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IRREDUCIBLE MODULES FOR POLYCYCLIC GROUP ALGEBRAS

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1. Introduction. If G is a polycyclic group and k an absolute field then every irreducible kG-module is finite dimensional [**10**], while if k is non-absolute every irreducible module is finite dimensional if and only if G is abelian-by-finite [**3**]. However something more can be said about the infinite dimensional irreducible modules. For example P. Hall showed that if G is a finitely generated nilpotent group and V an irreducible kG-module, then the image of kZ in $\mathrm{End}_{kG}V$ is algebraic over k [**3**]. Here $Z = Z(G)$ denotes the centre of G. It follows that the restriction V_Z of V to Z is generated by finite dimensional kZ-modules. In this paper we prove a generalization of this result to polycyclic group algebras.

We introduce some terminology. A free abelian subgroup, A, of a polycyclic-by-finite group is said to be a *plinth* if there exists a subgroup G_0 of G containing A such that

i) $|G:G_0| < \infty$ and $A \trianglelefteq G_0$

ii) $A \otimes_{\mathbf{Z}} \mathbf{Q}$ is an irreducible $\mathbf{Q}H$-module whenever H is a subgroup of finite index in G_0.

It is known that any infinite normal subgroup of a polycyclic-by-finite group contains a plinth [**10**, Lemma 2].

If $\dim_{\mathbf{Q}}(A \otimes_{\mathbf{Z}} \mathbf{Q}) = 1$, then A is said to be a *centric* plinth, and if $\dim_{\mathbf{Q}}(A \otimes_{\mathbf{Z}} \mathbf{Q}) > 1$ A is an *eccentric* plinth.

We consider the subgroup generated by all of the plinths and set

$$\mathrm{Pl\,soc}\,(G) = \langle A | A \text{ is a plinth in } G \rangle,$$

the *plinth socle* of G. Clearly this is a characteristic subgroup.

MAIN THEOREM. *If G is a polycyclic-by-finite group, A = Plsoc (G) and V is an irreducible kG-module, then V_A is a locally finite module.*

A module is said to be *locally finite* if every element generates a finite dimensional submodule.

This result is proved by induction on the Hirsch number, $h(G)$, of G using a result, Theorem 2.5 (essentially due to D. L. Harper) that under certain circumstances an irreducible kG-module is induced from a subgroup of smaller Hirsch number.

In Section 3 we consider some applications of the main theorem. Following the terminology of [**2**] we shall call a kG-module V *finitely*

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induced if $V \cong W \otimes_{kH} kG$ where H is a subgroup of G and W is a finite dimensional kH-module. If k is absolute then of course every irreducible kG-module is finite dimensional, and so we assume that k is non-absolute. Segal [**12**] has shown that if G is nilpotent-by-finite, then every irreducible kG-module is finitely induced if and only if G is abelian-by-finite. We prove

THEOREM 3.1. *Let G be polycyclic-by-finite and k non-absolute, then every irreducible kG-module is finitely induced if and only if every nilpotent subgroup of G is abelian-by-finite.*

We also consider essential extensions of irreducible kG-modules. If V is a finite dimensional kG-module, then K. A. Brown has shown that the injective hull $E_{kG}(V)$ of V is locally finite as a kG-module. Hence if U is an infinite dimensional irreducible kG-module then $\text{Ext}(U, V) = 0$, [**1**, Theorem 1.1.1]. We prove a dual result.

THEOREM 3.3. *If G is a polycyclic-by-finite group, k a field and V, U are irreducible kG-modules such that V is infinite dimensional and U is finite dimensional, then* $\text{Ext}(U, V) = 0$.

It is also known that if V is a finite dimensional irreducible module then $E_{kG}(V)$ is artinian ([**1**, Theorem 2.1.2], [**7**, Theorem B]). It seems likely that if V is an infinite dimensional irreducible module, then $E_{kG}(V)$ is not artinian. We can show this in particular cases.

COROLLARY 3.6. *Let G be a polycyclic-by-finite group and suppose that kG is a primitive ring with V a faithful irreducible module. Then $E_{kG}(V)$ is not locally artinian.*

Finally in Section 4 we discuss the relationship of the plinth socle to the Zalesskii subgroup and give some examples.

The results of this paper are taken from the author's Ph.D. thesis at the University of Warwick. I should like to thank D. L. Harper for sending me a copy of his thesis and my supervisor Brian Hartley for some helpful suggestions. Thanks are also due to the Science Research Council of Great Britain for their financial support.

2. Locally finite modules. We quote some results from [**6**].

THEOREM 2.1 ([**6**, Theorem 4.5]). *Suppose that G is a polycyclic-by-finite group, k any field and A an infinite abelian normal subgroup of G. Then no irreducible kG-module can be torsion free as a kA-module.*

This generalizes a result of Roseblade for absolute fields [**10**, Theorem E and remarks on p. 313].

If a subgroup B of a polycyclic group G is a plinth we do not require B

to be normal in G, only that $|G:N_G(B)| < \infty$. It is therefore useful to have the following result.

LEMMA 2.2 ([**6**, Lemma 4.3]). *If B is a plinth in the polycyclic-by-finite group G, then there exists an abelian normal subgroup A of G such that $A \cap B > 1$.*

In fact if Zal (G) denotes the Zalesskii subgroup of G it is not hard to see that Zal $(G) \cap B > 1$ and we may take A to be the centre of Zal (G).

If A is a subgroup of a group G, we denote by $\underline{\underline{a}}_s$ the augmentation ideal of the subgroup

$$A^s = \langle a^s | a \in A \rangle.$$

LEMMA 2.3 ([**6**, Lemma 2.2]). *Let A be an eccentric plinth in the polycyclic-by-finite group G, and suppose that $A \trianglelefteq G$. If k is a field and P a prime ideal of kA such that $|G:N_G(P)| < \infty$ then there is a positive integer s such that $\underline{\underline{a}}_s \subseteq P$.*

Since $A \trianglelefteq G$, G acts by conjugation on kA and we set

$$N_G(P) = \{g \in G | P^g = P\}.$$

We notice that a theorem of Bergman [**9**, Corollary 9.3.9] shows that $\dim_k kA/P < \infty$.

Harper shows that certain group algebras of polycyclic groups cannot have primitive irreducible modules (a module is said to be *primitive* if it cannot be induced from a module for the group algebra of a proper subgroup). The technique is to use a result of Roseblade in conjunction with the results stated above.

If S is any ring and V an S-module, then for any subset X of S we denote by $\text{ann}_V X$ the annihilator of X in V namely

$$\text{ann}_V X = \{v \in V | vx = 0 \text{ for all } x \in X\}.$$

We denote by $\Pi_S(V)$ the set of all ideals P of S which are maximal with respect to $\text{ann}_V P > 0$. If S satisfies the maximal condition on ideals, then $\Pi_S(V)$ is non-empty and it is easily seen that any member of $\Pi_S(V)$ is a prime ideal.

We shall consider the case where A is an abelian normal subgroup of the polycyclic group G and V is an irreducible kG-module. By Theorem 2.1, V cannot be torsion free as a kA-module, so if $P \in \Pi_{kA}(V)$, then $P > 0$.

LEMMA 2.4 ([**10**, Lemma 3]). *Let k be a field, and R the group ring of a group G over k. Let S be the group ring of a normal subgroup H of G over k, and V a kG-module. If $P \in \Pi_S(V)$ and T is a right transversal to $N_G(P)$ in G then*

$$(\text{ann}_V P)R = \bigoplus_{t \in T} (\text{ann}_V P)t.$$

If V is irreducible, then as $\mathrm{ann}_V P > 0$ we must have $(\mathrm{ann}_V P)R = V$ and hence

$$V = \mathrm{ann}_V P \otimes_{kN} kG,$$

where $N = N_G(P)$.

The next theorem is a slight extension of a result of D. L. Harper, ([**5**, Theorem 4.5] see also [**6**, Theorem A]).

THEOREM 2.5. *Let G be a polycyclic-by-finite group, k a field, V an irreducible kG-module, and suppose that B is an eccentric plinth in G such that $B \cap C_G(V) = 1$. Then there exists a subgroup K of G such that $h(K) < h(G)$, $|B:B \cap K| < \infty$ and $V = V_1 \otimes_{kK} kG$ where V_1 is an irreducible kK-module.*

Proof. By Lemma 2.2 there exists an abelian normal subgroup A of G such that $A \cap B > 1$. Therefore rank (B) = rank $(A \cap B)$, and if K is any subgroup of G such that $|A \cap B:A \cap B \cap K| < \infty$, then $|B:A \cap B \cap K| < \infty$ and so $|B:B \cap K| < \infty$.

Therefore by replacing B by $A \cap B$, we may assume that $B \subseteq A$.

Let $H = \mathrm{core}_G\,(N_G(B))$, that is the largest normal subgroup of G contained in $N_G(B)$. Then $|G:H| < \infty$, and $A \subseteq H$. Now by Clifford's theorem $V_H = U_1 \oplus \ldots \oplus U_r$ where the U_i are the homogeneous components of V as a kH-module. If $B \cap C_H(U_i) \neq 1$ for each i then $B \cap C_H(V) \neq 1$ which goes against our hypothesis. Hence $B \cap C_H(U_i) = 1$ for some i.

Let $H_1 = \{g \in G | U_i g = U_i\}$, the stabiliser of U_i. Then again by Clifford's theorem $V = U_i \otimes_{kH_1} kG$, and we may replace G with H_1 to assume that V has only one homogeneous component as a kH-module. Therefore $V_H = W_1 \oplus \ldots W_s$ where the W_i are irreducible kH-modules, which are all isomorphic to a fixed module W, say.

By Theorem 2.1 W is not torsion free as a kB-module. Choose $P \in \Pi_{kB}(W)$, then $P \neq 0$ and $\mathrm{ann}_V P \neq 0$. Therefore $\mathrm{ann}_V(PkA) > 0$ and we can choose $Q \in \Pi_{kA}(V)$ such that $PkA \subseteq Q$. Then $P \subseteq Q \cap kB$.

Since the W_i are isomorphic, and Q has non-zero annihilator in V, we deduce that V has non-zero annihilator in W. Hence $Q \cap kB$ has non-zero annihilator in W.

By the maximality of P we have $P = Q \cap kB$. Now let $K = N_G(Q)$, $L = N_H(P)$. Then by Lemma 2.4 we have

$$V = \mathrm{ann}_V Q \otimes_{kK} kG \text{ and } W = \mathrm{ann}_W P \otimes_{kL} kH.$$

Let $V_1 = \mathrm{ann}_V Q$, an irreducible kK-module. We have $B \subseteq A$ and Q is an ideal of kA, so $B \subseteq N_G(Q) = K$.

It only remains to show that $h(K) < h(G)$. If this is not the case then $|G:K| < \infty$. Now $H \cap K \subseteq L$ and so $|H:L| < \infty$, so by Lemma 2.3, $\underline{b}_s \subseteq P$ for some $s \geqq 1$.

Therefore $(\text{ann}_V P)\underline{\underline{b}}_s = 0$ and since B^s is characteristic in B, it is normal in H and so $(\text{ann}_V P \otimes h)\underline{\underline{b}}_s = 0$ for each $h \in H$. Therefore $W\underline{\underline{b}}_s = 0$, but this contradicts the fact that $B \cap C_H(W) = 1$.

LEMMA 2.6. *Let B_1, B_2 be subgroups of a group G and V a kG-module.*

i) *If $B_i \trianglelefteq G$ and V_{B_i} is locally finite for $i = 1, 2$ then V_B is locally finite where $B = B_1B_2$.*

ii) *If $B_1 \subseteq B_2$, $|B_2:B_1| < \infty$ and V_{B_1} is locally finite then V_{B_2} is locally finite.*

Proof. i) It suffices to show that if $v \in V$, then $\dim_k vkB < \infty$. By assumption $\dim_k vkB_1 < \infty$, so let $v_1, \ldots, v_n$ be a k-basis of vkB_1. Then

$$vkB \subseteq \sum_{i=1}^{n} v_i kB_2$$

which is finite dimensional.

ii) Let $v \in V$. We must show that $\dim_k vkB_2 < \infty$. If $g_1, \ldots, g_n$ is a left transversal to B_1 in B_2, then any element of kB_2 can be written as $\sum_{i=1}^n g_i\beta_i$, where $\beta_i \in kB_1$. Therefore

$$vkB_2 \subseteq \sum_{i=1}^{n} vg_i kB_1$$

and this is finite dimensional.

PROPOSITION 2.7. *Let B be a plinth in the polycyclic-by-finite group G and V an irreducible kG-module. Then V_B is locally finite.*

Proof. We use induction on $h(G)$. If $h(G) \leqq 2$, G is abelian-by-finite and the result is trivial.

Suppose first that $B \cap C_G(V) \neq 1$. Then since B is a plinth, we have $|B:B \cap C_G(V)| < \infty$. Now $V_{B\cap C_G(V)}$ is a direct sum of trivial modules, so is certainly locally finite, and the result follows from Lemma 2.6 (ii).

If rank $(B) = 1$, the result follows by methods of P. Hall [**3**]. There is a normal subgroup H of finite index in G which centralises B. As a kH-module $V = \bigoplus_{i=1}^r W_i$ where W_i is an irreducible kH-module.

Let $\langle z\rangle = B \cap H$. Then by Lemma 2.1 or [**9**, Lemma 12.2.8] there is a non-zero polynomial $f_i(z)$ in $k\langle z\rangle$ such that $W_if_i(z) = 0$. Hence each W_i is a locally finite $k\langle z\rangle$-module and the result follows from Lemma 2.6 since $|B:\langle z\rangle| < \infty$.

Now, we can in fact assume that $B \triangleleft G$. For, if $H = \text{core}_G\,(N_G(B))$, then $|G:H| < \infty$ and V is completely reducible as a kH-module, and it suffices to show that each kH-submodule of V is locally finite as a $k(H \cap B)$-module.

Let us suppose that $B \cap C_G(V) = 1$, and rank $(B) \geqq 2$. Then by Theorem 2.5 there is a subgroup K of G such that $h(K) < h(G)$, $|B:B \cap K| < \infty$ and $V = V_1 \otimes_{kK} kG$ where V_1 is an irreducible kK-

module. Since $h(K) < h(G)$ we can apply the inductive hypothesis to the plinths of K.

Consider the $\mathbf{Q}G$-module $B \otimes_{\mathbf{Z}} \mathbf{Q}$. It is a consequence of Mal'cev's theorem on soluble linear groups, [**9**, Theorem 12.1.3] that as B is a plinth $N_G(B)/C_G(B)$ is abelian-by-finite. Hence there is a normal subgroup G_0 of finite index in G such that G_0/C is abelian where $C = C_{G_0}(B)$.

Now if $K_0 = K \cap G_0$ then

$$K_0C/C \lhd G_0/C$$

and by Clifford's theorem it follows that $B \otimes_{\mathbf{Z}} \mathbf{Q}$ is completely reducible as a $\mathbf{Q}K_0$-module.

Hence there is a subgroup $B_1 \times B_2 \times \ldots \times B_n$ of finite index in B such that each B_i is a plinth in K. By induction $V_1|_{B_i}$ is locally finite for each $i = 1, \ldots, n$. Therefore by Lemma 2.6 (i) $V_1|_B$ is locally finite.

Since $V_1|_B$ is locally finite and $B \trianglelefteq G$ we conclude that $V_1 \otimes g|_B$ is locally finite for all $g \in G$ and therefore $V = V_1 \otimes_{kK} kG$ is locally finite as a kB-module.

Proof of the main theorem. We have a polycyclic-by-finite group G, and an irreducible kG-module V and we must show that V_A is locally finite where $A = \text{Plsoc}\,(G)$. Now

$$A = \langle B | B \text{ is a plinth in } G \rangle$$

and since A is finitely generated there are finitely many plinths $B_1, \ldots, B_r$ such that $A = \langle B_1, \ldots, B_r \rangle$.

Let $G_0 = \text{core}_G(\bigcap_{i=1}^r N_G(B_i))$, so that G_0 is a normal subgroup of finite index in G which normalises each B_i, and let $C_i = B_i \cap G_0 \trianglelefteq G_0$, and $C = \langle C_1, \ldots, C_r \rangle$. Then $|B_i : C_i| < \infty$ and by [**11**, Lemma 1], $|A : C| < \infty$.

Hence by Lemma 2.6 (ii) it suffices to show that V is locally finite as a kC-module. Now by Clifford's theorem V is completely reducible as a kG_0-module and by Proposition 2.7 V_{C_i} is locally finite. Since $C_i \trianglelefteq G_0$, $C = C_1C_2 \ldots C_r$ and Lemma 2.6 (i) shows that V_C is locally finite. This completes the proof.

In fact the restriction of an irreducible kG-module to the plinth socle is completely reducible.

COROLLARY 2.8. *Let G be a polycyclic-by-finite group, k a field and V an irreducible kG-module. If $A = \text{Pl soc}\,(G)$ then V_A is completely reducible.*

Proof. Since V_A is locally finite it contains an irreducible kA-submodule. It is easily seen that the socle of V as a kA-module is a proper kG-submodule of V.

We have, as a consequence of the main theorem an 'intersection

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theorem' for maximal right ideals which could be compared with [**9**, Lemma 7.4.9] and Bergman's theorem [**9**, Corollary 9.3.9].

COROLLARY 2.9. *If G is a polycyclic-by-finite group, k a field, M a maximal right ideal of kG and A = Pl soc (G), then*

$$\dim_k kA/M \cap kA < \infty.$$

Proof. Since kG/M is an irreducible kG-module it is locally finite as a kA-module. Hence for all $\alpha \in kG$, $\dim_k (M + \alpha kA)/M < \infty$. Putting $\alpha = 1$ gives the result.

3. Applications. We first study finitely induced modules. Let $\mathscr{X}$ be the class of polycyclic-by-finite groups all of whose nilpotent subgroups are abelian-by-finite. We need to know that the class $\mathscr{X}$ is closed under taking homomorphic images. Suppose that $G \in \mathscr{X}$, $H \trianglelefteq G$ and K/H is a nilpotent subgroup of G/H. Then K is a subgroup of G and so $K \in \mathscr{X}$. It suffices to show that if $G \in \mathscr{X}$, then every nilpotent factor group of G is abelian-by-finite. This follows from a result of Zaitsev, [**13**] which implies that if G is polycyclic-by-finite, $H \triangleleft G$ with G/H nilpotent, then there is a nilpotent subgroup X of G such that $|G:HX| < \infty$. Now, if $G \in \mathscr{X}$, then X is abelian-by-finite, and so G/H is abelian-by-finite.

THEOREM 3.1. *Let G be a polycyclic-by-finite group and k a non-absolute field. Every irreducible kG-module is finitely induced if and only if $G \in \mathscr{X}$.*

Proof. Suppose that $G \in \mathscr{X}$ and V is an irreducible kG-module. We use induction on $h(G)$ to show that V is finitely induced. Since the class $\mathscr{X}$ is closed under taking homomorphic images we may assume $C_G(V) = 1$.

Now Fit (G), the Fitting subgroup of G is abelian-by-finite and hence G is metabelian-by-finite. If every plinth in G is centric, then by [**8**, Corollary 4.5] or [**6**, Proposition 5.2], G is nilpotent-by-finite and so abelian-by-finite, and V is finite dimensional.

If B is an eccentric plinth in G, then since $B \cap C_G(V) = 1$, there is a subgroup K of G such that $h(K) < h(G)$ and $V = V_1 \otimes_{kK} kG$ by Theorem 2.5. Now V_1 is an irreducible kK-module, and hence by induction V_1 and therefore V are finitely induced.

Conversely suppose that the polycyclic-by-finite group G does not belong to $\mathscr{X}$. We claim that G has a subgroup H isomorphic to a free nilpotent group of class two on two generators,

$$\langle x, y, z | (x, y) = z, (x, z) = (y, z) = 1 \rangle.$$

Now G has a nilpotent subgroup H_1 which is not abelian-by-finite, and H_1 has a subgroup H_2 of finite index which is torsion free, but not abelian-by-finite. Let $Z_1 = Z(H_2)$, $Z_2 = Z(H_2/Z_1)$ and choose $x \in Z_2 \backslash Z_1$. Then $(x, y) \neq 1$ for some $y \in H_2$ and $(x, y) = z \in Z_1$. Take $H = \langle x, y, z \rangle$ as the required subgroup.

We use Segal's construction [**12**] of a primitive irreducible kH-module. Let ζ be an element of k^* which is not a root of one, and

$$M = (z - \zeta)kH + (x + y + 1)kH$$

a right ideal of kH. In [**12**], Segal shows that $U = kH/M$ is a primitive irreducible module (see also [**4**]).

Let V be an irreducible kG-module such that $U \subseteq V_H$. Suppose if possible that $V = W \otimes_{kK} kG$, where $K \subseteq G$, and W is a finite dimensional kK-module. Then by Mackey's theorem

$$V_H = \bigoplus (W_a|_{K_a})^H,$$

where $K_a = K^a \cap H$, $W_a = W \otimes a$ a module for kK^a, and the sum is taken over all double cosets KaH. Now since U is irreducible and $U \subseteq V_H$, we see that

$$U \subseteq (W_a|_{K_a})^H \text{ for some } a \in G.$$

Let $L = K_a$ and $W' = W_a$ so that $U \subseteq W' \otimes_{kL} kH$.

We now analyze the possibilities for L. Let $Z = \langle z \rangle$. If $L \cap Z = 1$, let $\{g_i | i \in I\}$ be a transversal to LZ in H. Then as a transversal to L in H, we may take $\{z^j g_i | j \in \mathbf{Z}, i \in I\}$ and so as a kZ-module, $W' \otimes_{kL} kH$ is free of rank $|I| \cdot \dim_k W'$. Since U has non-zero annihilator in kZ it cannot be embedded in a free kZ-module. Therefore $|LZ:L| < \infty$ and we may assume that $Z \subseteq L$.

Now U is generated by an element u such that

$$u(z - \zeta) = u(x + y + 1) = 0.$$

Suppose that $x_1 = x^n y^m \in L$ where $x_1 \neq 1$, then as W' is finite dimensional $W' f(x_1) = 0$, for some non-zero polynomial $f(x_1) \in k[x_1]$. Now x_1 is contained in the abelian normal subgroup $A = \langle x_1, z \rangle$ of H, and so commutes with each of its conjugates in H, so as $u \in W' \otimes_{kL} kH$, $ug(x_1) = 0$, for some non-zero polynomial $g(x_1)$. Therefore the ideal

$$I = g(x_1)kA + (z - \zeta)kA$$

has non-zero annihilator in U, and we may choose $P \in \Pi_{kA}(U)$ containing I.

Then by Lemma 2.4 $U = \operatorname{ann}_V P \otimes_{kN} kH$, where $N = N_H(P)$. Since U is a primitive module $N = H$, but then PkH is a two-sided ideal of kH properly containing $(z - \zeta)kH$, which is impossible as $(z - \zeta)kH$ is primitive and so by [**11**, Theorem $G1$] a maximal ideal of kH. Therefore $L = Z$.

If $w_1, \ldots, w_t$ is a vector space basis for W', then

$$\{w_i x^j y^k | i = 1, \ldots, t; j, k \in \mathbf{Z}\}$$

is a basis for $W' \otimes_{kZ} kH$.

Now if we express u as linear combination of these basis elements and we use the fact that $u(x + y + 1) = 0$, we easily obtain a contradiction to their linear independence. This shows that the irreducible kG-module V is not finitely induced.

Particular cases of the above result have been obtained by Harper, [**5**, Corollary 4.9] and Segal [**12**, Theorem A]. In [**2**] Farkas and Snider show that any primitive ideal in the group algebra of a polycyclic-by-finite group is the annihilator of a finitely induced module.

We now examine essential extensions of irreducible modules. The following sufficient condition for a short exact sequence to split is taken from [**8**, Proposition 4.6].

PROPOSITION 3.2. *Let G be any group, k a field and*

(*) $\quad 0 \to V \to W \to U \to 0$

an exact sequence of kG-modules. Suppose that A is a finitely generated nilpotent normal subgroup of G such that V_A *is locally finite. If* $C_V(A) = 0$ *and* $C_U(A) = U$, *then the sequence of kG-modules* (*) *splits.*

Here $C_V(A)$ denotes the fixed points of V as a kA-module, that is the submodule

$$\{v \in V | va = v \text{ for all } a \in A\}.$$

THEOREM 3.3. *If G is a polycyclic-by-finite group, k a field and U, V are irreducible kG-modules such that V is infinite dimensional and U is finite dimensional, then* Ext $(U, V) = 0$.

Proof. Let

(**) $\quad 0 \to V \to W \to U \to 0$

be a sequence of kG-modules with V and U as in the statement of the theorem. We have to show this sequence splits.

Clearly we may assume $C_G(W) = 1$. Also, if $H \trianglelefteq G$, with $|G:H| < \infty$ and (**) splits as a sequence of kH-modules, then $W = V \oplus U'$ where $U' \cong_{kH} U$ and $U'kG$ is a finite dimensional kG-submodule of W, and therefore $V \cap U'kG = 0$ and (**) splits as a sequence of kG-modules.

Let $C = C_G(U)$, $D = C_G(V)$. If $a, b \in C \cap D$ and $w \in W$, then

$$w((a - 1)(b - 1) - (b - 1)(a - 1)) = 0$$

so $(a, b) \in C_G(W) = 1$ and so $C \cap D$ is abelian. Hence if F denotes the Fitting subgroup of G, then $C \cap D \subseteq C \cap F$. Now G/C and G/F are abelian-by-finite and so $G/(C \cap F)$ is abelian-by-finite. If $|C \cap F: C \cap D| < \infty$ then $G/(C \cap D)$ would be abelian-by-finite. However, this contradicts the fact that W is an infinite dimensional irreducible module.

Therefore $(C \cap F)/(C \cap D)$ is an infinite normal subgroup of

$G/(C \cap D)$ and so contains a plinth $A/(C \cap D)$. Dropping to a subgroup of finite index we may assume that A is a normal subgroup of G.

Now A is a nilpotent normal subgroup of G, and it is easily seen that $C_V(A) = 0$, and $C_U(A) = U$. Moreover since V is an irreducible $kG/(C \cap D)$-module and $A/(C \cap D)$ is a plinth in $G/(C \cap D)$, Proposition 2.7 shows that V_A is locally finite. In other words the conditions of Proposition 3.2 are satisfied and so the sequence (**) splits.

As our final application, we consider the injective hull $E_{kG}(V)$ of an irreducible kG-module V, for G a polycyclic-by-finite group.

If $\dim_k V$ is finite, then as we have remarked $E_{kG}(V)$ is artinian and it seems likely that the converse holds.

If A is a plinth in G, then by [**9**, Lemma 12.3.1] we can find an element $x \in N_G(A)$ such that A is a plinth in $\langle A, x\rangle$. In contrast with Theorem 2.1 we have the following result.

LEMMA 3.4. *With G, A and x as above, suppose that V is an irreducible kG-module and A is an eccentric plinth in G such that $A \cap C_G(V) = 1$. Then $V_{\langle x\rangle}$ is torsion free.*

Proof. Let $H = \mathrm{core}_G(N_G(A))$. Then H is a normal subgroup with finite index in G which normalizes A. Clifford's theorem shows that $V_H = \oplus_{i=1}^{r} W_i$, a direct sum of irreducible kH-modules, and it suffices to show each W_i is torsion free. Hence we may suppose that $A \trianglelefteq G$.

If $V_{\langle x\rangle}$ is not torsion free, it has a finite dimensional $k\langle x\rangle$-submodule. Since $B = \langle A, x\rangle$ is a split extension of A by x, and V_A is locally finite it follows that V has a finite dimensional kB-submodule, by arguments similar to those used in proving Lemma 2.6.

Hence V has a finite dimensional irreducible kB-submodule V_1. Let $P = \mathrm{Ann}_{kA} V_1 \neq 0$ since $\dim_k V_1 < \infty$. Hence by Lemma 2.3 $\underline{a}_s \subseteq P$ for some $s \geqq 1$. Therefore $V_1\underline{a}_s = 0$, and since $A \trianglelefteq G$, $V_1 g\underline{a}_s = 0$ for all $g \in G$. Since V is an irreducible kG-module this shows that $V\underline{a}_s = 0$ contradicting our assumption that $A \cap C_G(V) = 1$.

THEOREM 3.5. *If G is a polycyclic-by-finite group, k a field, V an irreducible kG-module, and A an eccentric plinth in G such that $A \cap C_G(V) = 1$, then $E_{kG}(V)$ is not locally artinian.*

Proof. Since V_A is locally finite it contains a finite dimensional irreducible kA-submodule $U_0 = u_0 kA$ for some $u_0 \in U_0$. Now there exists $x \in N_G(A)$ such that A is a plinth in $\langle A, x\rangle = H$. We consider the kH-submodule $U = u_0 kH$ of V.

By Lemma 3.4, the elements $\{u_0 x^i | i \in \mathbf{Z}\}$ are linearly independent. Let $u_i = u_0 x^i$ and $U_i = u_i kA = U_0 x^i$. Suppose that

$$U_{r+1} \cap \sum_{i=0}^{n} U_i \neq 0.$$

Then $U_{n+1} \subseteq \sum_{i=0}^{n} U_i$ since U_{n+1} is an irreducible kA-module. Hence

$$u_0 x^m \in \sum_{i=0}^{n} U_i \quad \text{for all } m \geqq 0$$

and so these elements cannot be linearly independent.

It follows that the sum $\sum_{i \in \mathbf{Z}} U_i$ is direct and therefore $U = U_0 \otimes_{kA} kH$ and if $v_1, \ldots, v_n$ is a basis for U_0, then

$$\{v_i x^j | i = 1, \ldots, n, j \in \mathbf{Z}\}$$

is a basis for U.

Now since $(x - 1)$ is a non-zero divisor, there exists $w \in E_{kH}(U)$ such that $w(x - 1) = v_1$. It is easily seen that w cannot belong to U since we have an explicit vector space basis for U. Hence we have an essential extension

$$0 \rightarrow U \rightarrow W \rightarrow \bar{W} \rightarrow 0$$

of kH-modules and using Bergman's theorem, as in [**8**] we see that either $\dim_k \bar{W} < \infty$ or $\bar{W}$ is free as a kA-module. The first alternative is impossible by Theorem 3.3.

Now $w \in E_{kH}(U) \subseteq E_{kH}(V) \subseteq E_{kG}(V)$. Let $W_1 = wkG$, a finitely generated essential extension of V. If W_1 is artinian, then it has a composition series of finite length, and hence by Proposition 2.7 the restriction of W_1 to A is locally finite. This is patently not the case since W_1 has a non-zero free kA-module as a factor module of a submodule.

We have shown that $E_{kG}(V)$ is not locally artinian.

COROLLARY 3.6. *Let G be a non-trivial polycyclic-by-finite group and suppose that kG is a primitive ring with V a faithful irreducible module. Then $E_{kG}(V)$ is not locally artinian.*

Proof. Clearly $C_G(V) = 1$. Also since kG is primitive no plinth in G can be centralized by a subgroup of finite index. Hence any plinth in G is eccentric and the result follows.

4. Properties of the plinth socle. Throughout this section, G will denote a polycyclic-by-finite group, $F = F(G)$ the finite radical of G, H/F the Fitting radical of G/F and Zal $(G)/F$ the centre of H/F. The subgroup Zal (G) is the Zalesskii subgroup of G.

As noted by Roseblade [**11**, p. 390] every non-trivial normal subgroup of G meets Zal (G) non-trivially. Since any infinite normal subgroup of G contains a plinth in G, any infinite normal subgroup of G has non-trivial intersection with Pl soc (G), and so any non-trivial normal subgroup of G meets Pl soc $(G)F(G)$ non-trivially.

LEMMA 4.1. *If G is a finitely generated nilpotent group, then*

$$\text{Zal}(G) = \text{Pl soc}(G)F(G).$$

Proof. Take $b \in \text{Zal}(G)$. If b has finite order then $b \in F(G)$. Otherwise $B = \langle b \rangle$ is a plinth in G.

Conversely, suppose that B is a plinth in G. Then BF/F is a plinth in G/F, and so is centralized by a subgroup of finite index in G. However centralizers are isolated in a torsion free nilpotent group and so $BF/F \subseteq \text{Zal}(G)/F$. Therefore $BF \subseteq \text{Zal}(G)$.

LEMMA 4.2. *If B is a plinth in a polycyclic-by-finite group G, and $G_0 = N_G(B)$, then $B \subseteq \text{Zal}(G_0)$.*

Proof. Since the Zalesskii subgroup meets any non-trivial normal subgroup of G_0, we have

$$|B:B \cap \text{Zal}(G_0)| < \infty.$$

Let $F = F(G_0)$ and H/F be the Fitting subgroup of G_0/F.

If $(B, H) \neq 1$, then $(B, h) \neq 1$ for some $h \in H$. The map $\theta:b \to (b, h)$ is an endomorphism of B. Moreover

$$B^n \subseteq B \cap \text{Zal}(G_0)$$

for some $n \geqq 1$, and if $b \in B \cap \text{Zal}(G_0)$ and $h \in H$, then $(b, h) \in B \cap F = 1$.

Therefore $B^n \subseteq \ker \theta$ and hence (B, h) has finite order. Hence $(B, h) = 1$.

We remark that if L is any subgroup of a polycyclic-by-finite group G having the property of the plinth socle expressed in the main theorem, then L must be abelian-by-finite. For any irreducible kL-module V can be embedded in an irreducible kG-module W, and if W_L is locally finite then V must be finite dimensional. If this is true for all fields k, then L must be abelian-by-finite.

Finally, we give two examples of polycyclic groups G such that $\text{Zal}(G)/\text{Pl soc}(G)$ is infinite.

Example 4.3. Let

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

act on a free $\mathbf{Z}$-module $A = \langle a, b, c, d \rangle$ of rank 4. Thus

$$a \cdot x = b, \; b \cdot x = a + b, \; c \cdot x = c + d, \; d \cdot x = d.$$

We form the split extension $G = A \rtimes_{\mathscr{J}} \langle x \rangle$ and regard A as a subgroup of G. Let $A_1 = \langle a, b \rangle$, $A_2 = \langle c, d \rangle$ and $A_3 = \langle d \rangle = Z(G)$.

Since A is an abelian normal subgroup of G, we have $A \subseteq \text{Fit}\,(G)$. Since no power of x acts nilpotently on A_1 (this can be checked by computing the eigenvalues of x), $A = \text{Fit}\,(G) = \text{Zal}\,(G)$.

Now if we regard $A \otimes_{\mathbf{Z}} \mathbf{Q}$ as a $\mathbf{Q}\langle x^n \rangle$-module for $n \geqq 1$, we see that there are just two irreducible submodules $A_1 \otimes_{\mathbf{Z}} \mathbf{Q}$ and $A_3 \otimes_{\mathbf{Z}} \mathbf{Q}$. Hence $\text{Pl soc}\,(G) = A_1 \times A_3$, with A_1 an eccentric and A_2 a centric plinth.

We remark that in this example it is possible to construct irreducible kG-modules V such that V_A is not locally finite where $A = \text{Zal}\,(G)$.

Example 4.4. Let H be a free abelian group of finite rank $\geqq 1$, and A_1 a free abelian group of rank $\geqq 2$ on which H and its subgroups of finite index act rationally irreducibly.

It is easily seen that $V = A_1 \otimes_{\mathbf{Z}} \mathbf{Q}$ cannot be injective as a $\mathbf{Q}H$-module. However the injective hull $E_{\mathbf{Q}H}(V)$ is artinian and has all its composition factors isomorphic to V. Hence there is an essential extension W of V such that $W/V \cong V$. Now choose a $\mathbf{Z}H$-submodule A_2 of W such that $W = A_2 \otimes_{\mathbf{Z}} \mathbf{Q}$ and let G be the split extension of A_2 by H. It is easily seen that

$$\text{Fit}\,(G) = \text{Zal}\,(G) = A_2 \quad \text{and} \quad \text{Pl soc}\,(G) = A_1.$$

This method yields many explicit examples.

Let $H = \langle x \rangle$ be infinite cyclic and let A_1 be the $\mathbf{Z}[x]$-module $\mathbf{Z}[x]/(x^2 - x - 1)$. Then $V = A_1 \otimes_{\mathbf{Z}} \mathbf{Q}$ is irreducible as a $Q[x^n]$-module for each $n \geqq 1$, and we can choose a $\mathbf{Z}$-basis $\{a, b\}$ such that

$$a \cdot x = b \quad \text{and} \quad b \cdot x = a + b.$$

There exists an element $c \in E_{\mathbf{Q}[x]}(V)$ such that $c(x^2 - x - 1) = a$. If $d = c \cdot x$ then $d \cdot x = a + c + d$. Let A be the $\mathbf{Z}[x]$-module $\langle a, b, c, d \rangle$ and $G = A \rtimes_{\mathscr{J}} \langle x \rangle$. Then

$$\text{Zal}\,(G) = A \quad \text{and} \quad \text{Pl soc}\,(G) = A_1.$$

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