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ABSTRACT

We determine the real counting function $N(q)$ ($q \in [1, \infty)$) for the hypothetical ‘curve’ $C = \overline{\text{Spec } \mathbb{Z}}$ over \mathbb{F}_1 , whose corresponding zeta function is the complete Riemann zeta function. We show that such a counting function exists as a distribution, is positive on $(1, \infty)$ and takes the value $-\infty$ at $q = 1$ as expected from the infinite genus of C . Then, we develop a theory of functorial \mathbb{F}_1 -schemes which reconciles the previous attempts by Soulé and Deitmar. Our construction fits with the geometry of monoids of Kato, is no longer limited to toric varieties and it covers the case of schemes associated with Chevalley groups. Finally we show, using the monoid of adèle classes over an arbitrary global field, how to apply our functorial theory of \mathfrak{M} -schemes to interpret conceptually the spectral realization of zeros of L -functions.

1. Introduction

In this paper we develop three correlated aspects pertaining to the broad theory of \mathbb{F}_1 . The appearance, in the printed literature, of some explicit remarks related to this (hypothetical) degenerate algebraic structure is due to Tits, who proposed its existence to explain the limit case of the algebraic structure underlying the geometry of a Chevalley group over a finite field \mathbb{F}_q , as q tends to 1 (cf. [Tit57, § 13] and [CC08]). A suggestive comment pointing out to a finite geometry inherent to the limit case $q = 1$ is also contained in an earlier paper by Steinberg (cf. [Ste51, p. 279]), in relation to a geometric study of the representation theory of the general linear group over a finite field.

In more recent years, the classical point of view that adjoining roots of unity is analogous to producing extensions of a base field has also been applied in the process of developing a suitable arithmetic theory over \mathbb{F}_1 . This idea leads to the introduction of the notion of algebraic field extensions \mathbb{F}_{1^n} of \mathbb{F}_1 which are not defined *per se*, but are described by the following equation (cf. Kapranov and Smirnov [KS] and Soulé [Sou04, § 2.4])

$$\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} := \mathbb{Z}[T]/(T^n - 1), \quad n \in \mathbb{N}.$$

The need for the ‘absolute point’ $\text{Spec } \mathbb{F}_1$ has also emerged in Arakelov’s geometry, especially in the context of an absolute motivic interpretation of the zeros of zeta and L -functions (cf. Manin [Man95, § 1.5]). In [Sou04, § 6], Soulé introduced the zeta function of a variety X over \mathbb{F}_1 by considering the *polynomial integer* counting function of the associated functor \underline{X} .

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In this paper we take up the following central question formulated in [Man95] which originally motivated the development of the study of the arithmetic over \mathbb{F}_1 .

Question. Can one find a ‘curve’ $C = \overline{\text{Spec } \mathbb{Z}}$ over \mathbb{F}_1 (defined in a suitable sense) whose zeta function $\zeta_C(s)$ is the complete Riemann zeta function $\zeta_{\mathbb{Q}}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$?

After transforming the limit definition (for $q \rightarrow 1$) of the zeta function given in [Sou04] into an integral formula, which is more suitable in the cases of general types of counting functions and distributions, we show how to determine the *real* counting distribution $N_C(q) = N(q)$, $q \in [1, \infty)$, associated with the hypothetical curve C over \mathbb{F}_1 .

A convincing solution to this problem is a fundamental preliminary test for any arithmetic theory over \mathbb{F}_1 . The difficulty inherent to the above question can be easily understood by considering the following facts. First of all, note that the value $N(1)$ is conjectured to take the meaning of the Euler characteristic of the curve $C = \overline{\text{Spec } \mathbb{Z}}$. Since one expects C to be of infinite genus (cf. [Man95]), $N(1)$ is supposed to take the value $-\infty$, thus precluding any easy use of the limit definition of the zeta and a naive approach to the definition of C , by generalizing the constructions of [Sou04]. On the other hand, the counting function $N(q)$ is also supposed to be positive for q real, $q > 1$, since it should detect the cardinality of the set of points of C defined over various ‘field extensions’ of \mathbb{F}_1 . This requirement creates an apparent contradiction with the earlier condition $N(1) = -\infty$.

The precise statement of our result (cf. Theorem 2.2 and Remark 2.3) is as follows.

THEOREM 1.1.

- (1) *The counting function $N(q)$ satisfying the above requirements exists as a distribution and is given by the formula*

$$N(q) = q - \frac{d}{dq} \left(\sum_{\rho \in Z} \text{order}(\rho) \frac{q^{\rho+1}}{\rho + 1} \right) + 1 \tag{1}$$

where Z is the set of non-trivial zeros of the Riemann zeta function and the derivative is taken in the sense of distributions.

- (2) *The function $N(q)$ is positive (as a distribution) for $q > 1$.*
- (3) *The value $N(1)$ is equal to $-\infty$ and reflects precisely the distribution of the zeros of zeta in $E \log E$.¹*

This result supplies a strong indication on the coherence of the quest for an arithmetic theory over \mathbb{F}_1 . For an irreducible, smooth and projective algebraic curve X over a prime field \mathbb{F}_p , the counting function is of the form

$$\#X(\mathbb{F}_q) = N(q) = q - \sum_{\alpha} \alpha^r + 1, \quad q = p^r,$$

where the α are the complex roots of the characteristic polynomial of the Frobenius acting on the étale cohomology $H^1(X \otimes \overline{\mathbb{F}}_p, \mathbb{Q}_{\ell})$ of the curve ($\ell \neq p$). By writing these roots in the form $\alpha = p^{\rho}$, for ρ a zero of the Hasse–Weil zeta function of X , the above equality reads as

$$\#X(\mathbb{F}_q) = N(q) = q - \sum_{\rho} \text{order}(\rho)q^{\rho} + 1. \tag{2}$$

¹ Here $E = 1/\epsilon$, $\epsilon > 0$, appears when taking the derivative $\lim_{\epsilon \rightarrow 0} ((J(1 + \epsilon) - J(1))/\epsilon)$ of the primitive J of $N(q)$.

Equations (1) and (2) are now completely identical, except for the fact that in (2) the values of q are restricted to the discrete set of powers of p and that (2) involves only a finite sum, which allows one to differentiate term by term.

Equation (1) is a typical application of the Riemann–Weil explicit formulae. These formulae become natural when lifted to the idèle class group. This fact supports the expectation that, even if a definition of the hypothetical curve C is at this time still out of reach, its counterpart, through the application of the class-field theory isomorphism, can be realized by a space of adelic nature and this is in agreement with earlier constructions of Connes *et al.* (cf. [CCM07, CCM08, CCM09]).

A second topic that we develop in this paper is centered on the definition of a suitable geometric theory of algebraic schemes over \mathbb{F}_1 . The viewpoint that we introduce in this article is an attempt at unifying the theories developed on the one side in [Sou04] and in our paper [CC08], and on the other side by Deitmar in [Dei05, Dei08] (following Kurokawa *et al.* [KOW03]), by Kato in [Kat94] (with the geometry of logarithmic structures) and by Töen and Vaquié in [TV09].

In [CC08], we introduced a refinement of the original notion (cf. [Sou04]) of an affine variety over \mathbb{F}_1 , and following this path we proved that Chevalley group schemes are examples of affine varieties defined over the field extension \mathbb{F}_{1^2} . While in the process of assembling this construction, we realized that the functors (from finite abelian groups to graded sets) describing these affine schemes fulfill stronger properties than those required in [Sou04]. In this paper we develop this approach and show that the functors underlying the structure of the most common examples of schemes (of finite type) over \mathbb{F}_1 extend from (finite) abelian groups to a larger category obtained by gluing together the category $\mathfrak{M}\mathfrak{o}$ of commutative monoids (used in [Dei05, Kat94, KOW03, TV09]) with the category $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$ of commutative rings. This process uses a natural pair of adjoint functors relating $\mathfrak{M}\mathfrak{o}$ to $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$ and follows an idea we learnt from P. Cartier. The resulting category $\mathfrak{M}\mathfrak{R}$ (cf. §4 for details) defines an ideal framework in which the above two approaches are combined together to determine a very natural notion of variety (and of scheme) \mathcal{X} over \mathbb{F}_1 . In particular, the conditions imposed in the original definition of a variety over \mathbb{F}_1 in [Sou04] are now applied to a covariant functor $\mathcal{X} : \mathfrak{M}\mathfrak{R} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}$ to the category of sets. Such a functor determines a scheme (of finite type) over \mathbb{F}_1 if it also fulfills the following three properties (cf. Definition 4.7).

- (i) The restriction $X_{\mathbb{Z}}$ of \mathcal{X} to $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$ is a scheme in the sense of [DG70].
- (ii) The restriction \underline{X} of \mathcal{X} to $\mathfrak{M}\mathfrak{o}$ is locally representable.
- (iii) The natural transformation connecting \underline{X} to $X_{\mathbb{Z}}$, when applied to a field, yields a bijection (of sets).

The category $\mathfrak{A}\mathfrak{b}$ of abelian groups embeds as a full subcategory in $\mathfrak{M}\mathfrak{o}$. This fact allows one, in particular, to restrict a covariant functor from $\mathfrak{M}\mathfrak{o}$ to sets to the subcategory (isomorphic to) $\mathfrak{A}\mathfrak{b}$. In §3.7 we prove that if the $\mathfrak{M}\mathfrak{o}$ -functor is locally representable, then the restriction to $\mathfrak{A}\mathfrak{b}$ yields a functor to *graded* sets. This result shows that the grading structure that we assumed in [CC08] is now derived as a byproduct of this new refined approach.

In particular, we deduce that Chevalley groups are \mathbb{F}_{1^2} -schemes in our new sense; the group law exists on the set of points of *lowest degree* and is given by Tits' functorial construction of the normalizer of a maximal split torus.

As an arithmetic application of our new theory of \mathbb{F}_1 -schemes, we compute the zeta function of a Noetherian \mathbb{F}_1 -scheme \mathcal{X} . Theorem 4.13 extends Theorem 1 of [Dei06] beyond the toric case² and states the following results.

THEOREM 1.2. *Let \mathcal{X} be a Noetherian \mathbb{F}_1 -scheme which is locally torsion free.*

(a) *There exists a polynomial $N(x + 1)$ with positive integral coefficients such that*

$$\#\underline{X}(\mathbb{F}_{1^n}) = N(n + 1) \quad \text{for all } n \in \mathbb{N}.$$

(b) *For each finite field \mathbb{F}_q , the cardinality of the set of points of the \mathbb{Z} -scheme $X_{\mathbb{Z}}$ which are rational over \mathbb{F}_q is equal to $N(q)$.*

(c) *The zeta function of \mathcal{X} in the sense of [Sou04] is given by*

$$\zeta_{\mathcal{X}}(s) = \prod_{x \in X} \frac{1}{(1 - (1/s))^{\otimes n(x)}}.$$

Here, the \otimes -product is the Kurokawa tensor product and $n(x)$ denotes the local dimension at the point $x \in X$ of the geometric realization of \underline{X} (cf. Definition 3.23).

The study of the zeta function of an arbitrary Noetherian \mathbb{F}_1 -scheme, i.e. when the ‘no torsion’ hypothesis is removed is developed in our forthcoming paper [CC09].

The geometric theory of schemes over \mathbb{F}_1 that we have developed in §§ 3 and 4 also reveals the importance to replace, when necessary, an abelian group H by a naturally associated commutative monoid M (with a zero element), so that $H = M^\times$ is interpreted as the group of invertible elements in the monoid. This idea applies, in particular, to the idèle class group $C_{\mathbb{K}}$ of a global field \mathbb{K} , since by construction $C_{\mathbb{K}}$ is the group of invertible elements in the multiplicative monoid of the adèle classes

$$M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^\times, \quad \mathbb{K}^\times = \text{GL}_1(\mathbb{K}). \tag{3}$$

This application of the theory of \mathfrak{Mo} -schemes to the study of geometric objects pertinent to the realm of non-commutative geometry determines the third aspect of the theory of \mathbb{F}_1 that we have developed in this paper. In our previous work, the adèle class space has been considered mostly as a non-commutative space and its algebraic structure as a monoid did not play any role. One of the goals of the present paper is to promote this additional structure by pointing out how and where it provides a precious guide.

In § 5, we consider the particular case of the \mathfrak{Mo} -scheme $\mathbb{P}_{\mathbb{F}_1}^1$ describing a projective line over \mathbb{F}_1 . It turns out that this scheme provides a perfect geometric framework to understand, simultaneously and at a conceptual level, the spectral realization of zeros of L -functions, the functional equation and the explicit formulae. All of these statements are deduced by simply computing the cohomology of a natural sheaf Ω of functions on the set $\mathbb{P}_{\mathbb{F}_1}^1(M)$: the projective adèles class space. The sheaf Ω is a sheaf of complex vector spaces over the geometric realization $\mathbb{P}_{\mathbb{F}_1}^1$ of the \mathfrak{Mo} -scheme $\mathbb{P}_{\mathbb{F}_1}^1$. To define this sheaf, we use a specific property of an \mathfrak{Mo} -scheme, namely the existence, for each monoid M , of a natural projection $\pi_M : \underline{X}(M) \rightarrow X$, connecting the \mathfrak{Mo} -scheme \underline{X} (understood as a functor from the category \mathfrak{Mo} of monoids to sets) to its associated geometric space X , that is, its geometric realization. For the \mathfrak{Mo} -scheme $\mathbb{P}_{\mathbb{F}_1}^1$ the geometric realization $\mathbb{P}_{\mathbb{F}_1}^1$ is a very simple space [Dei05] which consists of three points

$$\mathbb{P}_{\mathbb{F}_1}^1 = \{0, u, \infty\}, \quad \overline{\{0\}} = \{0\}, \quad \overline{\{u\}} = \mathbb{P}_{\mathbb{F}_1}^1, \quad \overline{\{\infty\}} = \{\infty\}.$$

² The affine varieties underlying Chevalley groups (e.g. $\text{SL}(2)$) are not toric varieties in general.

A striking fact is that in spite of the apparent simplicity of this space, the computation of $H^0(\mathbb{P}_{\mathbb{F}_1}^1, \Omega)$ already yields the graph of the Fourier transform (Lemma 5.3). Thus, while the singularity of the operation $x \mapsto x^{-1}$ on the space of adèles prevents one from obtaining any interesting global function on the projective space of the adèles, this difficulty disappears at the level of the quotient space M of the adèle classes. In fact, the Fourier transform at the level of the adèles depends upon the choice of a basic character, but this dependence disappears at the level of the quotient space M of adèle classes, that is, for its action on the space $\mathcal{S}(\mathbb{A}_{\mathbb{K}})/\overline{\{f - f_q\}}$ of coinvariants for the action of \mathbb{K}^\times on $\mathcal{S}(\mathbb{A}_{\mathbb{K}})$. Finally, the first cohomology group $H^1(\mathbb{P}_{\mathbb{F}_1}^1, \Omega)$ of the sheaf Ω over $\mathbb{P}_{\mathbb{F}_1}^1$ of complex valued functions on $\mathbb{P}_{\mathbb{F}_1}^1(M)$, provides the space of the spectral realization of the zeros of L -functions (Theorem 5.5). The complete result is stated as follows.

THEOREM 1.3. *The cohomology $H^0(\mathbb{P}_{\mathbb{F}_1}^1, \Omega)$ is given, up to a finite-dimensional space, by the graph of the Fourier transform acting on coinvariants.*

The spectrum of the natural action of the idèle class group $C_{\mathbb{K}}$ on the cohomology $H^1(\mathbb{P}_{\mathbb{F}_1}^1, \Omega)$ gives the spectral realization of zeros of Hecke L -functions.

The symmetry associated with the functional equation is derived as a simple consequence of the inversion $x \mapsto x^{-1}$ holding on $\mathbb{P}_{\mathbb{F}_1}^1$. Finally, we want to stress the point that the most interesting aspect of this final result does not rely on its technical part, since for instance the aforementioned spectral realization is identical to that obtained in several earlier works (cf. [CCM07, CM08, Mey05]) and initiated in [Con99]. The novelty of our statement is that of proposing a new conceptual explanation for some fundamental constructions of non-commutative arithmetic geometry, in a way that the Fourier transform, the Poisson formula and the cokernel of the restriction map to the idèles all appear in an effortless and natural manner on the projective line $\mathbb{P}_{\mathbb{F}_1}^1(M)$, $M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^\times$.

2. Zeta functions over \mathbb{F}_1 and $C = \overline{\text{Spec } \mathbb{Z}}$

In [Sou04] (cf. §6), Soulé introduced the zeta function of a variety X over \mathbb{F}_1 using the *polynomial* counting function $N(x) \in \mathbb{Z}[x]$ of the associated functor \underline{X} . After correcting a sign misprint (which is also reproduced in [Dei06]), the precise definition of the zeta function is as follows

$$\zeta_X(s) := \lim_{q \rightarrow 1} Z(X, q^{-s})(q-1)^{N(1)}, \quad s \in \mathbb{R}, \quad (4)$$

where $Z(X, q^{-s})$ denotes the evaluation at $T = q^{-s}$ of the Hasse–Weil exponential series

$$Z(X, T) := \exp\left(\sum_{r \geq 1} N(q^r) \frac{T^r}{r}\right). \quad (5)$$

Note, incidentally, that the function $\zeta_X(s)$ as in (4) fulfills the properties of an absolute motivic zeta function as predicted by Manin in [Man95] (cf. §1.5).

In this section we first transform the limit (4) into an integral formula, since this latter description is more suitable when one works with general counting functions and distributions. Then, we shall determine a precise formula for the counting function associated to the hypothetical curve $C = \overline{\text{Spec } (\mathbb{Z})}$.

2.1 An integral formula for $\partial_s \zeta_N(s)/\zeta_N(s)$

Let $N(q)$ be a real-valued continuous function on $[1, \infty)$ satisfying a polynomial bound $|N(q)| \leq Cq^k$, for some finite positive integer k and a fixed positive constant C . Then, the corresponding generating function takes the following form

$$Z(q, T) = \exp\left(\sum_{r \geq 1} N(q^r) \frac{T^r}{r}\right)$$

and one knows that the power series $Z(q, q^{-s})$ converges for $\Re(s) > k$. The zeta function over \mathbb{F}_1 associated with $N(q)$ is defined as follows

$$\zeta_N(s) := \lim_{q \rightarrow 1} Z(q, q^{-s})(q - 1)^\chi, \quad \chi = N(1).$$

This definition requires some care to ensure its convergence. To eliminate the ambiguity in the extraction of the finite part, one works with the logarithmic derivative

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = -\lim_{q \rightarrow 1} F(q, s), \tag{6}$$

where

$$F(q, s) = -\partial_s \sum_{r \geq 1} N(q^r) \frac{q^{-rs}}{r}. \tag{7}$$

LEMMA 2.1. *With the above notation and for $\Re(s) > k$, one has*

$$\lim_{q \rightarrow 1} F(q, s) = \int_1^\infty N(u)u^{-s} d^*u, \quad d^*u = du/u \tag{8}$$

and

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = -\int_1^\infty N(u)u^{-s} d^*u. \tag{9}$$

Proof. The proof follows immediately by noting that

$$F(q, s) = \sum_{r \geq 1} N(q^r) q^{-rs} \log q$$

is a Riemann sum for the integral $\int_1^\infty N(u)u^{-s} d^*u$. □

Let us now assume that $N(q)$ is smooth at the point $q = 1$. Then, if $N(1) = 0$ we integrate in s and we obtain the following expression (c is a constant of integration)

$$\log(\zeta_N(s)) = \int_1^\infty \frac{N(u)}{\log u} u^{-s} d^*u + c. \tag{10}$$

If $N(1) \neq 0$, one has to choose a principal value in the expression (10) near $u = 1$, since the term $N(u)/\log u$ is singular. The normalization used in [Sou04] corresponds to the principal value

$$\log(\zeta_N(s)) = \lim_{\epsilon \rightarrow 0} \left(\int_{1+\epsilon}^\infty \frac{N(u)}{\log u} u^{-s} d^*u + N(1) \log \epsilon \right). \tag{11}$$

Note that this choice does not alter (9). This fact is quite important since we use (9) to investigate the analytic nature of $\zeta_N(s)$.

2.2 The counting function of $C = \overline{\text{Spec } \mathbb{Z}}$

It is natural to wonder about the existence of a ‘curve’ $C = \overline{\text{Spec } \mathbb{Z}}$ suitably defined over \mathbb{F}_1 , whose zeta function $\zeta_C(s)$ is the complete Riemann zeta function $\zeta_{\mathbb{Q}}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ (cf. also [Man95]). In this section, we prove that the integral equation (9) produces a precise formula for the counting function $N_C(q) = N(q)$ associated with C . In fact, (9) shows in this case that

$$\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = - \int_1^{\infty} N(u) u^{-s} d^*u. \tag{12}$$

This integral formula appears in the Riemann–Weil explicit formulae and when $\Re(s) > 1$, one derives that

$$- \frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} + \int_1^{\infty} \kappa(u) u^{-s} d^*u. \tag{13}$$

Here, $\Lambda(n)$ is the von-Mangoldt function³ and $\kappa(u)$ is the distribution which is defined, for any test function f , by

$$\int_1^{\infty} \kappa(u) f(u) d^*u = \int_1^{\infty} \frac{u^2 f(u) - f(1)}{u^2 - 1} d^*u + c f(1), \quad c = \frac{1}{2}(\log \pi + \gamma)$$

where $\gamma = -\Gamma'(1)$ is the Euler constant. This distribution is positive on $(1, \infty)$ by construction. Hence, we derive the consequence that the counting function $N(q)$ of the hypothetical curve C over \mathbb{F}_1 , is the *distribution* given by the sum of $\kappa(q)$ with the discrete term equal to the derivative $d\varphi(q)/dq$, taken in the sense of distributions, of the function⁴

$$\varphi(u) = \sum_{n < u} n \Lambda(n). \tag{14}$$

Indeed, since $d^*u = du/u$, one has for any test function f ,

$$\int_1^{\infty} f(u) \left(\frac{d}{du} \varphi(u) \right) d^*u = \int_1^{\infty} \frac{f(u)}{u} d\varphi(u) = \sum \Lambda(n) f(n).$$

Thus, one can write (13) as

$$- \frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = \int_1^{\infty} \left(\frac{d}{du} \varphi(u) + \kappa(u) \right) u^{-s} d^*u. \tag{15}$$

If one compares (15) and (12), one derives the following formula for $N(u)$

$$N(u) = \frac{d}{du} \varphi(u) + \kappa(u). \tag{16}$$

The above expression encloses in a very subtle and intrinsic form some fundamental information on the description of the counting function as a geometric ‘trace type’ formula. To substantiate this statement, we recall the well-known equation (cf. [Ing90, ch. IV, Theorems 28 and 29] and use $\varphi(u) = u\psi_0(u) - \psi_1(u)$, where $\psi_0(u)$ is the Chebyshev function $\psi_0(u) = \sum_{n < u} \Lambda(n)$ and $\psi_1(u) = \int_0^u \psi_0(x) dx$ is its primitive) valid for $u > 1$ (and not a prime power)

$$\varphi(u) = \frac{u^2}{2} - \sum_{\rho \in \mathbb{Z}} \text{order}(\rho) \frac{u^{\rho+1}}{\rho+1} + a(u). \tag{17}$$

³ With value $\log p$ for powers p^ℓ of primes and zero otherwise.

⁴ The value at the points of discontinuity does not affect the distribution.

Here, one sets

$$a(u) = \text{ArcTanh}\left(\frac{1}{u}\right) - \frac{\zeta'(-1)}{\zeta(-1)} \tag{18}$$

and Z denotes the set of non-trivial zeros of the Riemann zeta function. Note that the sum over Z in (17) has to be taken in a symmetric manner to ensure convergence, i.e. as a limit of the partial sums over the symmetric set Z_m of first $2m$ zeros. When one differentiates (17) in a formal way, the term in $a(u)$ gives

$$\frac{d}{du}a(u) = \frac{1}{1-u^2}.$$

Hence, at the formal level (i.e. disregarding the principal value), one obtains

$$\frac{d}{du}a(u) + \kappa(u) = 1.$$

Thus, after a formal differentiation of (17), one obtains

$$N(u) = \frac{d}{du}\varphi(u) + \kappa(u) \sim u - \sum_{\rho \in Z} \text{order}(\rho)u^\rho + 1. \tag{19}$$

This formula for the counting function is now formally similar to formula (2) describing the counting function of the number of points of an irreducible, smooth and projective curve C over a finite field.

Note that in the above formal computations we have neglected to consider the principal value for the distribution $\kappa(u)$. By taking this into account, we obtain the following more precise result (cf. also Figure 1).

THEOREM 2.2. *The tempered distribution $N(u)$ satisfying the equation*

$$-\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = \int_1^\infty N(u)u^{-s} d^*u,$$

is positive on $(1, \infty)$ and is given on $[1, \infty)$ by

$$N(u) = u - \frac{d}{du} \left(\sum_{\rho \in Z} \text{order}(\rho) \frac{u^{\rho+1}}{\rho+1} \right) + 1 \tag{20}$$

where the derivative is taken in the sense of distributions, and the value at $u = 1$ of the term $\omega(u) = \sum_{\rho \in Z} \text{order}(\rho)u^{\rho+1}/(\rho+1)$ is given by

$$\omega(1) = \frac{1}{2} + \frac{\gamma}{2} + \frac{\log 4\pi}{2} - \frac{\zeta'(-1)}{\zeta(-1)}. \tag{21}$$

Proof. The function $\varphi(u)$ is non-decreasing and the positivity of the distribution $N(u)$ on $(1, \infty)$ follows from (16). For $u > 1$ we define

$$\omega(u) = \sum_{\rho \in Z} \text{order}(\rho) \frac{u^{\rho+1}}{\rho+1}. \tag{22}$$

By (17) one has (for $u > 1$)

$$\omega(u) = -\varphi(u) + \frac{u^2}{2} + a(u). \tag{23}$$

In a neighborhood of 1 one has $\varphi(u) = 0$ and $a(u) \sim -(1/2) \log(u-1)$ when $u \rightarrow 1+$. Thus, $\omega(u)$ diverges when $u \rightarrow 1$ although it is locally integrable and defines a distribution. Since $[1, \infty)$ has

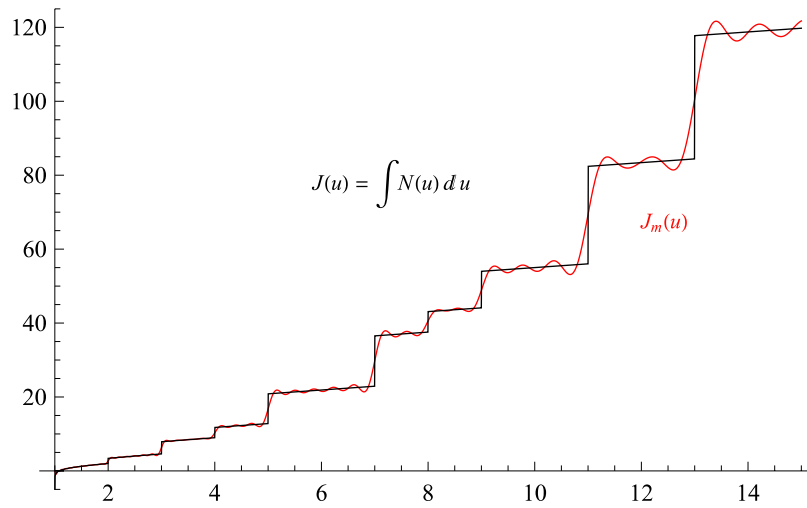


FIGURE 1. Primitive $J(u)$ of $N(u)$ and approximation using the symmetric set Z_m of first $2m$ zeros, by $J_m(u) = u^2/2 - \sum_{Z_m} \text{order}(\rho)u^{\rho+1}/(\rho + 1) + u$. Note that $J(u) \rightarrow -\infty$ when $u \rightarrow 1+$.

a boundary, the derivative of the distribution depends on its boundary value and is defined, for f smooth and of fast enough decay at ∞ , as

$$\left\langle \frac{d}{du} \omega(u), f(u) \right\rangle = - \int_1^\infty \omega(u) \frac{d}{du} f(u) du - \omega(1)f(1). \tag{24}$$

We apply this to the function $f(u) = u^{-s-1}$, for $\Re(s) > 1$. One has $-df(u)/du = (s + 1)u^{-s-2}$ and one obtains

$$\left\langle \frac{d}{du} \omega(u), f(u) \right\rangle = (s + 1) \int_1^\infty \left(-\varphi(u) + \frac{u^2}{2} + a(u) \right) u^{-s-2} du - \omega(1).$$

By applying some results from [Ing90] (cf. ch. I, (17): use $\varphi(u) = u\psi_0(u) - \psi_1(u)$, $\psi'(u) = \psi_0(u)$), one deduces that

$$-\frac{\partial_s \zeta(s)}{\zeta(s)} = (s + 1) \int_1^\infty \varphi(u) u^{-s-2} du \tag{25}$$

and by using

$$\int_1^\infty (u + 1)f(u) du = \frac{1}{s} + \frac{1}{s - 1}, \quad -(s + 1) \int_1^\infty \frac{u^2}{2} u^{-s-2} du = -\frac{1}{2} - \frac{1}{s - 1}$$

one concludes that

$$\left\langle \left(u - \frac{d}{du} \omega(u) + 1 \right), f(u) \right\rangle = \frac{1}{s} - \frac{1}{2} - \frac{\partial_s \zeta(s)}{\zeta(s)} + \omega(1) - (s + 1) \int_1^\infty a(u) u^{-s-2} du.$$

Finally, we claim that the following equation holds

$$\frac{1}{s} - (s + 1) \int_1^\infty a(u) u^{-s-2} du = -\frac{\partial_s \Gamma(s/2)}{\Gamma(s/2)} + \frac{\zeta'(-1)}{\zeta(-1)} - \log 2 - \frac{\gamma}{2}.$$

Indeed, using a process of integration by parts one has

$$-(s + 1) \int_{1+\epsilon}^\infty \left(\text{ArcTanh} \left(\frac{1}{u} \right) - u \right) u^{-s-2} du = \int_{1+\epsilon}^\infty \frac{u^{-s+1}}{u^2 - 1} du + b(\epsilon)$$

with

$$b(\epsilon) = -\left(\text{ArcTanh}\left(\frac{1}{1+\epsilon}\right) - (1+\epsilon)\right)(1+\epsilon)^{-s-1} = -\int_{1+\epsilon}^{\infty} \frac{u^{-1}}{u^2-1} du + c + O(\epsilon \log(1/\epsilon))$$

and $c = 1 - \log 2$. Moreover, a simple change of variables in the Gauss formula for the logarithmic derivative $\Gamma'(s)/\Gamma(s)$ of the Gamma function gives

$$\frac{\partial_s \Gamma(s/2)}{\Gamma(s/2)} = -\frac{\gamma}{2} + \int_1^{\infty} \frac{u^{-1} - u^{-s+1}}{u^2 - 1} du.$$

Thus, one obtains

$$\left\langle \left(u - \frac{d}{du} \omega(u) + 1\right), f(u) \right\rangle = -\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)},$$

provided that

$$\omega(1) = \frac{1}{2} + \frac{\gamma}{2} + \frac{\log 4\pi}{2} - \frac{\zeta'(-1)}{\zeta(-1)}. \tag{26}$$

To check this latter equality one cannot use the explicit formula (17) which is not valid at $u = 1$, since the term $\text{ArcTanh}(1/u)$ is infinite, thus displaying the discontinuity of the function $\omega(u)$ at $u = 1$. To verify (26), we use instead the following formula (taken from [Ing90]; cf. III, (26))

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = \sum_{\mathcal{Z}} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{s}{2} + 1\right) + \log(2\pi) - 1 - \frac{\gamma}{2}$$

when $s \rightarrow 1$. We note that the left-hand side of the above formula tends to γ while the right-hand side, using the symmetry $\rho \rightarrow 1 - \rho$ of the zeros (and a symmetric summation and the formula $\Gamma'/\Gamma(3/2) = 2 - \gamma - 2 \log 2$) tends to

$$2 \sum_{\mathcal{Z}} \frac{1}{\rho} - 2 + \log(4\pi).$$

Thus, one obtains

$$\sum_{\mathcal{Z}} \frac{1}{\rho} = \frac{\gamma}{2} + 1 - \frac{1}{2} \log(4\pi).$$

One then concludes by using the equalities (cf. [Ing90, IV, Theorem 28])

$$\sum_{\mathcal{Z}} \frac{1}{\rho(\rho+1)} = \frac{1}{2} - \log(4\pi) + \frac{\zeta'(-1)}{\zeta(-1)}$$

and the formula (using a symmetric summation)

$$\sum_{\mathcal{Z}} \frac{1}{\rho+1} = \sum_{\mathcal{Z}} \frac{1}{\rho} - \sum_{\mathcal{Z}} \frac{1}{\rho(\rho+1)}. \quad \square$$

Remark 2.3. In agreement with [Sou04], the value $N(1)$ should be thought of as the Euler characteristic of the hypothetical curve C over \mathbb{F}_1 . Since C is expected to have infinite genus, one would deduce that $N(1) = -\infty$, in apparent conflict with the expected positivity of $N(q)$ for $q > 1$. This apparent contradiction is resolved in the proof of Theorem 2.2, since the distribution $N(q)$ is positive for $q > 1$ but its value at $q = 1$ is formally given by

$$N(1) = 2 - \lim_{\epsilon \rightarrow 0} \frac{\omega(1+\epsilon) - \omega(1)}{\epsilon} \sim -\frac{1}{2} E \log E, \quad E = \frac{1}{\epsilon}$$

also reflecting, when $\epsilon \rightarrow 0$, the density of the zeros.

Remark 2.4. One may wonder how to extend Theorem 2.2 to Hecke L -functions with Grössencharakteren. For individual L -functions, the positivity of the distribution $N(u)$ no longer holds: L -functions of non-trivial characters provide a first example. However, the positivity of $N(u)$ is restored when one combines together all Hecke L -functions with Grössencharakteren. The distribution $N(u)$ can then be lifted to the idèle class group $C_{\mathbb{K}}$ of a global field \mathbb{K} . Formula (17), which is an essential component in the proof of Theorem 2.2, is a typical application of the Riemann–Weil explicit formulae. These formulae become natural once they are lifted to the idèle class group. It seems therefore natural to expect that also the hypothetical curve $C = \overline{\text{Spec}(\mathbb{Z})}$ should be of adèlic nature and possessing an action of the idèle class group. This speculation is in agreement with the interpretation of the explicit formulae as a trace formula, by using the non-commutative geometric formalism of the adèle class space (cf. [Con99, CCM07, CCM08, CCM09, Mey05]). Using this interpretation of the explicit formulae, one can show that the counting distribution $N(u)$, once lifted to the idèle class group $C_{\mathbb{K}}$, is simply given by the distributional trace of the natural representation of $C_{\mathbb{K}}$ on functions on the adèle class space. We shall return on this adèlic interpretation of C in § 5 of this paper.

3. $\mathfrak{M}\mathfrak{o}$ -schemes

In this section we describe, following a functorial approach similar to that of [DG70], a generalization of the theory of \mathbb{Z} -functors and schemes which is obtained by enlarging the category of rings to that of commutative monoids. This functorial construction will be applied in § 4.3, after gluing together the categories of monoids and rings, to derive a new notion of \mathbb{F}_1 -schemes and associated zeta functions. Our construction has evident connections with the theory of schemes over \mathbb{F}_1 developed by Deitmar in [Dei05, Dei08], with the theory of logarithmic structures of Kato in [Kat94], with the arithmetic theory over \mathbb{F}_1 described by Kurokawa *et al.* in [KOW03], and with the algebro-topological approach followed by Töen and Vaquié in [TV09].

3.1 Monoids: the category $\mathfrak{M}\mathfrak{o}$

Throughout the paper we denote by $\mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}$, $\mathfrak{A}\mathfrak{b}$ and $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$ the categories of sets, abelian groups and commutative rings with unit, respectively.

We let $\mathfrak{M}\mathfrak{o}$ be the category of commutative monoids M denoted multiplicatively, with a neutral element 1 (i.e. unit) and an absorbing element 0 ($0 \cdot x = x \cdot 0 = 0$, for all $x \in M$). For a monoid M , we write M^\times for the group of its invertible elements.

A homomorphism $\varphi : M \rightarrow N$ in $\mathfrak{M}\mathfrak{o}$ is unital (i.e. $\varphi(1) = 1$) and satisfying $\varphi(0) = 0$.

Remark 3.1. Given a commutative group H in $\mathfrak{A}\mathfrak{b}$, we set

$$\mathbb{F}_1[H] = H \cup \{0\} \quad (0 \cdot h = h \cdot 0 = 0, \forall h \in H).$$

Following the analogy with the category of rings, one sees that in $\mathfrak{M}\mathfrak{o}$ a monoid of the form $\mathbb{F}_1[H]$ corresponds to a *field* F ($F = F^\times \cup \{0\}$) in $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$. The collection of monoids like $\mathbb{F}_1[H]$, for $H \in \text{Obj}(\mathfrak{A}\mathfrak{b})$, forms a full subcategory of $\mathfrak{M}\mathfrak{o}$ isomorphic to the category of abelian groups (cf. Proposition 3.21).

DEFINITION 3.2. An $\mathfrak{M}\mathfrak{o}$ -functor \mathcal{F} is a covariant functor from the category $\mathfrak{M}\mathfrak{o}$ to $\mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}$.

To a monoid M in $\mathfrak{M}\mathfrak{o}$ one associates the covariant functor $\mathfrak{spec} M$ defined as follows

$$\mathfrak{spec} M : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}, \quad N \mapsto \mathfrak{spec} M(N) = \text{Hom}_{\mathfrak{M}\mathfrak{o}}(M, N). \tag{27}$$

Note that by applying Yoneda’s lemma, a morphism of functors (natural transformation) $\varphi : \mathbf{spec} M \rightarrow \mathcal{F}$, with $\mathcal{F} : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}$ is completely determined by the element $\varphi(\text{id}_M) \in \mathcal{F}(M)$, moreover any such element gives rise to a morphism $\mathbf{spec} M \rightarrow \mathcal{F}$. By applying this fact to the functor $\mathcal{F} = \mathbf{spec} N$, for $N \in \text{Obj}(\mathfrak{M}\mathfrak{o})$, one obtains an *inclusion* of $\mathfrak{M}\mathfrak{o}$ as a full subcategory of the category of $\mathfrak{M}\mathfrak{o}$ -functors.

Morphisms in the category of $\mathfrak{M}\mathfrak{o}$ -functors are natural transformations.

An ideal I of a monoid M is a subset $I \subset M$ such that $0 \in I$ and $x \in I \implies xy \in I$, for all $y \in M$ (cf. [Gil80]). As for rings, an ideal $I \subset M$ defines an interesting subfunctor $\underline{D}(I) \subset \mathbf{spec} M$:

$$\underline{D}(I) : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}, \quad \underline{D}(I)(N) = \{\rho \in \mathbf{spec}(M)(N) \mid \rho(I)N = N\}. \tag{28}$$

We recall that an ideal $\mathfrak{p} \subset M$ is said to be *prime* if $1 \notin \mathfrak{p}$ and its complement $\mathfrak{p}^c = M \setminus \mathfrak{p}$ is a multiplicative subset of M i.e.

$$x \notin \mathfrak{p}, \quad y \notin \mathfrak{p} \implies xy \notin \mathfrak{p}.$$

For an ideal $I \subset M$, one denotes by $D(I)$ the set of prime ideals $\mathfrak{p} \subset M$ which do not contain I . These subsets are the open sets for the natural topology on the set $X = \text{Spec}(M)$ of prime ideals of M (cf. [Kat94]). The smallest ideal containing a collection of ideals $\{I_\alpha\}_{\alpha \in A}$ (A an index set) of a monoid M is just the union $I = \bigcup_{\alpha \in A} I_\alpha$ and the corresponding open subset $D(I) \subset \text{Spec}(M) = \{\mathfrak{p} \subset M \mid \mathfrak{p} \text{ prime ideal}\}$ satisfies the property $D(\bigcup_{\alpha \in A} I_\alpha) = \bigcup_{\alpha \in A} D(I_\alpha)$. It is a standard fact that the inverse image of a prime ideal by a morphism of monoids is a prime ideal. Moreover, it is also straightforward to verify that the complement of the set of invertible elements in a monoid M , $\mathfrak{p}_M = (M^\times)^c$, is a prime ideal in M which contains all other prime ideals of the monoid.

3.2 Automatic locality

An interesting property fulfilled by any $\mathfrak{M}\mathfrak{o}$ -functor is that of being *local*. Locality is not automatically satisfied by \mathbb{Z} -functors, essentially it corresponds to state the exactness, on an open (finite) covering of an affine scheme $\text{Spec}(R) = \bigcup_{i \in \mathcal{I}} D(f_i)$ ($f_i \in R$, \mathcal{I} index set) of sequences such as (29) below. On the other hand, we shall see that an $\mathfrak{M}\mathfrak{o}$ -functor is local by construction. We recall the following result (cf. [Dei05])

LEMMA 3.3. *Let M be an object in $\mathfrak{M}\mathfrak{o}$ and let $\{W_\alpha\}_{\alpha \in A}$ (A an index set) be a (finite) open cover of the topological space $X = \text{Spec}(M)$. Then $W_\alpha = \text{Spec}(M)$, for some index $\alpha \in A$.*

Proof. The point $\mathfrak{p}_M = (M^\times)^c \in \text{Spec}(M)$ must be contained in at least one W_α , for some index $\alpha \in A$. One has $W_\alpha = D(I_\alpha)$ for some ideal $I_\alpha \subset M$, hence $\mathfrak{p}_M \in D(I_\alpha)$, for some $\alpha \in A$ and this means $I_\alpha \cap M^\times \neq \emptyset$, that is, $I_\alpha = M$. □

Let M be an object of $\mathfrak{M}\mathfrak{o}$. For $S \subset M$ a *multiplicative* subset we recall that the *monoid* $S^{-1}M$ is the quotient of the set made by all expressions $a/s = (a, s) \in A \times S$, by the following equivalence relation

$$a/s \sim b/t \iff \exists u \in S \quad uta = usb.$$

One checks that the product $a/s \cdot b/t = ab/st$ is well-defined on the quotient $S^{-1}M$. For $f \in M$ and $S = \{f^n; n \in \mathbb{Z}_{\geq 0}\}$ one denotes $S^{-1}M$ by M_f .

For any $\mathfrak{M}\mathfrak{o}$ -functor \mathcal{F} and any monoid M one defines a sequence of maps of sets (\mathcal{I} an index set)

$$\mathcal{F}(M) \xrightarrow{u} \prod_{i \in \mathcal{I}} \mathcal{F}(M_{f_i}) \xrightleftharpoons[w]{v} \prod_{(i,j) \in \mathcal{I} \times \mathcal{I}} \mathcal{F}(M_{f_i f_j}) \quad (29)$$

which is obtained by using the (finite) open covering of $\text{Spec}(M)$ made by the open sets $D(f_i M)$ ($f_i \in M$), the natural morphisms $M \rightarrow M_{f_i}$ and the functoriality of \mathcal{F} .

The following lemma shows that any $\mathfrak{M}\mathfrak{o}$ -functor is local.

LEMMA 3.4. *For any $\mathfrak{M}\mathfrak{o}$ -functor \mathcal{F} and any monoid M , the sequence (29) is exact.*

Proof. By Lemma 3.3, there exists an index $i = i_0 \in \mathcal{I}$ such that $f_i \in M^\times$. Then, the map $\rho_{i_0} : M \rightarrow M_{f_{i_0}}$ is invertible, thus u is injective. Let $(x_i) \in \prod_{i \in \mathcal{I}} \mathcal{F}(M_{f_i})$ be a family, with $x_i \in \mathcal{F}(M_{f_i})$ such that $(x_i)_{f_j} = (x_j)_{f_i}$, for all $i, j \in \mathcal{I}$. This gives in particular the equality between the image of $x_i \in \mathcal{F}(M_{f_i})$ under the isomorphism $\mathcal{F}(\rho_{i_0}) : \mathcal{F}(M_{f_i}) \rightarrow \mathcal{F}(M_{f_i f_{i_0}})$ and $\mathcal{F}(\rho_{i_0 i})(x_1) \in \mathcal{F}(M_{f_{i_0} f_i}) = \mathcal{F}(M_{f_i f_{i_0}})$. By writing $x_{i_0} = \rho_{i_0}(x)$ one finds that $u(x)$ is equal to the family (x_i) . \square

3.3 Open $\mathfrak{M}\mathfrak{o}$ -subfunctors

In analogy with the theory of \mathbb{Z} -schemes, we introduce the notion of an open subfunctor of an $\mathfrak{M}\mathfrak{o}$ -functor and describe a few relevant examples.

DEFINITION 3.5. A subfunctor $\mathcal{G} \subset \mathcal{F}$ of an $\mathfrak{M}\mathfrak{o}$ -functor \mathcal{F} is open if for any object M of $\mathfrak{M}\mathfrak{o}$ and any morphism of $\mathfrak{M}\mathfrak{o}$ -functors $\varphi : \text{spec } M \rightarrow \mathcal{F}$, there exists an ideal $I \subset M$ satisfying the following property: for any object N of $\mathfrak{M}\mathfrak{o}$ and for any $\rho \in \text{spec } M(N) = \text{Hom}_{\mathfrak{M}\mathfrak{o}}(M, N)$:

$$\varphi(\rho) \in \mathcal{G}(N) \subset \mathcal{F}(N) \Leftrightarrow \rho(I)N = N. \quad (30)$$

To clarify the meaning of this definition we develop a few examples.

Example 3.6. The functor

$$\mathcal{G} : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}, \quad N \rightarrow \mathcal{G}(N) = N^\times$$

is an open subfunctor of the (identity) functor \mathcal{D}^1

$$\mathcal{D}^1 : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}, \quad N \rightarrow \mathcal{D}^1(N) = N.$$

In fact, let M be a monoid, then by Yoneda's lemma a morphism of functors $\varphi : \text{spec } M \rightarrow \mathcal{D}^1$ is determined by an element $z \in \mathcal{D}^1(M) = M$. For any monoid N and $\rho \in \text{Hom}(M, N)$, one has $\varphi(\rho) = \rho(z) \in \mathcal{D}^1(N) = N$, thus the condition $\varphi(\rho) \in \mathcal{G}(N) = N^\times$ means that $\rho(z) \in N^\times$. One takes for I the ideal generated by z in M : $I = zM$. Then it is straightforward to check that (30) is fulfilled.

Example 3.7. Let $I \subset M$ be an ideal of a monoid M and consider the subfunctor $\underline{D}(I) \subset \text{spec}(M)$ as defined in (28). Then, $\underline{D}(I)$ is an open subfunctor of $\text{spec } M$.

Indeed, for any object A of $\mathfrak{M}\mathfrak{o}$ and $\varphi : \text{spec } A \rightarrow \text{spec } M$ one has $\varphi(\text{id}_A) = \eta \in \text{spec}(M)(A) = \text{Hom}_{\mathfrak{M}\mathfrak{o}}(M, A)$. One takes in A the ideal $J = \eta(I)A$. This ideal fulfills the condition (30) for any object N of $\mathfrak{M}\mathfrak{o}$ and $\rho \in \text{Hom}_{\mathfrak{M}\mathfrak{o}}(A, N)$. In fact, one has $\varphi(\rho) = \rho \circ \eta \in \text{Hom}_{\mathfrak{M}\mathfrak{o}}(M, N)$ and $\varphi(\rho) \in \underline{D}(I)(N)$ means that $\rho(\eta(I))N = N$. This latter equality holds if and only if $\rho(J)N = N$.

3.4 Open covering by $\mathfrak{M}\mathfrak{o}$ -subfunctors

The next task is to introduce the notion of an *open cover* in the category of $\mathfrak{M}\mathfrak{o}$ -functors. We use the fact (cf. Proposition 3.21) that the category \mathfrak{Ab} of abelian groups embeds as a full subcategory of $\mathfrak{M}\mathfrak{o}$, by means of the functor $H \rightarrow \mathbb{F}_1[H]$.

DEFINITION 3.8. Let \mathcal{F} be an $\mathfrak{M}\mathfrak{o}$ -functor and let $\{\mathcal{F}_\alpha\}_{\alpha \in S}$ be a family of open subfunctors of \mathcal{F} . Then one says that $\{\mathcal{F}_\alpha\}_{\alpha \in S}$ (with S an index set) is an open cover of \mathcal{F} if

$$\mathcal{F}(\mathbb{F}_1[H]) = \bigcup_{\alpha \in S} \mathcal{F}_\alpha(\mathbb{F}_1[H]) \quad \text{for all } H \in \text{Obj}(\mathfrak{Ab}). \tag{31}$$

Since commutative groups (with 0 added) replace fields in $\mathfrak{M}\mathfrak{o}$, the above definition is the natural transposition of the definition of open covers as in [DG70] within the category of $\mathfrak{M}\mathfrak{o}$ -functors. The following proposition gives a precise characterization of the open covers of an $\mathfrak{M}\mathfrak{o}$ -functor.

PROPOSITION 3.9. Let \mathcal{F} be an $\mathfrak{M}\mathfrak{o}$ -functor and let $\{\mathcal{F}_\alpha\}_{\alpha \in S}$ be a family of open subfunctors of \mathcal{F} . Then, the family $\{\mathcal{F}_\alpha\}_{\alpha \in S}$ forms an open cover of \mathcal{F} if and only if

$$\mathcal{F}(M) = \bigcup_{\alpha \in S} \mathcal{F}_\alpha(M) \quad \text{for all } M \in \text{Obj}(\mathfrak{M}\mathfrak{o}).$$

Proof. The condition is obviously sufficient. To show the converse, we assume that (31) holds. Let M be a monoid and let $\xi \in \mathcal{F}(M)$, one needs to show that $\xi \in \mathcal{F}_\alpha(M)$ for some $\alpha \in S$. Let φ be the morphism of functors from $\mathfrak{spec} M$ to \mathcal{F} such that $\varphi(id_M) = \xi$. Since each \mathcal{F}_α is an open subfunctor of \mathcal{F} , one can find ideals $I_\alpha \subset M$ such that for any object N of $\mathfrak{M}\mathfrak{o}$ and for any $\rho \in \mathfrak{spec} M(N) = \text{Hom}_{\mathfrak{M}\mathfrak{o}}(M, N)$ one has

$$\varphi(\rho) \in \mathcal{F}_\alpha(N) \subset \mathcal{F}(N) \Leftrightarrow \rho(I_\alpha)N = N. \tag{32}$$

One applies this to the morphism $\epsilon_M : M \rightarrow \mathbb{F}_1[M^\times] = \kappa$ given by

$$M \xrightarrow{\epsilon} \mathbb{F}_1[M^\times], \quad \epsilon_M(y) = \begin{cases} 0 & \text{for all } y \notin M^\times, \\ y & \text{for all } y \in M^\times. \end{cases} \tag{33}$$

One has $\epsilon_M \in \mathfrak{spec}(M)(\kappa)$ and $\varphi(\epsilon_M) \in \mathcal{F}(\kappa) = \bigcup_{\alpha \in S} \mathcal{F}_\alpha(\kappa)$. Thus, there exists α such that $\varphi(\epsilon_M) \in \mathcal{F}_\alpha(\kappa)$. By (32) one has $\epsilon_M(I_\alpha)\kappa = \kappa$ and $I_\alpha \cap M^\times \neq \emptyset$, hence $I_\alpha = M$. Then applying (32) to $\rho = id_M$ one obtains $\xi \in \mathcal{F}_\alpha(M)$ as required. \square

3.5 $\mathfrak{M}\mathfrak{o}$ -schemes

In view of the fact that any $\mathfrak{M}\mathfrak{o}$ -functor is local, the definition of an $\mathfrak{M}\mathfrak{o}$ -scheme simply involves the condition of local representability.

DEFINITION 3.10. An $\mathfrak{M}\mathfrak{o}$ -scheme is an $\mathfrak{M}\mathfrak{o}$ -functor which admits an open cover by representable subfunctors.

We consider several elementary examples of $\mathfrak{M}\mathfrak{o}$ -schemes.

Example 3.11 (The affine spaces \mathcal{D}^n). For a fixed $n \in \mathbb{N}$, we consider the following $\mathfrak{M}\mathfrak{o}$ -functor

$$\mathcal{D}^n : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{Sets}, \quad \mathcal{D}^n(M) = M^n.$$

This functor is representable since it is described by

$$\mathcal{D}^n(M) = \text{Hom}_{\mathfrak{M}\mathfrak{o}}(\mathbb{F}_1[T_1, \dots, T_n], M),$$

where

$$\mathbb{F}_1[T_1, \dots, T_n] := \{0\} \cup \{T_1^{a_1} \cdots T_n^{a_n} \mid a_j \in \mathbb{Z}_{\geq 0}\} \tag{34}$$

is the union of $\{0\}$ with the semi-group generated by the T_j .

Example 3.12. The projective line \mathbb{P}^1 . We consider the $\mathfrak{M}\mathfrak{o}$ -functor \mathbb{P}^1 which associates to an object M of $\mathfrak{M}\mathfrak{o}$ the set $\mathbb{P}^1(M)$ of complemented submodules E of rank one in M^2 , where the rank is defined locally. By definition, a complemented submodule is the range of an idempotent matrix $e \in M_2(M)$ (i.e. $e^2 = e$) with each line having at most⁵ one non-zero entry. To a morphism $\rho: M \rightarrow N$ one associates the following map $\mathbb{P}^1(\rho)$

$$E \rightarrow N \otimes_M E \subset N^2,$$

which replaces $e \in M_2(M)$ by $\rho(e) \in M_2(N)$. Let $\epsilon_{\mathfrak{p}}$ be the morphism from M to $\mathbb{F}_1[M_{\mathfrak{p}}^{\times}]$ (where $M_{\mathfrak{p}} = S^{-1}M$, with $S = \mathfrak{p}^c$) given by

$$\epsilon_{\mathfrak{p}}(y) = \begin{cases} 0 & \text{for all } y \in \mathfrak{p}, \\ y & \text{for all } y \notin \mathfrak{p}. \end{cases} \tag{35}$$

The condition of rank one means that for any prime ideal $\mathfrak{p} \in \text{Spec } M$ the matrix $\epsilon_{\mathfrak{p}}(e)$ obtained by applying $\epsilon_{\mathfrak{p}}$ to each matrix element, fulfills $\epsilon_{\mathfrak{p}}(e) \notin \{0, 1\}$ (i.e. $\epsilon_{\mathfrak{p}}(e)$ is neither the zero nor the unit matrix).

Now, we compare \mathbb{P}^1 with the following $\mathfrak{M}\mathfrak{o}$ -functor

$$\mathcal{P}(M) = M \sqcup_{M^{\times}} M \tag{36}$$

where the gluing map is given by $x \rightarrow x^{-1}$. In other words, we define on the disjoint union $M \sqcup M$ an equivalence relation given by (using the identification $M \times \{1, 2\} = M \sqcup M$)

$$(x, 1) \sim (x^{-1}, 2) \quad \text{for all } x \in M^{\times}.$$

We define a natural transformation e from \mathcal{P} to \mathbb{P}^1 by observing that the matrices

$$e_1(a) = \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix}, \quad e_2(b) = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}, \quad a, b \in M$$

are idempotent ($e^2 = e$) and their ranges also fulfill the following property

$$\text{Im } e_1(a) = \text{Im } e_2(b) \iff ab = 1.$$

LEMMA 3.13. *The natural transformation e is an isomorphism, that is,*

$$\mathcal{P}(M) = M \sqcup_{M^{\times}} M \cong \mathbb{P}^1(M).$$

Moreover, the two copies of M define an open cover of \mathbb{P}^1 by representable sub-functors \mathcal{D}^1 .

Proof. We show that an idempotent matrix $e \in M_2(M)$ of rank one, with each line having at most one non-zero entry is of the form $e_j(a)$ for some $j \in \{1, 2\}$. First we claim that one of the matrix elements of

$$e = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

must be invertible. Otherwise, by localizing M at the prime ideal $\mathfrak{p}_M = (M^{\times})^c$ one would obtain the zero matrix which contradicts the hypothesis of rank one. Assume first that a is invertible. Then $b = 0$, and from the idempotency condition on e one gets that $a^2 = a$ and hence $a = 1$.

⁵ Note that we need the 0-element to state this condition.

Now we show that $d = 0$. Again from the condition $e^2 = e$ one gets $d^2 = d$. Then, if $d \neq 0$ there exists a prime ideal $\mathfrak{p} \subset M$ such that $d \notin \mathfrak{p}$. This is because the intersection of all prime ideals is the set of nilpotent elements. More generally, one knows [Gil80] that given an ideal $I \subset M$, the intersection of the prime ideals $\mathfrak{p} \subset M$ with $\mathfrak{p} \supset I$ coincides with the radical of I

$$\bigcap_{\mathfrak{p} \supset I} \mathfrak{p} = \sqrt{I} := \{x \in M \mid \exists n \in \mathbb{N}, x^n \in I\}.$$

Thus, by localizing M at \mathfrak{p} one obtains that e is the unit matrix at \mathfrak{p} which contradicts the hypothesis of rank one. Thus, $d = 0$ and $e = e_1(c)$.

If b is invertible, then $a = 0$, $bd = b$ so that $d = 1$ and $c = 0$, thus $e = e_2(b)$.

The two other cases are treated in a similar manner. The functor \mathcal{P} admits by construction two copies of the functor \mathcal{D}^1 embedded in it as a subfunctor. We need to show that these two subfunctors are open in \mathbb{P}^1 : we prove it for the first copy of \mathcal{D}^1 . Let N be an object of $\mathfrak{M}\mathfrak{o}$. A morphism $\text{spec}(N) \rightarrow \mathbb{P}^1$ (in the category of $\mathfrak{M}\mathfrak{o}$ -functors) is determined by an element $z \in \mathbb{P}^1(N)$. If z belongs to the first copy of N , it follows that for any $\rho \in \text{Hom}_{\mathfrak{M}\mathfrak{o}}(N, M)$, $\rho(z)$ is in the first copy of $\mathcal{D}^1(M)$. In this case one can take $I = N$. Otherwise, z belongs to the second copy of N and in this case, likewise in the above Example 3.6, one takes $I = zN$. The local representability follows since \mathcal{D}^1 is representable. \square

Example 3.14. Let M be a monoid and let $I \subset M$ be an ideal. Consider the $\mathfrak{M}\mathfrak{o}$ -functor $\underline{D}(I)$ of Example 3.7. The next proposition states that this is an $\mathfrak{M}\mathfrak{o}$ -scheme.

PROPOSITION 3.15.

- (1) Let $f \in M$ and $I = fM$. Then the subfunctor $\underline{D}(I) \subset \text{spec } M$ is represented by M_f .
- (2) For any ideal $I \subset M$, the $\mathfrak{M}\mathfrak{o}$ -functor $\underline{D}(I)$ is an $\mathfrak{M}\mathfrak{o}$ -scheme.

The proof is straightforward.

3.6 Geometric realization

As in the case of \mathbb{Z} -schemes (and following similar set-theoretic precautions to those stated in the preliminary chapter of [DG70]), it can be shown that any $\mathfrak{M}\mathfrak{o}$ -scheme \underline{X} can be represented in the form

$$\underline{X}(N) = \text{Hom}(\text{Spec}(N), X), \quad N \in \text{Obj}(\mathfrak{M}\mathfrak{o}), \tag{37}$$

where X is the associated geometric space, that is, the geometric realization of \underline{X} . In this framework, a geometric space is properly defined by:

- a topological space X ;
- a sheaf \mathcal{O}_X of monoids on X .

For details on the properties of the geometric spaces which are locally of the form $\text{Spec}(M)$ we refer to [Dei05, Dei08, Kat94]. Note that there is no need to require that the stalks of the structural sheaf of a geometric space are ‘local’ since any monoid already has a local algebraic structure. We recall only a few concepts from the basic terminology and we refer to [Dei05, Dei08, DG70, Kat94] for details. A homomorphism $\rho : M_1 \rightarrow M_2$ of monoids is said to be local if $\rho^{-1}(M_2^\times) = M_1^\times$. A morphism $\varphi : X \rightarrow Y$ between two geometric spaces is given by a pair $(\varphi, \varphi^\sharp)$ of a continuous map φ and a local morphism of sheaves of monoids

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{\varphi^\sharp} \Gamma(\varphi^{-1}(V), \mathcal{O}_X)$$

that is, the map of stalks $\mathcal{O}_{Y, \varphi(x)} \xrightarrow{\varphi^\sharp} \mathcal{O}_{X, x}$ is a local homomorphism of monoids.

The sheaf of monoids associated with the *prime spectrum* $\text{Spec}(M)$ satisfies the following properties.

- The stalk at $\mathfrak{p} \in \text{Spec}(M)$ is $\mathcal{O}_{\mathfrak{p}} = S^{-1}M$, with $S = \mathfrak{p}^c$.
- For any $f \in M$, the map $\varphi : M_f \rightarrow \Gamma(D(fM), \mathcal{O})$ defined by

$$\varphi(x)(\mathfrak{p}) = a/f^n \in \mathcal{O}_{\mathfrak{p}} \quad \text{for all } \mathfrak{p} \in D(fM), \text{ for all } x = a/f^n \in M_f$$

is an isomorphism.

- On an open set $U \subset \text{Spec}(M)$, a section $s \in \Gamma(U, \mathcal{O})$ is an element of $\prod_{\mathfrak{p} \in U} \mathcal{O}_{\mathfrak{p}}$ such that on any open set $D(f) \subset U$ its restriction agrees with an element in M_f .

For any geometric space (X, \mathcal{O}_X) , one defines (as in [DG70]) a canonical morphism $\psi_X : X \rightarrow \text{Spec}(\mathcal{O}(X))$.

DEFINITION 3.16. A geometric space (X, \mathcal{O}_X) is a *prime spectrum* if the morphism ψ_X is an isomorphism. (X, \mathcal{O}_X) is a *geometric $\mathfrak{M}\mathfrak{o}$ -scheme* if X admits an open covering by prime spectra.

The terminology is justified since the $\mathfrak{M}\mathfrak{o}$ -functor $\underline{X}(M) = \text{Hom}(\text{Spec } M, X)$ associated to a geometric $\mathfrak{M}\mathfrak{o}$ -scheme X is an $\mathfrak{M}\mathfrak{o}$ -scheme in the sense of Definition 3.10. We can now state the following proposition.

PROPOSITION 3.17. *Under the same set-theoretic conditions as in [DG70], any $\mathfrak{M}\mathfrak{o}$ -scheme \underline{X} can be represented in the form*

$$\underline{X}(N) = \text{Hom}(\text{Spec}(N), X), \quad N \in \text{Obj}(\mathfrak{M}\mathfrak{o}) \tag{38}$$

for a geometric $\mathfrak{M}\mathfrak{o}$ -scheme X which is unique up to isomorphism.

The proof of this proposition follows the lines of that for \mathbb{Z} -schemes exposed in [DG70].

Remark 3.18. The geometric realization $|\mathcal{F}|$ makes sense for any $\mathfrak{M}\mathfrak{o}$ -functor \mathcal{F} and is defined as an inductive limit of prime spectra: cf. Proposition 4.1 of [DG70]. The existence of a final object in the full subcategory \mathfrak{Ab} yields a canonical identification of $|\mathcal{F}|$ with the set $\mathcal{F}(\mathbb{F}_1)$ endowed with a suitable topology (whose open sets U correspond to open subfunctors \mathcal{F}_U of \mathcal{F}) and the sheaf of monoids given by the morphisms to the $\mathfrak{M}\mathfrak{o}$ -functor \mathcal{D} , that is, the affine line. Proposition 3.9 ensures that the natural map from $\text{Hom}(\text{Spec}(N), X)$ to $\underline{X}(N)$ is surjective. We refer to [CC09] for a detailed proof.

For any $\mathfrak{M}\mathfrak{o}$ -scheme \underline{X} and any monoid M , there is a natural map of sets connecting $\underline{X}(M)$ to the set underlying the geometric realization of the scheme \underline{X} . This map has no analogue in the theory of \mathbb{Z} -schemes.

DEFINITION 3.19. Let \underline{X} be an $\mathfrak{M}\mathfrak{o}$ -scheme and let X be its geometric realization. For any monoid M we define the canonical map of sets

$$\pi_M : \underline{X}(M) \rightarrow X, \quad \pi_M(\phi) = \phi(\mathfrak{p}_M), \quad \phi \in \text{Hom}(\text{Spec}(M), X). \tag{39}$$

An important property of the map π_M is described by the following.

PROPOSITION 3.20. *Let \underline{X} be an $\mathfrak{M}\mathfrak{o}$ -scheme and let X be its geometric realization. Let U be an open subset of X and let $\underline{U} \subset \underline{X}$ be the associated open subfunctor. Then*

$$\underline{U}(M) = \pi_M^{-1}(U) \subset \underline{X}(M). \tag{40}$$

Proof. Since U is open, for any $\phi \in \underline{X}(M) = \text{Hom}(\text{Spec}(M), X)$, it follows from the locality of the theory of $\mathfrak{M}\mathfrak{o}$ -schemes that

$$\pi_M(\phi) = \phi(\mathfrak{p}_M) \in U \iff \phi^{-1}(U) = \text{Spec}(M). \quad \square$$

3.7 Restriction to abelian groups

In this section we describe the functor obtained by restricting $\mathfrak{M}\mathfrak{o}$ -schemes to the category $\mathfrak{A}\mathfrak{b}$ of abelian groups. We first recall the definition of the natural functor-inclusion of $\mathfrak{A}\mathfrak{b}$ in $\mathfrak{M}\mathfrak{o}$.

PROPOSITION 3.21. *The covariant functor*

$$\mathbb{F}_1[\cdot] : \mathfrak{A}\mathfrak{b} \rightarrow \mathfrak{M}\mathfrak{o}, \quad H \mapsto \mathbb{F}_1[H]$$

embeds the category of abelian groups as a full subcategory of the category of commutative monoids.

Proof. We show that the group homomorphism

$$\text{Hom}_{\mathfrak{A}\mathfrak{b}}(H, K) \longrightarrow \text{Hom}_{\mathfrak{M}\mathfrak{o}}(\mathbb{F}_1[H], \mathbb{F}_1[K]), \quad \phi \mapsto \mathbb{F}_1[\phi]$$

is bijective. It is injective by restriction to $H \subset \mathbb{F}_1[H]$. Moreover, any unital monoid homomorphism in $\text{Hom}_{\mathfrak{M}\mathfrak{o}}(\mathbb{F}_1[H], \mathbb{F}_1[K])$ preserves the absorbing elements and sends invertible elements to invertible elements since it is unital. Thus, it arises from a group homomorphism. \square

We identify $\mathfrak{A}\mathfrak{b}$ with this full subcategory of $\mathfrak{M}\mathfrak{o}$. Of course, any $\mathfrak{M}\mathfrak{o}$ -functor $\underline{X} : \mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}$ can be restricted to $\mathfrak{A}\mathfrak{b}$ and it gives rise to a functor taking values in $\mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}$.

In fact, there is a pair of adjoint functors, $\mathfrak{A}\mathfrak{b} \rightarrow \mathfrak{M}\mathfrak{o}$, $H \mapsto \mathbb{F}_1[H]$ and $\mathfrak{M}\mathfrak{o} \rightarrow \mathfrak{A}\mathfrak{b}$, $M \mapsto M^\times$; these two functors are linked by the isomorphism

$$\text{Hom}_{\mathfrak{M}\mathfrak{o}}(\mathbb{F}_1[H], M) \cong \text{Hom}_{\mathfrak{A}\mathfrak{b}}(H, M^\times).$$

Moreover, for a monoid M , the *Weil restriction* of the functor $\text{spec } M$, is defined by

$$\mathfrak{A}\mathfrak{b} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}, \quad H \mapsto \text{Hom}_{\mathfrak{M}\mathfrak{o}}(M, \mathbb{F}_1[H]). \quad (41)$$

The next proposition shows that the restriction to $\mathfrak{A}\mathfrak{b}$ of an $\mathfrak{M}\mathfrak{o}$ -scheme is a direct sum of representable functors.

PROPOSITION 3.22. *Let \underline{X} be an $\mathfrak{M}\mathfrak{o}$ -scheme and X its geometric realization. Then the (restriction) functor*

$$\underline{X} : \mathfrak{A}\mathfrak{b} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}, \quad H \mapsto \text{Hom}(\text{Spec } \mathbb{F}_1[H], X) = \underline{X}(\mathbb{F}_1[H])$$

is the disjoint union

$$\underline{X}(\mathbb{F}_1[H]) = \bigsqcup_{x \in X} \underline{X}_x(H), \quad \underline{X}_x(H) = \text{Hom}_{\mathfrak{A}\mathfrak{b}}(\mathcal{O}_{X,x}^\times, H). \quad (42)$$

Proof. Let $\varphi \in \text{Hom}(\text{Spec } \mathbb{F}_1[H], X)$. The unique point $\mathfrak{p} \in \text{Spec } \mathbb{F}_1[H]$ corresponds to the ideal (0). Let $\varphi(\mathfrak{p}) = x \in X$ be its image; there is a corresponding map of the stalks

$$\varphi^\# : \mathcal{O}_{\varphi(\mathfrak{p})} \rightarrow \mathcal{O}_{\mathfrak{p}} = \mathbb{F}_1[H].$$

This homomorphism is local by hypothesis: this means that the inverse image of (0) by $\varphi^\#$ is the maximal ideal of $\mathcal{O}_{\varphi(\mathfrak{p})} = \mathcal{O}_{X,x}$. Therefore, the map $\varphi^\#$ is entirely determined by the group homomorphism $\rho \in \text{Hom}_{\mathfrak{A}\mathfrak{b}}(\mathcal{O}_{X,x}^\times, H)$ obtained as the restriction of $\varphi^\#$.

Thus, $\varphi \in \text{Hom}(\text{Spec } \mathbb{F}_1[H], X)$ is entirely specified by a point $x \in X$ and a group homomorphism $\rho \in \text{Hom}_{\mathfrak{Ab}}(\mathcal{O}_{X,x}^\times, H)$. □

Proposition 3.22 is the analogue of the description given in [DG70, I, §§ 1 and 4] of the restriction of a \mathbb{Z} -functor to the category of fields. The further restriction of an \mathfrak{Mo} -scheme \underline{X} to the category of finite abelian groups corresponds to Soulé’s construction in [Sou04]. We denote by $\underline{X}(\mathbb{F}_{1^n})$ the evaluation of \underline{X} on the cyclic group $\mathbb{Z}/n\mathbb{Z}$, and in particular we use the notation $\mathbb{F}_{1^n} = \mathbb{F}_1[\mathbb{Z}/n\mathbb{Z}]$.

In algebraic geometry, given an algebraic variety X one knows that the degree of transcendence of the residue field $\kappa(x)$ of a point $x \in X$ measures the *dimension* of the closure $\overline{\{x\}} \subset X$. For an \mathfrak{Mo} -scheme one has the following corresponding (local) notion.

DEFINITION 3.23. Let X be a geometric \mathfrak{Mo} -scheme and let $x \in X$ be a point. The *local dimension* $n(x)$ of x is the rank of the abelian group $\mathcal{O}_{X,x}^\times$.

The local dimension determines a natural grading on the restriction of an \mathfrak{Mo} -scheme \underline{X} to \mathfrak{Ab} by assigning the degree $n(x)$ to the subset $\underline{X}_x(H) \subset \underline{X}(\mathbb{F}_1[H])$ in the decomposition (42).

PROPOSITION 3.24. *The restriction of the following \mathfrak{Mo} -schemes \underline{X} to \mathfrak{Ab} coincides, as a functor to $\mathbb{Z}_{\geq 0}$ -graded sets, with the functors defined in [CC08].*

- (1) *Spectra of fields in \mathfrak{Mo} (cf. Remark 3.1) of type: $\underline{X} = \text{spec}(\mathbb{F}_1[H])$, H finite abelian group.*
- (2) *Tori \mathbb{G}_m^n : $\underline{X} = \text{spec}(\mathbb{F}_1[\mathbb{Z}^n])$.*
- (3) *Affine space \mathcal{D}^n : $\underline{X} = \text{spec}(\mathbb{F}_1[T_1, \dots, T_n])$.*
- (4) *Projective line: $\underline{X} = \mathbb{P}_{\mathbb{F}_1}^1$.*

Proof.

- (1) The space $\text{Spec}(\mathbb{F}_1[H])$ has a single point and the local dimension is zero which agrees with [CC08, Example 3.1].
- (2) For $n \in \mathbb{N}$, the space $\text{Spec}(\mathbb{F}_1[\mathbb{Z}^n])$ consists of a single point and the local dimension is n which agrees with [CC08, Example 3.2].
- (3) Let $M = \mathbb{F}_1[T_1, \dots, T_n] = \{0\} \cup \{T_1^{a_1} \cdots T_n^{a_n} \mid a_j \in \mathbb{Z}_{\geq 0}\}$. A prime ideal \mathfrak{p} of M is of the form $\mathfrak{p} = \bigcup_{i \in J} T_i M$, where $J \subseteq \{1, \dots, n\}$. One has $\mathcal{O}_{X,\mathfrak{p}}^\times \simeq \mathbb{Z}^{J^c}$, with $J^c = \{1, \dots, n\} \setminus J$, generated by the T_j , with $j \in J^c$. Thus, the local dimension of $\text{Spec}(M)$ at \mathfrak{p} is the cardinality of J^c and this agrees with [CC08, Example 3.3].
- (4) The geometric realization $\mathbb{P}_{\mathbb{F}_1}^1$ is obtained by gluing two affine lines (cf. [Dei05] and § 3) and consists of three points, that is, $\mathbb{P}_{\mathbb{F}_1}^1 = \{0, u, \infty\}$, where $n(0) = 0 = n(\infty)$ and $n(u) = 1$. This agrees with [CC08, Example 3.4]. □

4. The category \mathfrak{MR} and \mathbb{F}_1 -schemes

As we already remarked in [CC08] (cf. § 4), the definition of the (affine) variety over \mathbb{F}_1 for a Chevalley group is inclusive of the datum given by a covariant functor to the category \mathfrak{Sets} of sets, fulfilling much stronger properties than those required originally in [Sou04] (for affine varieties). The domain of such a functor is a category which contains both the category of commutative rings and that of monoids (these two categories being linked by a pair of adjoint functors) and moreover in the definition of the variety one also requires the existence of a suitable

natural transformation. In this section we develop the details of this construction following an idea we learnt from P. Cartier. The introduction in § 4.1 develops some generalities on the gluing process of two categories linked by a pair of adjoint functors. In § 4.2, we also treat in this generality the extension of functors. The specific case of interest is covered in § 4.3 where we show that Chevalley groups are schemes over \mathbb{F}_{12} . Finally, in § 4.4 we extend the computation of zeta functions of [Dei06, Theorem 1] to our new setup which is no longer restricted to toric varieties (as it covers in particular the case of Chevalley groups).

4.1 Gluing two categories using adjoint functors

We consider two categories \mathcal{C} and \mathcal{C}' and a pair of adjoint functors $\beta : \mathcal{C} \rightarrow \mathcal{C}'$ and $\beta^* : \mathcal{C}' \rightarrow \mathcal{C}$. Thus, by definition one has a canonical identification

$$\mathrm{Hom}_{\mathcal{C}'}(\beta(H), R) \stackrel{\Phi}{\cong} \mathrm{Hom}_{\mathcal{C}}(H, \beta^*(R)) \quad \text{for all } H \in \mathrm{Obj}(\mathcal{C}), R \in \mathrm{Obj}(\mathcal{C}'). \tag{43}$$

The naturality of Φ is expressed by the commutativity of the following diagram where the vertical arrows are given by composition, for all $f \in \mathrm{Hom}_{\mathcal{C}}(G, H)$ and for all $h \in \mathrm{Hom}_{\mathcal{C}'}(R, S)$.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}'}(\beta(H), R) & \xrightarrow{\Phi} & \mathrm{Hom}_{\mathcal{C}}(H, \beta^*(R)) \\ \mathrm{Hom}(\beta(f), h) \downarrow & & \downarrow \mathrm{Hom}(f, \beta^*(h)) \\ \mathrm{Hom}_{\mathcal{C}'}(\beta(G), S) & \xrightarrow{\Phi} & \mathrm{Hom}_{\mathcal{C}}(G, \beta^*(S)) \end{array} \tag{44}$$

We now define a category $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$ obtained by gluing \mathcal{C} and \mathcal{C}' . The collection⁶ of objects of \mathcal{C}'' is obtained as the *disjoint union* of the collection of objects of \mathcal{C} and \mathcal{C}' . For $R \in \mathrm{Obj}(\mathcal{C}')$ and $H \in \mathrm{Obj}(\mathcal{C})$, one sets $\mathrm{Hom}_{\mathcal{C}''}(R, H) = \emptyset$. On the other hand, one defines

$$\mathrm{Hom}_{\mathcal{C}''}(H, R) = \mathrm{Hom}_{\mathcal{C}'}(\beta(H), R) \cong \mathrm{Hom}_{\mathcal{C}}(H, \beta^*(R)). \tag{45}$$

The morphisms between objects contained in the same category are unchanged. The composition of morphisms in \mathcal{C}'' is defined as follows. For $\phi \in \mathrm{Hom}_{\mathcal{C}''}(H, R)$ and $\psi \in \mathrm{Hom}_{\mathcal{C}}(H', H)$, one defines $\phi \circ \psi \in \mathrm{Hom}_{\mathcal{C}''}(H', R)$ as the composite

$$\phi \circ \beta(\psi) \in \mathrm{Hom}_{\mathcal{C}'}(\beta(H'), R) = \mathrm{Hom}_{\mathcal{C}''}(H', R). \tag{46}$$

Using the commutativity of the diagram (44), one obtains

$$\Phi(\phi \circ \beta(\psi)) = \Phi(\phi) \circ \psi \in \mathrm{Hom}_{\mathcal{C}}(H', \beta^*(R)). \tag{47}$$

Similarly, for $\theta \in \mathrm{Hom}_{\mathcal{C}'}(R, R')$ one defines $\theta \circ \phi \in \mathrm{Hom}_{\mathcal{C}''}(H, R')$ as the composite

$$\theta \circ \phi \in \mathrm{Hom}_{\mathcal{C}'}(\beta(H), R') = \mathrm{Hom}_{\mathcal{C}''}(H, R') \tag{48}$$

and using again the commutativity of (44), one obtains that

$$\Phi(\theta \circ \phi) = \beta^*(\theta) \circ \Phi(\phi) \in \mathrm{Hom}_{\mathcal{C}}(H, \beta^*(R')). \tag{49}$$

Moreover, one also defines specific morphisms α_H and α'_R as follows

$$\alpha_H = \mathrm{id}_{\beta(H)} \in \mathrm{Hom}_{\mathcal{C}'}(\beta(H), \beta(H)) = \mathrm{Hom}_{\mathcal{C}''}(H, \beta(H)) \tag{50}$$

$$\alpha'_R = \Phi^{-1}(\mathrm{id}_{\beta^*(R)}) \in \Phi^{-1}(\mathrm{Hom}_{\mathcal{C}}(\beta^*(R), \beta^*(R))) = \mathrm{Hom}_{\mathcal{C}''}(\beta^*(R), R). \tag{51}$$

⁶ It is not a set: we refer for details to the discussion contained in the preliminaries of [DG70].

By construction, one obtains

$$\text{Hom}_{\mathcal{C}''}(H, R) = \{g \circ \alpha_H \mid g \in \text{Hom}_{\mathcal{C}'}(\beta(H), R)\} \tag{52}$$

and for any morphism $\rho \in \text{Hom}_{\mathcal{C}}(H, K)$ the following equation holds

$$\alpha_K \circ \rho = \beta(\rho) \circ \alpha_H. \tag{53}$$

Similarly, it also turns out that

$$\text{Hom}_{\mathcal{C}''}(H, R) = \{\alpha'_R \circ f \mid f \in \text{Hom}_{\mathcal{C}}(H, \beta^*(R))\} \tag{54}$$

and the associated equalities hold

$$\alpha'_S \circ \beta^*(\rho) = \rho \circ \alpha'_R \quad \text{for all } \rho \in \text{Hom}_{\mathcal{C}'}(R, S) \tag{55}$$

$$g \circ \alpha_H = \alpha'_R \circ \Phi(g) \quad \text{for all } g \in \text{Hom}_{\mathcal{C}'}(\beta(H), R). \tag{56}$$

PROPOSITION 4.1. *We have that $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$ is a category which contains \mathcal{C} and \mathcal{C}' as full subcategories. Moreover, for any object H of \mathcal{C} and R of \mathcal{C}' , one has*

$$\text{Hom}_{\mathcal{C}''}(H, R) = \text{Hom}_{\mathcal{C}'}(\beta(H), R) \cong \text{Hom}_{\mathcal{C}}(H, \beta^*(R)).$$

Proof. One needs to check that the composition \circ'' of morphisms is associative in \mathcal{C}'' , that is, that $h \circ'' (g \circ'' f) = (h \circ'' g) \circ'' f$. The only relevant case to check is when the image of f is an object H of \mathcal{C} and the image of g is an object R of \mathcal{C}' . Then $f(G) = H$, with G an object of \mathcal{C} and $h(R) = S$ an object of \mathcal{C}' . One has $g \in \text{Hom}_{\mathcal{C}'}(\beta(H), R)$ and

$$g \circ'' f = g \circ \beta(f), \quad h \circ'' (g \circ'' f) = h \circ (g \circ \beta(f)) = (h \circ g) \circ \beta(f) = (h \circ'' g) \circ'' f. \quad \square$$

4.2 Extension of functors

We keep the notation introduced in § 4.1. In particular, (β, β^*) denotes a pair of adjoint functors linking \mathcal{C} and \mathcal{C}' , that is, $\beta: \mathcal{C} \rightarrow \mathcal{C}'$ and $\beta^*: \mathcal{C}' \rightarrow \mathcal{C}$, and the isomorphism (43) holds. Let $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{T}$ and $\mathcal{F}': \mathcal{C}' \rightarrow \mathcal{T}$ be covariant functors to the same category \mathcal{T} . It is a straightforward routine to verify that the assignment of a natural transformation $\mathcal{F} \rightarrow \mathcal{F}' \circ \beta$ is equivalent to giving a natural transformation $\mathcal{F} \circ \beta^* \rightarrow \mathcal{F}'$. By implementing in this setup the category $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$ defined in § 4.1, one obtains the following more precise result.

PROPOSITION 4.2.

- (1) *With the above notation, let $\mathcal{F}'' : \mathcal{C}'' \rightarrow \mathcal{T}$ denote a covariant functor. Then the assignment $H \mapsto \mathcal{F}''(\alpha_H)$ defines a natural transformation $\mathcal{F}''|_{\mathcal{C}} \rightarrow \mathcal{F}''|_{\mathcal{C}'} \circ \beta$ and analogously the assignment $R \mapsto \mathcal{F}''(\alpha'_R)$ defines a natural transformation $\mathcal{F}''|_{\mathcal{C}} \circ \beta^* \rightarrow \mathcal{F}''|_{\mathcal{C}'}$.*
- (2) *Let $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{T}$ and $\mathcal{F}': \mathcal{C}' \rightarrow \mathcal{T}$ be covariant functors. Then:*
 - (a) *given a natural transformation $\mathcal{F} \rightarrow \mathcal{F}' \circ \beta$, there exists a unique covariant functor \mathcal{F}'' which extends \mathcal{F} , \mathcal{F}' and agrees with the natural transformation on the morphisms α_H ;*
 - (b) *given a natural transformation $\mathcal{F} \circ \beta^* \rightarrow \mathcal{F}'$, there exists a unique covariant functor \mathcal{F}'' which extends \mathcal{F} , \mathcal{F}' and agrees with the natural transformation on the morphisms α'_R .*

Proof.

- (1) This follows from (53) and (55).
- (2) (a) A natural transformation $\mathcal{F} \rightarrow \mathcal{F}' \circ \beta$ determines, by (52), the extension from $\mathcal{C} \cup \mathcal{C}'$ to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$.
- (b) The proof is similar to the proof of (a). □

Let us assume now that we are given a representable functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{T}$, where $\mathcal{T} = \mathfrak{Sets}$. In the following we investigate the extensions of \mathcal{F} to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$.

Note first that \mathcal{F} admits a unique representable extension to \mathcal{C}'' . Indeed, the representability of \mathcal{F} amounts to the existence of an object G in \mathcal{C} such that

$$\mathcal{F}(H) = \text{Hom}_{\mathcal{C}}(G, H) \quad \text{for all } H \in \text{Obj}(\mathcal{C}).$$

If the extension of \mathcal{F} is represented by an object of \mathcal{C}'' , then this object must necessarily belong to \mathcal{C} , since by definition of \mathcal{C}'' there is no morphism of \mathcal{C}'' from an object of \mathcal{C}' to an object of \mathcal{C} . Moreover, by restriction to \mathcal{C} one obtains the uniqueness of the extension. Thus, any extension of \mathcal{F} to \mathcal{C}'' which is represented by an object of this latter category is unique (as a representable functor) and is given on \mathcal{C}' by

$$\mathcal{F}'(R) = \text{Hom}_{\mathcal{C}'}(\beta(G), R). \tag{57}$$

The natural transformation $\mathcal{F} \rightarrow \mathcal{F}' \circ \beta$ is simply given by the restriction of the functor β

$$\beta : \text{Hom}_{\mathcal{C}}(G, H) \rightarrow \text{Hom}_{\mathcal{C}'}(\beta(G), \beta(H)).$$

Similarly, the natural transformation $\mathcal{F} \circ \beta^* \rightarrow \mathcal{F}'$ is defined by the identity map

$$\text{Hom}_{\mathcal{C}}(G, \beta^*(R)) \rightarrow \mathcal{F}'(R) = \text{Hom}_{\mathcal{C}'}(\beta(G), R) \cong \text{Hom}_{\mathcal{C}}(G, \beta^*(R)).$$

Thus, the following result holds.

PROPOSITION 4.3. *Let $\mathcal{F} : \mathcal{C} \rightarrow \mathfrak{Sets}$ be a representable functor.*

- (1) *There exists a unique extension $\tilde{\mathcal{F}}$ of \mathcal{F} to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$ as a representable functor.*
- (2) *Let \mathcal{G} be any extension of \mathcal{F} to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$. Then, there exists a unique morphism of functors from $\tilde{\mathcal{F}}$ to \mathcal{G} which restricts to the identity on \mathcal{C} .*

Proof. For part (1), we note that the object of \mathcal{C} representing \mathcal{F} is unique up to isomorphism and it represents $\tilde{\mathcal{F}}$.

The proof of part (2) follows from the following facts. If $A \in \text{Obj}(\mathcal{C})$ represents $\tilde{\mathcal{F}}$, by applying Yoneda's lemma (i.e. $\text{Nat}(\tilde{\mathcal{F}}, \mathcal{G}) \simeq \mathcal{G}(A)$) we know that there exists a uniquely determined natural transformation $\phi : \tilde{\mathcal{F}} \rightarrow \mathcal{G}$ associated with any object ρ of $\mathcal{G}(A)$: the pair (ϕ, ρ) being linked by the formulae $\rho = \phi(A)(\text{id}_A) \in \mathcal{G}(A)$ and $\phi = \rho^\#(\rho^\#(R)(\beta) = \mathcal{G}(\beta)(\rho))$. However, since the restriction of ϕ to \mathcal{C} is the identity map from $\mathcal{F}(A)$ to $\mathcal{G}(A) = \mathcal{F}(A)$, one obtains the required uniqueness. \square

The following corollary shows that even though the extension $\tilde{\mathcal{F}}$ of \mathcal{F} to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$ is not unique, it is still universal.

COROLLARY 4.4. *Let $\mathcal{F} : \mathcal{C} \rightarrow \mathfrak{Sets}$ be a representable functor and let $\mathcal{F}' : \mathcal{C}' \rightarrow \mathfrak{Sets}$ be defined as in (57). Let \mathcal{G}' be a functor from \mathcal{C}' to \mathfrak{Sets} and let $\phi : \mathcal{F} \rightarrow \mathcal{G}' \circ \beta$ be a natural transformation. Then there exists a unique morphism of functors $\psi : \mathcal{F}' \rightarrow \mathcal{G}'$ such that*

$$\phi_H = \psi_{\beta(H)} \circ \tilde{\mathcal{F}}(\alpha_H) \quad \text{for all } H \in \text{Obj}(\mathcal{C}). \tag{58}$$

Proof. Given ϕ and \mathcal{G}' , there exists by Proposition 4.2 a unique extension \mathcal{G} of \mathcal{F} to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$ which restricts to \mathcal{G}' on \mathcal{C}' and is such that

$$\phi_H = \mathcal{G}(\alpha_H) \quad \text{for all } H \in \text{Obj}(\mathcal{C}).$$

A morphism of functors from $\tilde{\mathcal{F}}$ to \mathcal{G} extending the identity on \mathcal{C} is entirely specified by its restriction to \mathcal{C}' which is a morphism of functors ψ from \mathcal{F}' to \mathcal{G}' and it must be compatible with

the morphisms α_H . This compatibility is given by (58). Thus, the existence and uniqueness of ψ follows from Proposition 4.3. □

The next proposition states a similar, but simpler, result for extensions of functors from \mathcal{C}' to the larger category $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$.

PROPOSITION 4.5. *Let $\mathcal{F}' : \mathcal{C}' \rightarrow \mathbf{Sets}$ be a functor.*

- (1) *There exists a unique extension $\tilde{\mathcal{F}}$ of \mathcal{F}' to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$ given by $\mathcal{F}' \circ \beta$ on \mathcal{C} and such that $\tilde{\mathcal{F}}(\alpha_H) = \text{id}_{\beta(H)}$ for all objects H of \mathcal{C} .*
- (2) *Let \mathcal{G} be any extension of \mathcal{F}' to $\mathcal{C}'' = \mathcal{C} \cup_{\beta, \beta^*} \mathcal{C}'$, then there exists a unique morphism of functors from \mathcal{G} to $\tilde{\mathcal{F}}$ which is the identity on \mathcal{C}' .*

Proof. The first statement follows from Proposition 4.2(2) by using the identity as a natural transformation. Similarly, for the second statement, again Proposition 4.2(2) determines a unique morphism of functors ϕ given by $\phi(H) = \mathcal{G}(\alpha_H)$ and obtained from the restriction of \mathcal{G} to \mathcal{C} to $\mathcal{G} \circ \beta = \mathcal{F}' \circ \beta$. We extend ϕ as the identity on \mathcal{C}' . The commutative diagram

$$\begin{array}{ccc}
 \mathcal{G}(H) & \xrightarrow{\mathcal{G}(\alpha_H)} & \mathcal{G}(\beta(H)) \\
 \downarrow \phi_H & & \downarrow \\
 \tilde{\mathcal{F}}(H) = \mathcal{F}'(\beta(H)) & \longrightarrow & \tilde{\mathcal{F}}(\beta(H))
 \end{array} \tag{59}$$

shows that one obtains a morphism of functors from \mathcal{G} to $\tilde{\mathcal{F}}$. The same diagram also gives the uniqueness of ϕ . □

4.3 \mathbb{F}_1 -schemes and Chevalley groups

We apply the construction described in §§ 4.1 and 4.2 to the following pair of adjoint covariant functors β and β^* . The functor

$$\beta : \mathfrak{M}o \rightarrow \mathfrak{R}ing, \quad M \mapsto \beta(M) = \mathbb{Z}[M] \tag{60}$$

associates to a monoid M the convolution ring $\mathbb{Z}[M]$ (the 0 element of M is sent to 0). The adjoint functor β^*

$$\beta^* : \mathfrak{R}ing \rightarrow \mathfrak{M}o, \quad R \mapsto \beta^*(R) = R \tag{61}$$

associates to a ring R the ring itself viewed as a multiplicative monoid (forgetful functor). The adjunction relation states that

$$\text{Hom}_{\mathfrak{R}ing}(\beta(M), R) \cong \text{Hom}_{\mathfrak{M}o}(M, \beta^*(R)). \tag{62}$$

We apply Proposition 4.1 to construct the category $\mathfrak{M}\mathfrak{A} = \mathfrak{R}ing \cup_{\beta, \beta^*} \mathfrak{M}o$. Thus, for every object R of $\mathfrak{R}ing$, one obtains a morphism

$$\alpha'_R \in \text{Hom}_{\mathfrak{M}\mathfrak{A}}(\beta^*(R), R) \tag{63}$$

and the following relation between the morphisms of $\mathfrak{M}\mathfrak{A}$:

$$f \circ \alpha'_R = \alpha'_S \circ \beta^*(f), \quad \text{for all } f \in \text{Hom}_{\mathfrak{R}ing}(R, S). \tag{64}$$

Similarly, for every monoid M one has a morphism

$$\alpha_M \in \text{Hom}_{\mathfrak{M}\mathfrak{A}}(M, \beta(M)) \tag{65}$$

together with the relation

$$\beta(f) \circ \alpha_M = \alpha_N \circ f \quad \text{for all } f \in \text{Hom}_{\mathfrak{M}\mathfrak{o}}(M, N). \tag{66}$$

DEFINITION 4.6. An \mathbb{F}_1 -functor is a covariant functor from the category $\mathfrak{M}\mathfrak{R} = \mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g} \cup_{\beta, \beta^*} \mathfrak{M}\mathfrak{o}$ to the category of sets.

Then, it follows from Proposition 4.2 that the assignment of an \mathbb{F}_1 -functor $\mathcal{X} : \mathfrak{M}\mathfrak{R} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}$ is equivalent to the specification of the following data.

- An $\mathfrak{M}\mathfrak{o}$ -functor \underline{X} .
- A \mathbb{Z} -functor $X_{\mathbb{Z}}$.
- A natural transformation $e : \underline{X} \rightarrow X_{\mathbb{Z}} \circ \beta$.

The third condition can be equivalently replaced by the assignment of a natural transformation $\underline{X} \circ \beta^* \rightarrow X_{\mathbb{Z}}$.

Now that we have at our disposal the category $\mathfrak{M}\mathfrak{R}$ obtained by gluing $\mathfrak{M}\mathfrak{o}$ and $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$ we introduce our notion of an \mathbb{F}_1 -scheme.

DEFINITION 4.7. An \mathbb{F}_1 -scheme is an \mathbb{F}_1 -functor $\mathcal{X} : \mathfrak{M}\mathfrak{R} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}\mathfrak{s}$, such that:

- the restriction $X_{\mathbb{Z}}$ of \mathcal{X} to $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$ is a \mathbb{Z} -scheme;
- the restriction \underline{X} of \mathcal{X} to $\mathfrak{M}\mathfrak{o}$ is an $\mathfrak{M}\mathfrak{o}$ -scheme;
- the natural transformation $e : \underline{X} \circ \beta^* \rightarrow X_{\mathbb{Z}}$ associated with a field is a bijection (of sets).

Morphisms of \mathbb{F}_1 -schemes are natural transformations of the corresponding functors.

Example 4.8. The first three examples of Proposition 3.24 are affine $\mathfrak{M}\mathfrak{o}$ -schemes given by representable functors $\underline{X} = \mathfrak{sp}\mathfrak{e}\mathfrak{c}(\mathbb{F}_1[M])$, for some monoid M . By applying Proposition 4.3(1), these schemes determine, canonically, representable \mathbb{F}_1 -functors \mathcal{X} . By construction, the \mathbb{Z} -scheme corresponding to \mathcal{X} (i.e. the restriction $X_{\mathbb{Z}}$ of \mathcal{X} to $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$) is $\mathfrak{sp}\mathfrak{e}\mathfrak{c}(\mathbb{Z}[M])$. For any object R of $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$, the natural transformation $e(R) : \underline{X} \circ \beta^*(R) \rightarrow X_{\mathbb{Z}}(R)$ which is given by the adjunction relation (62), is a *bijection* of sets.

Example 4.9. The projective line \mathbb{P}^1 is the $\mathfrak{M}\mathfrak{o}$ -scheme described in Example 3.12. It associates to an object M of $\mathfrak{M}\mathfrak{o}$ the set $\mathbb{P}^1(M)$ of complemented submodules E of rank one in M^2 (the rank is defined locally), that is, the range of an idempotent matrix $e \in M_2(M)$, of rank one, with each line having at most one non-zero entry. Let R be an object of $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$ and let $M = \beta^*(R)$ be the underlying monoid. Then, the idempotent matrix $e \in M_2(M)$ is an idempotent matrix $e \in M_2(R)$. By applying Lemma 3.13 one has a complete description of these matrices e . One uses the natural isomorphism

$$\mathcal{P}(M) = M \sqcup_{M \times} M \cong \mathbb{P}^1(M)$$

and one checks that the corresponding matrices $e \in M_2(R)$ are all of rank one in the local sense of the definition of projective space $\mathbb{P}_{\mathbb{Z}}^1$ given in [DG70]. Thus, one obtains a natural transformation $e : \mathbb{P}^1 \circ \beta^* \rightarrow \mathbb{P}_{\mathbb{Z}}^1$. Moreover, when the object R of $\mathfrak{R}\mathfrak{i}\mathfrak{n}\mathfrak{g}$ is a field this natural transformation is a bijection since for any field K the one-dimensional subspaces of K^2 are ranges of projections of the above form. Note that, unlike the cases of the three schemes considered in Example 4.8, for the projective line $\underline{X} = \mathbb{P}^1$ it is not true that the natural transformation $e : \underline{X} \circ \beta^* \rightarrow X_{\mathbb{Z}}$ is a bijection for arbitrary rings.

Remark 4.10. Even though the restriction \underline{X} of an \mathbb{F}_1 -scheme \mathcal{X} to \mathfrak{Mo} is an \mathfrak{Mo} -scheme, the composite $\underline{X} \circ \beta^*$ is not in general a \mathbb{Z} -scheme since $\underline{X} \circ \beta^*$ may fail to be a local \mathbb{Z} -functor. In Example 4.9, for instance, $\underline{X} \circ \beta^*$ determines only a smaller portion of the projective line as a \mathbb{Z} -scheme. However, one can associate to $\underline{X} \circ \beta^*$ a *unique* \mathbb{Z} -scheme $(\underline{X} \circ \beta^*)_{\text{loc}}$ which is obtained by localization. This amounts to assigning to a ring R the set of solutions of (29), using $\underline{X} \circ \beta^*$ and an arbitrary partition of unity in R . Then, Proposition 4.3 describes a *canonical* morphism of \mathbb{Z} -schemes $\tau : (\underline{X} \circ \beta^*)_{\text{loc}} \rightarrow X_{\mathbb{Z}}$.

It is quite important to point out here that Definition 4.7 of an \mathbb{F}_1 -scheme does not require τ to be an isomorphism. When it is so, the obtained scheme $X_{\mathbb{Z}}$ is toric (cf. [Dei08]). In the case of Chevalley groups as described in [CC08], the corresponding schemes $X_{\mathbb{Z}}$ are *not* toric in general (it suffices to consider the case of $\text{SL}(2)$). For a Chevalley group G , the \mathbb{Z} -scheme $(\underline{X} \circ \beta^*)_{\text{loc}}$ is by construction a *disjoint sum* of affine schemes corresponding to the cells of the Bruhat decomposition (cf. [CC08, Theorem 4.1]), while the \mathbb{Z} -scheme $X_{\mathbb{Z}}$ is the algebraic group-scheme (i.e. Chevalley–Demazure group scheme) \mathfrak{G} associated with an irreducible root system of G (cf. [DG77]). The canonical morphism τ is effecting the gluing of the various cells.

Proposition 4.5 describes the natural morphism from the \mathfrak{Mo} -scheme \underline{X} to the ‘gadget’ (cf. [CC08, Definition 2.5]) associated with the \mathbb{Z} -scheme $X_{\mathbb{Z}}$.

Definition 4.7 admits variants corresponding to the extensions \mathbb{F}_{1^n} . For Chevalley schemes one considers the case $n = 2$. One defines the category $\mathfrak{Mo}^{(2)}$ of pairs (M, ϵ) made by an object M of \mathfrak{Mo} and an element $\epsilon \in M$ of square one. Morphisms $(M, \epsilon) \rightarrow (M_1, \epsilon_1)$ in $\mathfrak{Mo}^{(2)}$ are morphisms in \mathfrak{Mo} mapping $\epsilon \mapsto \epsilon_1$. One defines the functor $\beta : \mathfrak{Mo}^{(2)} \rightarrow \mathfrak{Ring}$ as $\beta(M, \epsilon) = \mathbb{Z}[M, \epsilon]$, where

$$\mathbb{Z}[M, \epsilon] = \mathbb{Z}[M]/J, \quad J = (1 + \epsilon)\mathbb{Z}[M]. \tag{67}$$

The adjoint functor $\beta^* : \mathfrak{Ring} \rightarrow \mathfrak{Mo}^{(2)}$ is given by $\beta^*(A) = (A, -1)$, where $(A, -1)$ is the object of $\mathfrak{Mo}^{(2)}$ given by the ring A viewed as a (multiplicative) monoid and the element $-1 \in A$. Then, the following adjunction relation holds for any commutative ring A

$$\text{Hom}_{\mathfrak{Ring}}(\beta(M, \epsilon), A) \cong \text{Hom}_{\mathfrak{Mo}^{(2)}}((M, \epsilon), \beta^*(A)). \tag{68}$$

Finally, we have the following result.

THEOREM 4.11. *The algebraic group-scheme \mathfrak{G} over \mathbb{Z} associated with (an irreducible root system of) a Chevalley group G extends to a scheme \mathcal{G} over \mathbb{F}_{1^2} .*

Proof. The proof follows from [CC08, Theorem 4.1], where we showed that:

- the construction of the functor \underline{G} extends from the category $\mathcal{F}_{ab}^{(2)}$ of pairs (D, ϵ) of a finite abelian group and an element of order two in D to the category $\mathfrak{Mo}^{(2)}$;
- the construction of the natural transformation e_G extends from $\mathcal{F}_{ab}^{(2)}$ to the category $\mathfrak{Mo}^{(2)}$; here e_G associates to any $A \in \text{Obj}(\mathfrak{Ring})$ a map

$$e_{G,A} : \underline{G}(A, -1) \rightarrow \mathfrak{G}(A); \tag{69}$$

- when A is a field the map $e_{G,A}$ is a bijection. □

The map $e_{G,A}$ of (69) is constructed in [CC08, proof of Theorem 4.1], and it yields the natural transformation $\underline{G} \circ \beta^* \rightarrow \mathfrak{G}$ (and using (56) the corresponding natural transformation $\underline{G} \rightarrow \mathfrak{G} \circ \beta$). For $H \in \text{Obj}(\mathfrak{Ab})$, this natural transformation is compatible with the group structure on the subset $\underline{G}^{(\ell)}(H) \subset \underline{G}(H)$ in *lowest degree* ℓ ($\ell = \text{rk } \mathfrak{G}$), for the grading on \underline{G} , as in [CC08, Definition 3.23].

4.4 Zeta function of Noetherian \mathbb{F}_1 -schemes

We recall that a congruence on a monoid M is an equivalence relation which is compatible with the semigroup operation. A monoid is Noetherian when any strictly increasing sequence of congruences is finite (cf. [Gil80, p. 30]). The following conditions on a monoid M are equivalent (cf. [Gil80, Theorems 7.7, 7.8 and 5.10]):

- M is Noetherian;
- M is finitely generated;
- $\mathbb{Z}[M]$ is a Noetherian ring.

In [Gil80, Theorem 5.1] it is proven that if M is a Noetherian monoid, then for any prime ideal $\mathfrak{p} \subset M$ the localized monoid $M_{\mathfrak{p}}$ is also Noetherian (the semigroup \mathfrak{p}^c is finitely generated and thus $M_{\mathfrak{p}}$ is also finitely generated). The same theorem also shows that the abelian group $(M_{\mathfrak{p}})^{\times}$ is finitely generated.

DEFINITION 4.12. An $\mathfrak{M}\mathfrak{o}$ -scheme is Noetherian if it admits a finite open cover by representable subfunctors $\text{spec}(M)$, where M are Noetherian monoids.

An \mathbb{F}_1 -scheme is Noetherian if the associated $\mathfrak{M}\mathfrak{o}$ and \mathbb{Z} -schemes are Noetherian.

A geometric $\mathfrak{M}\mathfrak{o}$ -scheme X is said to be *torsion free* if the groups $\mathcal{O}_{X,x}^{\times}$ of invertible elements of the monoids $\mathcal{O}_{X,x}$, for $x \in X$, are torsion free. The following result is related to Theorem 1 of [Dei06], but it applies to a wider class of varieties (i.e. non-necessarily toric).

THEOREM 4.13. *Let \mathcal{X} be a Noetherian \mathbb{F}_1 -scheme and let X be the geometric realization of its restriction \underline{X} to $\mathfrak{M}\mathfrak{o}$. Then, if X is torsion free, the following results hold.*

- (i) *There exists a polynomial $N(x + 1)$ with positive integral coefficients such that*

$$\#\underline{X}(\mathbb{F}_{1^n}) = N(n + 1) \quad \text{for all } n \in \mathbb{N}.$$

- (ii) *For each finite field \mathbb{F}_q the cardinality of the set of points of the \mathbb{Z} -scheme $X_{\mathbb{Z}}$ which are rational over \mathbb{F}_q is equal to $N(q)$.*
- (iii) *The zeta function of \mathcal{X} has the following description*

$$\zeta_{\mathcal{X}}(s) = \prod_{x \in X} \frac{1}{(1 - 1/s)^{\otimes n(x)}}, \tag{70}$$

where \otimes denotes Kurokawa’s tensor product and $n(x)$ is the local (finite) dimension of X at the point x (cf. Definition 3.23).

In (70), when $n(x) = 0$, we write

$$\left(1 - \frac{1}{s}\right)^{\otimes n(x)} = s.$$

We refer to [Kur92, Man95] for the details of the definition of Kurokawa’s tensor products and zeta functions.

When $\mathcal{X} = \mathbb{P}_{\mathbb{F}_1}^1$, the associated geometric space X is made by three points $X = \{0, u, \infty\}$ (cf. proof of Proposition 3.24(4)), $n(0) = n(\infty) = 0$ and $n(u) = 1$. Thus, (70) gives

$$\prod_{x \in X} \frac{1}{(1 - 1/s)^{\otimes n(x)}} = \frac{1}{s^2} \frac{1}{1 - 1/s} = \frac{1}{s(s - 1)}.$$

Formula (70) continues to hold even in the presence of torsion on the structural sheaf \mathcal{O}_X and in that case it corresponds to the treatment of torsion given in [Dei06].

Proof. (1) By definition, $\underline{X}(\mathbb{F}_{1^n})$ is the set obtained by evaluating the restriction of \underline{X} (from the subcategory $\mathfrak{M}\mathfrak{o}$ of $\mathfrak{M}\mathfrak{A}$) on $\mathfrak{A}\mathfrak{b}$, at the cyclic group $H = \mathbb{Z}/n\mathbb{Z}$. By applying Proposition 3.22, one has the decomposition

$$\underline{X}(H) = \bigsqcup_{x \in X} \underline{X}_x(H), \quad \underline{X}_x(H) = \text{Hom}_{\mathfrak{A}\mathfrak{b}}(\mathcal{O}_{X,x}^\times, H). \tag{71}$$

Since \mathcal{X} is Noetherian, X is a finite topological space and, by hypothesis, for each $x \in X$ the abelian group $\mathcal{O}_{X,x}^\times$ is finitely generated and torsion free. The rank of $\mathcal{O}_{X,x}^\times$ is $n(x)$, thus the set $\underline{X}_x(H) = \text{Hom}_{\mathfrak{A}\mathfrak{b}}(\mathcal{O}_{X,x}^\times, H)$ has cardinality $n^{n(x)}$. It follows that the function of the indeterminate y

$$P(y) = \sum_{x \in X} y^{n(x)} \tag{72}$$

is a polynomial with positive integral coefficients. Here $P(y)$ is related to the counting function of \mathcal{X} by the equation $N(x+1) = P(x)$. Then part (1) follows.

Part (2) follows from part (1) and the fact that the natural transformation $e : \underline{X} \circ \beta^* \rightarrow X_{\mathbb{Z}}$ (which is part of the set of data describing an \mathbb{F}_1 -scheme, cf. Definition 4.7) evaluated at any field is a bijection. In the case of a finite field \mathbb{F}_q , the corresponding monoid is $\mathbb{F}_1[H]$, for the cyclic group $H = \mathbb{Z}/n\mathbb{Z}$ of order $n = q - 1$.

(3) By definition, the zeta function $\zeta_{\mathcal{X}}(s)$ of \mathcal{X} is given by the formula (4) applied to the polynomial counting function $N(x)$ of \mathcal{X} . By part (1), we have $N(x+1) = P(x)$, where $P(x)$ is given by (72). In other words one has

$$N(q) = \sum_{x \in X} (q-1)^{n(x)}.$$

By construction (cf. [Sou04]), (4) transforms a sum of counting functions into a product of zeta functions. Thus, it is enough to show that for a monomial $(q-1)^n$, equation (4) gives the zeta function $1/((1-1/s)^{\otimes n})$. In order to check that, we start by making explicit the Kurokawa's tensor product (for $n > 0$) as follows

$$\left(1 - \frac{1}{s}\right)^{\otimes n} = \frac{\prod_{j \text{ even}} (s - n + j)^{\binom{n}{j}}}{\prod_{j \text{ odd}} (s - n + j)^{\binom{n}{j}}}. \tag{73}$$

The above equality is a straightforward consequence of the definition of Kurokawa's tensor product since the divisor of zeros of $(1 - 1/s)$ is $\{1\} - \{0\}$ and its n th power is given by the binomial formula

$$(\{1\} - \{0\})^n = \sum_k (-1)^k \binom{n}{k} \{n - k\}.$$

Then, we apply the simple fact (compare [Sou04])

$$N(x) = \sum_{k=0}^d a_k x^k \implies \zeta_N(s) = \prod_{k=0}^d (s - k)^{-a_k} \tag{74}$$

and (73) to conclude that for $(q-1)^n$ the zeta function is the inverse of $(1 - 1/s)^{\otimes n}$. □

5. The projective adèle class space

In this section we develop an application of the functorial approach to the theory of $\mathfrak{M}\mathfrak{o}$ -schemes to explain, at a conceptual level, the spectral realization of zeros of L -functions for an arbitrary global field \mathbb{K} .

5.1 Vanishing result for $\mathfrak{M}\mathfrak{o}$ -schemes

In this section, we first briefly review some standard facts on sheaf cohomology and then we show that for sheaves of abelian groups over (the geometric realization of) an $\mathfrak{M}\mathfrak{o}$ -scheme, sheaf cohomology and Čech cohomology agree.

Given a topological space X , we denote by $\mathfrak{Ab}(X)$ the category of sheaves of abelian groups on X . It is a well-known fact that $\mathfrak{Ab}(X)$ is an abelian category with enough injectives (cf. [Gro57, Propositions 3.1.1 and 3.1.2]). For any open subset $U \subset X$ the functor

$$\Gamma(U, \cdot) : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}, \quad F \mapsto \Gamma(U, F) = F(U)$$

describes the space of sections of the sheaf F on U . It is well known that the functor $\Gamma(U, \cdot)$ is left-exact. Its derived functor defines the sheaf cohomology $H(U, F)$. Moreover, for any point $x \in X$ the functor ‘stalk of F at x ’

$$\mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}, \quad F \mapsto \varinjlim_{x \in U} \Gamma(U, F) =: F_x \tag{75}$$

is exact.

PROPOSITION 5.1. *Let X be the geometric realization of an $\mathfrak{M}\mathfrak{o}$ -scheme, then the following results hold.*

- (1) *For any open affine set $U \subset X$*

$$H^p(U, F) = 0 \quad \text{for all } p > 0, \text{ for all } F \in \text{Obj } \mathfrak{Ab}(X). \tag{76}$$

- (2) *Let $\mathcal{U} = \{U_j\}_{j \in J}$ be an open cover of X such that all finite intersections $\bigcap_{j_k} U_{j_k}$ are affine, then for any sheaf F of abelian groups on X , one has*

$$H^p(X, F) = \check{H}^p(\mathcal{U}, F) \quad \text{for all } p \geq 0, \tag{77}$$

where the cohomology on the right-hand side is the Čech cohomology relative to the covering \mathcal{U} .

- (3) *Let $Y = U^c$ be the complement of an affine open set $U \subset X$. Then, for any sheaf F of abelian groups on X one has the exact sequence*

$$0 \rightarrow H_Y^0(X, F) \rightarrow H^0(X, F) \rightarrow H^0(U, F|_U) \rightarrow H_Y^1(X, F) \rightarrow H^1(X, F) \rightarrow 0 \tag{78}$$

where $H_Y^*(X, F)$ denotes the cohomology with support on Y .

Proof. (1) Let $U = \text{Spec } M$, where M is a monoid in $\mathfrak{M}\mathfrak{o}$. Then, any open set $V \subset U$ which contains the closed point $\mathfrak{p} = (M^\times)^c$ of U coincides with U (cf. § 3.2, proof of Lemma 3.3). Thus, the stalk $F_{\mathfrak{p}}$ is equal to $\Gamma(U, F) = F(U)$. Then, the result follows from the exactness of the functor ‘stalk at \mathfrak{p} ’ (75).

Part (2) follows from part (1) in view of the equality of $H^p(X, F)$ with the Čech cohomology relative to the covering \mathcal{U} , under the assumption that for all finite intersections $V = \bigcap_{j_k} U_{j_k}$ of opens in \mathcal{U} one has $H^p(V, F) = 0$ for all $p > 0$ (cf. [Har77, Exercice III, 4.11]).

(3) For any sheaf F of abelian groups on X , one has a long exact sequence of cohomology groups (cf. [Har77, III 2.3])

$$\begin{aligned} 0 \rightarrow H_Y^0(X, F) \rightarrow H^0(X, F) \rightarrow H^0(U, F|_U) \rightarrow H_Y^1(X, F) \rightarrow H^1(X, F) \\ \rightarrow H^1(U, F|_U) \rightarrow H_Y^2(X, F) \rightarrow \dots \end{aligned} \tag{79}$$

where $F|_U$ denotes the restriction of the sheaf F on the open set U . Thus, part (3) follows from (79) and the vanishing of $H^1(U, F|_U)$ shown in part (1). \square

5.2 The monoid $M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^\times$ of adèle classes

Throughout this section and until the end of the paper, we denote by \mathbb{K} a global field and by $\mathbb{A}_{\mathbb{K}}$ the space of adèles of \mathbb{K} . The idèle class group $C_{\mathbb{K}}$ of \mathbb{K} is the group M^\times of the invertible elements of the monoid

$$M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^\times, \quad \mathbb{K}^\times = \text{GL}_1(\mathbb{K}). \tag{80}$$

We consider the $\mathfrak{M}o$ -functor $\mathbb{P}_{\mathbb{F}_1}^1$ associated with the projective line (cf. Example 3.12). The geometric realization of $\mathbb{P}_{\mathbb{F}_1}^1$ (cf. [Dei05] and §3) is the finite topological space $\mathbb{P}_{\mathbb{F}_1}^1$ whose underlying set is made by three points $\mathbb{P}_{\mathbb{F}_1}^1 = \{0, u, \infty\}$ with

$$\overline{\{0\}} = \{0\}, \quad \overline{\{u\}} = \mathbb{P}_{\mathbb{F}_1}^1, \quad \overline{\{\infty\}} = \{\infty\}. \tag{81}$$

The topology on $\mathbb{P}_{\mathbb{F}_1}^1$ is described by the following three open sets

$$U_+ = \mathbb{P}_{\mathbb{F}_1}^1 \setminus \{\infty\}, \quad U_- = \mathbb{P}_{\mathbb{F}_1}^1 \setminus \{0\}, \quad U = U_+ \cap U_-. \tag{82}$$

DEFINITION 5.2. The projective adèle class space is the set

$$\mathbb{P}_{\mathbb{F}_1}^1(M) \quad \text{for } M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^\times.$$

Then, Definition 3.19 (cf. (39)) describes a canonical surjection of sets

$$\pi_M : \mathbb{P}_{\mathbb{F}_1}^1(M) = M \cup_{M^\times} M \rightarrow \mathbb{P}_{\mathbb{F}_1}^1. \tag{83}$$

By following the line of proof of Lemma 3.13, one sees that π_M maps an element of $M^\times = C_{\mathbb{K}}$ to $u \in \mathbb{P}_{\mathbb{F}_1}^1$ and the complement of M^\times to 0 or ∞ accordingly to the two copies of $M \setminus M^\times$ inside $\mathbb{P}_{\mathbb{F}_1}^1(M)$.

5.3 The space of functions on the projective adèle class space

To define a natural space $\mathcal{S}(M)$ of functions on the quotient space $M = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^\times$ of the adèle classes (cf. [Con99]), we consider the Bruhat–Schwartz space $\mathcal{S}(\mathbb{A}_{\mathbb{K}})$ of the locally compact abelian group $\mathbb{A}_{\mathbb{K}}$ and the space of its coinvariants under the action of \mathbb{K}^\times . More precisely, we introduce the exact sequence associated with the kernel of the \mathbb{K}^\times -invariant linear mapping $\epsilon(f) = (f(0), \int_{\mathbb{A}_{\mathbb{K}}} f(x) dx) \in \mathbb{C} \oplus \mathbb{C}[1]$, that is,

$$0 \rightarrow \mathcal{S}(\mathbb{A}_{\mathbb{K}})_0 \rightarrow \mathcal{S}(\mathbb{A}_{\mathbb{K}}) \xrightarrow{\epsilon} \mathbb{C} \oplus \mathbb{C}[1] \rightarrow 0. \tag{84}$$

Then, one lets

$$\mathcal{S}(M) = \mathcal{S}_0(M) \oplus \mathbb{C} \oplus \mathbb{C}[1], \quad \mathcal{S}_0(M) = \mathcal{S}(\mathbb{A}_{\mathbb{K}})_0 / \overline{\{f - f_q\}} \tag{85}$$

where $\overline{\{f - f_q\}}$ denotes the closure of the subspace of $\mathcal{S}(\mathbb{A}_{\mathbb{K}})_0$ generated by the differences $f - f_q$, with $q \in \mathbb{K}^\times$ ($f_q(x) = f(qx)$ for all $x \in \mathbb{A}_{\mathbb{K}}$).

We now introduce the functions on the projective adèle class space $\mathbb{P}_{\mathbb{F}_1}^1(M) = M \cup_{M^\times} M$. We define the following sheaf Ω on $\mathbb{P}_{\mathbb{F}_1}^1$. Here Ω is *uniquely* defined by the following spaces of sections

and by the restriction maps as in (87):

$$\begin{aligned} \Gamma(U_+, \Omega) &= \mathcal{S}(M) \\ \Gamma(U_-, \Omega) &= \mathcal{S}(M) \\ \Gamma(U_+ \cap U_-, \Omega) &= \mathcal{S}_\infty(C_{\mathbb{K}}) \end{aligned}$$

where, for a number field \mathbb{K} , $\mathcal{S}_\infty(C_{\mathbb{K}})$ is defined as follows

$$\mathcal{S}_\infty(C_{\mathbb{K}}) = \bigcap_{\beta \in \mathbb{R}} \mu^\beta \mathcal{S}(C_{\mathbb{K}}) = \{f \in \mathcal{S}(C_{\mathbb{K}}) \mid \mu^\beta(f) \in \mathcal{S}(C_{\mathbb{K}}), \forall \beta \in \mathbb{R}\}. \tag{86}$$

Here, $\mu \in C(C_{\mathbb{K}})$ denotes the module $\mu : C_{\mathbb{K}} \rightarrow \mathbb{R}_+^\times$, $\mu^\beta(g) = \mu(g)^\beta$ and $\mathcal{S}(C_{\mathbb{K}})$ is the Bruhat–Schwartz space over $C_{\mathbb{K}}$ (cf. [CM08, Mey05]). The natural restriction maps $\Gamma(U_\pm, \Omega) \rightarrow \Gamma(U_+ \cap U_-, \Omega)$ vanish on the component $\mathbb{C} \oplus \mathbb{C}[1]$ of $\mathcal{S}(M)$ while on $\mathcal{S}_0(M)$ they are defined as follows

$$\begin{aligned} (\text{Res}_+ f)(g) &= \sum_{q \in \mathbb{K}^\times} f(qg), \quad f \in \mathcal{S}_0(M) \subset \Gamma(U_+, \Omega) \\ (\text{Res}_- h)(g) &= |g|^{-1} \sum_{q \in \mathbb{K}^\times} h(qg^{-1}), \quad h \in \mathcal{S}_0(M) \subset \Gamma(U_-, \Omega). \end{aligned} \tag{87}$$

It is convenient to introduce the following mapping which will be used to simplify the notation in the following sections:

$$\Sigma : \mathcal{S}_0(M) \rightarrow \mathcal{S}_\infty(C_{\mathbb{K}}), \quad \Sigma(f)(g) = \sum_{q \in \mathbb{K}^\times} f(qg). \tag{88}$$

5.4 $H^0(\mathbb{P}_{\mathbb{F}_1}^1, \Omega)$ and the graph of the Fourier transform

The Čech complex of the covering $\mathcal{U} = \{U_\pm\}$ of $\mathbb{P}_{\mathbb{F}_1}^1$ has two terms

$$\begin{aligned} C^0 &= \Gamma(U_+, \Omega) \times \Gamma(U_-, \Omega) \\ C^1 &= \Gamma(U_+ \cap U_-, \Omega). \end{aligned}$$

The coboundary $\partial : C^0 \rightarrow C^1$ is given by

$$\partial(f, h)(g) = (\text{Res}_+ f)(g) - (\text{Res}_- h)(g) = \Sigma(f)(g) - |g|^{-1} \Sigma(h)(g^{-1}). \tag{89}$$

LEMMA 5.3. *The kernel of the coboundary $\partial : C^0 \rightarrow C^1$ coincides with the graph of the Fourier transform F on $\mathcal{S}_0(M)$ i.e.*

$$H^0(\mathbb{P}_{\mathbb{F}_1}^1, \Omega) = \{(f, F(f)) \mid f \in \mathcal{S}_0(M)\} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}[1] \oplus \mathbb{C}[1] \tag{90}$$

$$F(f)(a) = \int_{\mathbb{A}_{\mathbb{K}}} f(x) \alpha(ax) dx \tag{91}$$

where α is a non-trivial character of the additive group $\mathbb{A}_{\mathbb{K}}/\mathbb{K}$.

Proof. The lattice $\mathbb{K} \subset \mathbb{A}_{\mathbb{K}}$ coincides with its own dual. Note that the Fourier transform F on $\mathcal{S}(\mathbb{A}_{\mathbb{K}})_0$ depends on the choice of the character α but it becomes canonical modulo the subspace $\{f - f_q\}$ and *a fortiori* modulo its closure $\overline{\{f - f_q\}}$ i.e. on $\mathcal{S}_0(M)$. We recall that the Poisson formula gives the equality

$$\sum_{q \in \mathbb{K}} f(q) = \sum_{q \in \mathbb{K}} F(f)(q) \quad \text{for all } f \in \mathcal{S}(\mathbb{A}_{\mathbb{K}}).$$

When this equality is applied to the elements of $\mathcal{S}(\mathbb{A}_{\mathbb{K}})_0$ it gives

$$\sum_{q \in \mathbb{K}^\times} h(g^{-1}q) = |g| \sum_{q \in \mathbb{K}^\times} F(h)(gq). \tag{92}$$

In particular, for $(f, h) \in \text{Ker } \partial = H^0(\mathbb{P}_{\mathbb{F}_1}^1, \Omega)$, one obtains

$$\Sigma(Fh)(g) = |g|^{-1} \Sigma(h)(g^{-1}). \tag{93}$$

Thus, one obtains

$$\Sigma(f - Fh) = 0 \tag{94}$$

which shows, by applying Lemma 5.4 of [Mey05], that $f - F(h) \in \overline{\{f - f_q\}}$. □

5.5 The spectral realization on $H^1(\mathbb{P}_{\mathbb{F}_1}^1, \Omega)$

The idèle class group $C_{\mathbb{K}}$ acts on the sheaf Ω on $\mathbb{P}_{\mathbb{F}_1}^1$ as follows. For $\lambda \in C_{\mathbb{K}}$, set

$$\begin{aligned} \vartheta_+(\lambda)f(x) &= f(\lambda^{-1}x) \quad \text{for all } f \in \Gamma(U_+, \Omega) \\ \vartheta_-(\lambda)f(x) &= |\lambda|f(\lambda x) \quad \text{for all } f \in \Gamma(U_-, \Omega) \\ \vartheta(\lambda)f(x) &= f(\lambda^{-1}x) \quad \text{for all } f \in \Gamma(U_+ \cap U_-, \Omega). \end{aligned} \tag{95}$$

To check that formulae (95) determine a well-defined action, we need to show that (95) are compatible with the restriction maps (87). This is clear for the restriction from U_+ to $U = U_+ \cap U_-$. For the restriction from U_- to U , one has

$$\text{Res}(\vartheta_-(\lambda)f)(g) = |g|^{-1} \sum_{q \in \mathbb{K}^\times} (\vartheta_-(\lambda)f)(gq^{-1}) = |g|^{-1} \sum_{q \in \mathbb{K}^\times} |\lambda|f(\lambda gq^{-1}).$$

Since

$$\vartheta(\lambda)\text{Res}(f)(g) = \text{Res}(f)(\lambda^{-1}g) = |\lambda^{-1}g|^{-1} \sum_{q \in \mathbb{K}^\times} f(q(\lambda^{-1}g)^{-1}),$$

the required compatibility follows.

Let w be the element of the Weyl group W of PGL_2 given by the matrix

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This element acts on $C_{\mathbb{K}}$ by the automorphism $g \mapsto g^{-1}$ and this action defines the semi-direct product $N = C_{\mathbb{K}} \rtimes W$. Moreover, w acts on $\mathbb{P}_{\mathbb{F}_1}^1$ by exchanging 0 and ∞ .

We lift the action of w to the sheaf Ω on $\mathbb{P}_{\mathbb{F}_1}^1$ as follows. We consider the direct image sheaf $w_*\Omega$

$$\Gamma(V, w_*\Omega) = \Gamma(w^{-1}(V), \Omega) \quad \text{for all } V \text{ open, } V \subset \mathbb{P}_{\mathbb{F}_1}^1. \tag{96}$$

Then, we define the following morphism of sheaves $w_{\#} : \Omega \rightarrow w_*\Omega$

$$\begin{aligned} w_{\#}f &= f \in \Gamma(U_-, \Omega) \quad \text{for all } f \in \Gamma(U_+, \Omega) \\ w_{\#}f &= f \in \Gamma(U_+, \Omega) \quad \text{for all } f \in \Gamma(U_-, \Omega) \\ w_{\#}f(g) &= |g|^{-1}f(g^{-1}) \quad \text{for all } f \in \Gamma(U_+ \cap U_-, \Omega). \end{aligned} \tag{97}$$

The geometric action defined in the next proposition immediately implies the functional equation and in fact lifts the equation at the level of the representation.

PROPOSITION 5.4.

- (1) *The equalities (97) define an action of the Weyl group W of PGL_2 on the sheaf Ω . This action fulfills the following compatibility property with respect to the action (95)*

$$\vartheta(\lambda)w_{\#}\xi = |\lambda|w_{\#}\vartheta(\lambda^{-1})\xi. \tag{98}$$

- (2) *There is a unique action of $N = C_{\mathbb{K}} \rtimes W$ on the sheaf Ω which agrees with (97) on W and restricts on $C_{\mathbb{K}}$ to the twist $\vartheta[-1/2] = \vartheta \otimes \mu^{-1/2}$.*

Proof. We need to show that the map $w_{\#} : \Omega \rightarrow w_*\Omega$ defined in (97) is compatible with the restriction maps (87). For $f \in \Gamma(U_+, \Omega)$, one has

$$\mathrm{Res}(w_{\#}f)(g) = |g|^{-1} \sum_{q \in \mathbb{K}^{\times}} f(qg^{-1})$$

which agrees with $w_{\#}\mathrm{Res}(f)$, using (97). A similar result holds for $f \in \Gamma(U_-, \Omega)$. The full statement follows from the involutive property of the transformation $f \mapsto w_{\#}f$, $w_{\#}f(g) = |g|^{-1}f(g^{-1})$. \square

THEOREM 5.5. *The cohomology $H^1(\mathbb{P}_{\mathbb{F}_1}^1, \Omega)$ gives the spectral realization of zeros of Hecke L -functions with Grössencharakter. The spectrum of the action $\vartheta[-1/2]$ of $C_{\mathbb{K}}$ on $H^1(\mathbb{P}_{\mathbb{F}_1}^1, \Omega)$ is invariant under the symmetry $\chi(g) \mapsto \chi(g^{-1})$ of Grössencharakteres of \mathbb{K} .*

Proof. Consider the affine open set $U_- \subset \mathbb{P}_{\mathbb{F}_1}^1$ and its complement $Y = \{0\}$. One checks directly that the cohomology with support $H_Y^1(\mathbb{P}_{\mathbb{F}_1}^1, \Omega)$ describes the cokernel of the map $\Sigma : \mathcal{S}_0(M) \rightarrow \mathcal{S}_{\infty}(C_{\mathbb{K}})$ of (88), that is, the spectral realization of [CCM07, CM08, Mey05] and initiated in [Con99]. We refer to [CM08, IV, Theorem 4.116] for the detailed statement and proof. Since U_- is affine, the exact sequence (78) reduces to the isomorphism

$$0 \rightarrow H_Y^1(\mathbb{P}_{\mathbb{F}_1}^1, \Omega) \rightarrow H^1(\mathbb{P}_{\mathbb{F}_1}^1, \Omega) \rightarrow 0. \tag{99}$$

In fact, the homomorphism

$$H^0(\mathbb{P}_{\mathbb{F}_1}^1, \Omega) \rightarrow H^0(U_-, \Omega|_{U_-}) \tag{100}$$

is surjective: cf. Lemma 5.3. The symmetry then follows from the existence of the action of $N = C_{\mathbb{K}} \rtimes W$ on the sheaf Ω and hence on the cohomology $H^1(\mathbb{P}_{\mathbb{F}_1}^1, \Omega)$. \square

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