange ($T_{\rm eff} \simeq$ 5805K) the alternations indeed do not decrease to zero. Instead, the branch of the period-two solutions extends beyond that point and for 5799K$\leq T_{\rm eff} <$

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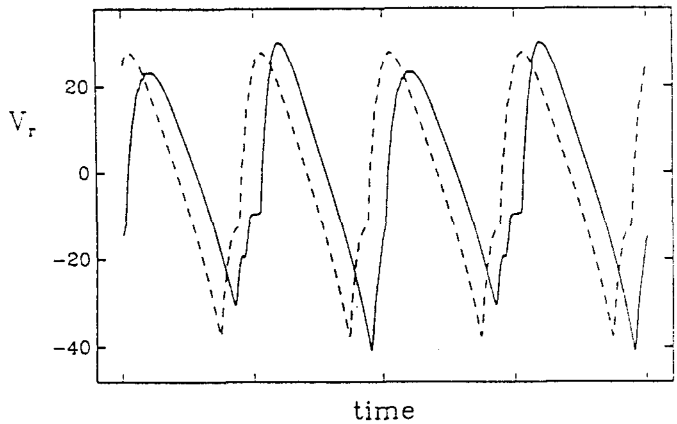

Fig. 3. Pulsational velocity curves (corrected for the limb darkening) for the model with $M = 0.55M_{\odot}$, $L = 125L_{\odot}$, $X = 0.7$, $Z = 0.001$ and $T_{\text{eff}} = 5800$K; *solid line*: the period-two limit cycle; *dashed line*: the period-one limit cycle. Both solutions are stable.

5805K two different stable solutions can *coexits*: the period-one limit cycle and the period-two limit cycle. That means that a *hysteresis* occurs over that narrow interval of T_{eff} and that the pulsational state of the star will depend here on its evolutionary history. We have checked that this hysteresis is robust with respect to the numerical parameters (*i.e.*, number of timesteps) and it is certainly not a computational artifact. In Fig. 3 we plot the velocity curves of the two different stable solutions for the model of $T_{\text{eff}} = 5800$K. Both curves are quite similar, nevertheless the alternations in one of them are clearly visible, whereas the other one repeats after each cycle.

2. Amplitude Equations

The behavior of the hydrodynamical stellar models and their bifurcations can be captured qualitatively and quantitatively by the amplitude equations (*cf.* Buchler 1992). The apposite set of equations for the 3:2 resonance case has the form (MB90)

$$\frac{dA}{dt} = \kappa_0 A + Re Q_0 A^3 + Re T_0 A B^2 + Re(\Pi_0 e^{i\Gamma}) A^2 B^2$$
$$\frac{dB}{dt} = \kappa_1 B + Re Q_1 B^3 + Re T_1 A^2 B + Re(\Pi_1 e^{-i\Gamma}) A^3 B$$
$$\frac{d\Gamma}{dt} = 2\Delta\omega + Im(2T_1 - 3Q_0)A^2 + Im(2Q_1 - 3T_0)B^2 + \\ + 2Im(\Pi_1 e^{-i\Gamma})A^3 - 3Im(\Pi_0 e^{i\Gamma})AB^2 \quad (1)$$

where A and B are the (real) amplitudes of the fundamental mode and of the first overtone, and the quantity Γ is a linear combination of the respective modal phases: $\Gamma = 2\phi_1 - 3\phi_0$. The coefficients Q_j and T_j describe the

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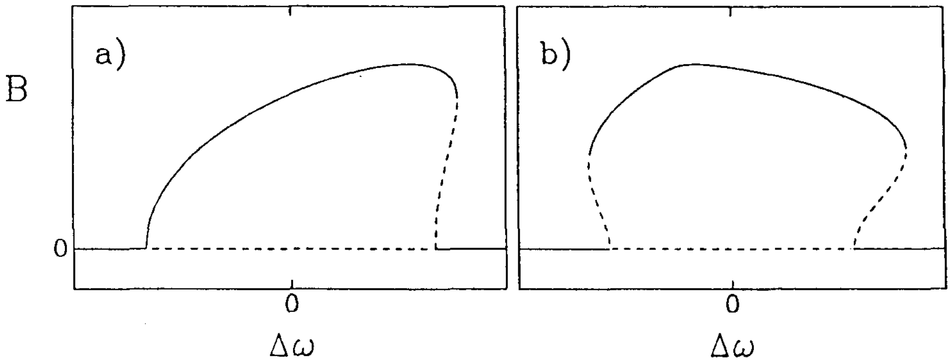

Fig. 4. Amplitude of the first overtone, B, for the fixed points of Eqs.(1) *vs.* resonance distance parameter $\Delta\omega$. Adopted parameters are: (a)$\Pi_0 = 0$, $\kappa_1 = \kappa_0$, $ReQ_0 = -2^{-1/3}\kappa_0^{1/3}|\Pi_1|^{2/3}$, $ReT_0 = 0$, $ReT_1 = -2^{2/3}\kappa_0^{1/3}|\Pi_1|^{2/3}$, $Im(2T_1 - 3Q_0) = 0$, $Im(2Q_1 - 3T_0) = 4ReQ_1$ and (b) $\kappa_1 = 3\kappa_0$, $ReQ_0 = -2^{-1/3}\kappa_0^{1/3}|\Pi_1|^{2/3}$, $ReT_0 = 0$, $ReQ_1 = -5 \times 2^{2/3}\kappa_0^{1/3}|\Pi_0||\Pi_1|^{-1/3}$, $ReT_1 = -2^{2/3}\kappa_0^{1/3}|\Pi_1|^{2/3}$, $Im(2T_1 - 3Q_0) = 0$, $Im(2Q_1 - 3T_0) = 0$, $\arg(\Pi_0\Pi_1) = 2.5$.

nonresonant nonlinear effects (namely saturation of the driving mechanism) and the coupling coefficients Π_j measure the strength of the resonant interaction. The driving rates κ_j as well as $\Delta\omega = \omega_1 - \frac{3}{2}\omega_0$ are given by the linear pulsation theory. The last parameter measures the distance to the resonance center. Within the framework of the amplitude equation formalism, the surface radius displacement can be expressed as

$$\delta R(t) = Ae^{i\omega_0 t} + Be^{i\frac{\Gamma}{2}}e^{\frac{3}{2}i\omega_0 t} + higher\ order\ terms. \tag{2}$$

According to this formula, a fixed point of Eqs. (1) corresponds to a limit cycle oscillation of the stellar model. In particular, the fixed point with $B \neq 0$ represents the period-two cycle, since $\delta R(t)$ varies then with the period of $2P_0 (P_0 = 2\pi/\omega_0)$. To the lowest order, the amplitude B measures the size of alternations in such a cycle. The solution with $B = 0$ corresponds to the period-one oscillations.

The fixed points of Eqs.(1) and their stability has been discussed in MB90. The authors have shown that the equations can reproduce very well the period doubling behavior found in the classical Cepheid models. In those models, though, the bifurcation is of the supercritical type, and the alternations in the period-two solutions vanish at the points of the stability exchange. It is interesting to check whether Eqs.(1) can also capture the subcritical bifurcation (*i.e.*, hysteresis) encountered in the BL Her models.

Fig. 4 shows that Eqs.(1) are indeed capable of reproducing such a behavior. The amplitude of the first overtone, B, has been calculated and plotted here as a function of $\Delta\omega$, with all other coefficients in the equations kept constant. Solutions displayed in Fig. 4a have been obtained for the simplified

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system with $ReT_0 = 0$ and $\Pi_0 = 0$, although in contrast to MB90 we have assumed that $Im(2T_1 - 3Q_0)$ and $Im(2Q_1 - 3T_0)$ are in general nonzero. The unstable fixed points are marked with the dashed lines. We see, that for $\Delta\omega > 0$, which corresponds to the low temperature side of the period doubling window, the period-two solution ($B \neq 0$) extends beyond the bifurcation point, bending back subsequently to join that point through the unstable branch. Thus, over the narrow interval of $\Delta\omega$ the stable period-two and period-one solutions coexist, just like in the hydrodynamical models. According to the amplitude equations, they are accompanied by the *unstable* period-two solution. In the simplified case considered in Fig. 4a it can be shown analytically, that the hysteresis will occur only if

$$|Im(2Q_1 - 3T_0)| > 2ReQ_1 \frac{\kappa_1 + ReT_1 A_0^2}{\sqrt{|\Pi_1|^2 A_0^6 - (\kappa_1 + ReT_1 A_0^2)^2}} \tag{3}$$

where $A_0^2 = -\kappa_0/ReQ_0$. Depending on the sign of $Im(2Q_1 - 3T_0)$, it will occur either for positive for negative $\Delta\omega$, but never for both. This latter property does not hold in the general case (*i.e.*, $ReT_0 \neq 0$, $\Pi_0 \neq 0$), when the hysteresis can be found on both sides of the period doubling window. An example of such a behavior is presented in Fig. 4b.

References

Buchler, J. R.: 1992, this conference.
Buchler, J. R. and Moskalik, P.: 1992, *Astrophysical Journal*, in press.
Moskalik, P. and Buchler, J. R.: 1990, *Astrophysical Journal* **355**, 590 (MB90).
Moskalik, P. and Buchler, J. R.: 1991, *Astrophysical Journal* **366**, 300.
Stellingwerf, R. A.: 1974, *Astrophysical Journal* **192**, 139.

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