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# FINITE $p$-GROUPS WITH HOMOCYCLIC CENTRAL FACTORS

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**1.** If $G$ has nilpotence class $c(G) = c$, let $G = L_1(G) > L_2(G) > \ldots > L_{c+1}(G) = 1$ and $1 = Z_0(G) < Z_1(G) < \ldots < Z_c(G) = G$ denote the lower central series and upper central series of $G$, respectively. When there is no possibility of confusion we use $L_i$ for $L_i(G)$ and $Z_i$ for $Z_i(G)$. Throughout the paper we assume that $G$ is a finite $p$-group of class greater than two. Let $B(c, p^r)$ denote the collection of all $G$ of class $c$ for which $L_i/L_{i+1}$ is cyclic of order $p^r$ for $i = 2, \ldots, c$ and $UC(c, p^r)$ the collection of all $G$ of class $c$ for which $Z_i/Z_{i-1}$ is cyclic of order $p^r$ for $i = 1, \ldots, c - 1$. In Section 2 we generalize some of the results obtained by Blackburn in [**2**] about members of $B(c, p)$ to the class $B(c, p^r)$ and in Section 3 we show how the two classes $B(c, p^r)$ and $UC(c, p^r)$ are related. We also find conditions on $G$ which guarantee that the upper central series and the lower central series of $G$ coincide.

**2.** LEMMA 2.1. *If $G$ belongs to $B(c, p^r)$ and $i < c - 1$, then $Z_i \cap L_{c-i} = L_{c-i+1}$.*

*Proof.* For $i = 0$ the result is obvious. It is clear that $Z_1 \cap L_{c-1} \geqq L_c$. Let $L_{c-1} = \langle x, L_c\rangle$ and suppose $Z_1 \cap L_{c-1} > L_c$. Then $Z_1 \cap L_{c-1} = \langle x^{p^k}, L_c\rangle$ for some $k$ with $0 < k < r$. For any $g \in G$, $[g, x]^{p^k} = [g, x^{p^k}] = 1$ [**2**, p. 49]. But $L_c = [L_{c-1}, G] = \langle [x, G]\rangle$ so that $\exp L_c = p^k$ with $k < r$. This contradicts the assumption that $L_c$ is cyclic of order $p^r$. Thus the lemma holds for $i = 1$.

Inductively, suppose $Z_i \cap L_{c-i} = L_{c-i+1}$. We will show $Z_{i+1} \cap L_{c-i-1} = L_{c-i}$. If $H$ is any subgroup of $G$, we let $H^*$ denote the image of $H$ under the canonical mapping from $G$ to $G/L_{c-i+1}$. Since $[G, Z_{i+1} \cap L_{c-i-1}] \leqq Z_i \cap L_{c-i} = L_{c-i+1}$ we have

$$(Z_{i+1} \cap L_{c-i-1})^* \leqq Z(G^*) \cap L_{c-i-1}(G^*).$$

Because $G^*$ belongs to $B(c - i, p^r)$ we have

$$Z(G^*) \cap L_{c-i-1}(G^*) = L_{c-i}(G^*) = (L_{c-i}(G))^*.$$

Hence $Z_{i+1} \cap L_{c-i-1} \leqq L_{c-i}$. The other inclusion follows from a well-known theorem [**5**, p. 262].

THEOREM 2.1. *If $G$ belongs to $B(c, p^r)$, then $Z_i \cap L_2 = L_{c+1-i}$ for $i = 0, 1, \ldots, c - 1$.*

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*Proof.* We proceed by induction on $j = c + 1 - i$. For $j = 2$ we have $i = c - 1$ and $Z_{c-1} \cap L_2 = L_2$ because $L_2 \leqq Z_{c-1}$. Assume $Z_i \cap L_2 = L_{c+1-i}$ holds. Clearly, $L_{c+1-i} = Z_i \cap L_2 \geqq Z_{i-1} \cap L_2$ so that $Z_{i-1} \cap L_{c+1-i} \geqq Z_{i-1} \cap L_2$. Thus, according to Lemma 2.1, $L_{c+2-i} \geqq Z_{i-1} \cap L_2$. The other inclusion follows as before.

We remark that Example 4.2 shows there is a class 3 $p$-group for which $L_2/L_3$ and $L_3$ are cyclic but $Z_1 \cap L_2 > L_3$ so the requirement in Theorem 2.1 that the cyclic factors have the same order is essential.

Theorem 2.1 will be used in Section 3 to show how the classes $B(c, p^r)$ and $UC(c, p^r)$ are related and to obtain a necessary and sufficient condition for members of $B(c, p^r)$ to have coincident upper and lower central series. Another consequence of Theorem 2.1 is that many of the results which Blackburn obtained about members of $B(c, p)$ can be generalized to the class $B(c, p^r)$. Blackburn remarks [**2**, p. 62] that some of his results could be generalized to nilpotent groups with cyclic central factors of arbitrary order, but he refrained from doing this because "the results are rather complicated and the price of generality is a very considerable loss of clarity". However, we show below that if one narrows his attention to the class $B(c, p^r)$ a greater number of Blackburn's results can be generalized and it can be done without complication or loss of clarity. We refer the reader to [**3**] for additional results of this type.

Let $C(L_i:L_{i+2}) = \{x \in G | [x, L_i] \leqq L_{i+2}\}$. Also let $L_1{}^* = C(L_2:L_4)$ and $L_i{}^* = L_i$ for $i = 2, \ldots, c + 1$. If $G$ belongs to $B(c, p^r)$, we say $G$ has degree of commutativity greater than 0 if $[L_i{}^*, L_j{}^*] \leqq L_{i+j+1}$ for $i, j = 1, 2, \ldots, c - 1$.

THEOREM 2.2 (cf. [**5**, p. 363]). *If $G$ belongs to $B(c, p^r)$, then the following are equivalent*:

(a) $[L_1{}^*, L_i{}^*] \leqq L_{i+2}$ *for* $i = 1, 2, \ldots, c - 1$;

(b) *$G$ has degree of commutativity greater than* 0;

(c) *the abelian factor groups* $L_i/L_{i+2} (2 \leqq i \leqq c - 1)$ *of order* $p^{2r}$ *are pairwise isomorphic as $G$-groups*;

(d) $C(L_2:L_4) = C(L_i:L_{i+2})$ *for* $i = 2, 3, \ldots, c - 1$.

*Proof.* Blackburn's proof in the case $r = 1$ is based on a series of lemmas which depend on the facts that the factors of the lower central series are cyclic and $Z_i \cap L_2(G) = L_{c+1-i}(G)$ both of which are true for members of $B(c, p^r)$. Thus his proof, with minor modifications, is valid for the case that $G$ belongs to $B(c, p^r)$.

LEMMA 2.2. *If $G$ belongs to $B(c, p^r)$, then there is an element $s$ in $G$ with the property that*

$$|sC(L_2:L_4)| = p^r = |sC(L_{c-1})|.$$

*Proof.* By a result of Blackburn [**2**, p. 54], $G/C(L_2:L_4)$ is cyclic of order $p^r$ so if $C(L_2:L_4) = C(L_{c-1})$ the result is trivial. Hence we may assume $C(L_2:L_4) \neq C(L_{c-1})$. We claim that there is an element $y$ in $G \backslash C(L_{c-1})$ such that $|yC(L_2:L_4)| = p^r$.

Let $G = \langle x, C(L_2:L_4)\rangle$. If $x \notin C(L_{c-1})$, the claim is established. Thus we may assume $x \in C(L_{c-1})$. By a result of Blackburn [**2**, p. 54], $C(L_2:L_4)$ and $C(L_{c-1})$ have the same order so let $u \in C(L_2:L_4)\backslash C(L_{c-1})$. Let $y = xu$. Then $|yC(L_2:L_4)| = |xuC(L_2:L_4)| = |xC(L_2:L_4)| = p^r$ and $y \notin C(L_{c-1})$. This proves the claim.

If $|yC(L_{c-1})| = p^r$, the theorem is proved. Thus we may assume that $|yC(L_{c-1})| = p^k$ with $0 < k < r$. By suitably modifying the proof given by Blackburn [**2**, p. 55] for the case $r = 1$, it can be shown [**3**, p. 11] that $C(L_2:L_4)/C(L_1:L_3)$ is cyclic of order $p^r$, so let $C(L_2:L_4) = \langle v, C(L_1:L_3)\rangle$. If $C(L_{c-1}) \cap C(L_2:L_4) \neq C(L_1:L_3)$, then

$$C(L_{c-1}) \cap C(L_2:L_4) = \langle v^{p^n}, C(L_1:L_3)\rangle$$

for some $n$ with $0 < n < r$. Also $G = \langle y, v, C(L_1:L_3)\rangle$ and $m = \max(k, n) < r$. Since $v^{p^n}$ and $y^{p^k}$ belong to $C(L_{c-1})$ we have $[v^{p^n}, L_{c-1}] = [v, L_{c-1}]^{p^n} = 1$ and $[y^{p^k}, L_{c-1}] = [y, L_{c-1}]^{p^k} = 1$ [**2**, p. 49]. Therefore, since $C(L_1:L_3) \leqq C(L_{c-1})$ [**2**, p. 47] we have $[G, L_{c-1}]^{p^m} = 1$. But $[G, L_{c-1}] = L_c$ and $L_c$ is cyclic of exponent $p^r$, a contradiction. Therefore, we may assume $C(L_{c-1}) \cap C(L_2:L_4) = C(L_1:L_3)$.

With $y$ and $v$ defined as above let $s = yv$. Then $|sC(L_2:L_4)| = |yC(L_2:L_4)| = p^r$. Now $y^{p^{r-1}} \in C(L_{c-1})$ and $s^{p^{r-1}} = y^{p^{r-1}} v^{p^{r-1}} \pmod{L_2}$ so if $s^{p^{r-1}} \in C(L_{c-1})$ then

$$v^{p^{r-1}} \in C(L_{c-1}) \cap C(L_2:L_4) = C(L_1:L_3).$$

But $|vC(L_1:L_3)| = p^r$ so that $s^{p^{r-1}} \notin C(L_{c-1})$. Thus $|sC(L_{c-1})| = p^r$ also and the result is proved.

Let $EB(c, p^r)$ denote the class of groups $G$ which belong to $B(c, p^r)$ and have $\exp(G/L_2) = p^r$. If $G$ belongs to $B(c, p^r)$, then the minimum order of $G$ is $p^{(c+1)r}$. Hence we adopt the notation $MB(c, p^r)$ for members of $B(c, p^r)$ which have order $p^{(c+1)r}$. When $r = 1$ members of $MB(c, p^r)$ are commonly called $p$-groups of maximal class.

THEOREM 2.3. *Suppose $G$ belongs to $EB(c, p^r)$ and, when $c > 4$, $G/L_c$ has degree of commutativity greater than 0. Then $G$ possesses a subgroup $K$ which belongs to $MB(c-1, p^r)$, and $L_i^*(K) = L_{i+1}(G)$ for $i = 1, 2, \ldots, c-1$.*

*Proof.* By Lemma 2.2 there is an element $s$ of $G$ with the property that $|sC(L_{c-1})| = |sC(L_2:L_4)| = p^r$. By modifying the proofs of Blackburn for the case $r = 1$ [**2**, pp. 55–56], it can be shown [**3**, pp. 11–13] that $C(L_1:L_3) = Z_{c-1}(G)$ and $C(L_2:L_4)/C(L_1:L_3)$ is cyclic of order $p^r$. So let $s_1$ and $Z_{c-1}(G)$ generate $C(L_2(G):L_4(G))$ and for $i = 2, \ldots, c$, let $s_i = [s_{i-1}, s]$. It follows from a well-known theorem [**5**, p. 258] that $L_{i+1}(G)$ and $s_i$ generate $L_i(G)$ for $i = 2, \ldots, c-1$. This is also true for $i = c$; for $L_{c-1} = \langle s_{c-1}, L_c\rangle$ so if $[s_{c-1}, s]^{p^k} = 1$ then $[s, L_{c-1}]^{p^k} = [s^{p^k}, L_{c-1}] = 1$ and therefore $s^{p^k}$ is in $C(L_{c-1})$ so $k \geqq r$ by choice of $s$.

Now let $K$ be the group generated by $L_2(G)$ and $s$. Then $|K| = p^{cr}$. Also $s_2$ and $s$ are in $K$ and $[s_2, s, \ldots, s] = s_c \neq 1$. Thus $c(K) \geqq c - 1$. Since $K$ is generated by $s$ and $L_2(G)$ we have $L_2(K) = [K, L_2(G)] \leqq L_3(G)$ [**2**, p. 52] and therefore $L_i(K) \leqq L_{i+1}(G)$ for $i \geqq 2$. In particular, $L_c(K) = 1$ so that $c(K) = c - 1$. Now since $|s_c| = p^r$ and $s_c$ is in $L_{c-1}(K)$ we have exp $(L_i(K)/L_{i+1}(K)) \geqq p^r$ [**2**, p. 49] for $i = 1, 2, \ldots, c - 1$. Since $|K| = p^{cr}$, exp $(L_i(K)/L_{i+1}(K)) \geqq p^r$ and $c(K) = c - 1$ we must have $L_i(K)/L_{i+1}(K)$ is cyclic of order $p^r$ for $i = 2, \ldots, c - 1$. Thus for $i \geqq 2$, $|L_i(K)| = |L_{i+1}(G)|$ and therefore $L_i(K) = L_{i+1}(G)$ for $i \geqq 2$. Also

$$[L_2(G), L_2(K)] = [L_2(G), L_3(G)] \leqq L_5(G) = L_4(K)$$

so that $L_2(G) \leqq C(L_2(K):L_4(K)) = L_1^*(K)$. Since these two groups have the same order they are equal. Finally, since $|K/L_2(K)| = p^{2r}$ and $L_i(K)/L_{i+1}(K)$ is cyclic of order $p^r$ for $i = 2, \ldots, c - 1$ we have $K$ belongs to $MB(c - 1, p^r)$. This proves the theorem.

LEMMA 2.3. *Suppose $G = \langle a \rangle \oplus \langle b \rangle$ where $|a| = |b| = p^r$ $(p \neq 2)$ and $G$ contains a subgroup $C$ of order $p^r$ such that $|aC| = p^r$. Let $A = \langle a \rangle$ and $B = \langle b \rangle$. Then there is a cyclic subgroup $D$ of $G$ of order $p^r$ with the property that $A \cap D = B \cap D = C \cap D = 1$.*

*Proof.* It is not difficult to show that one of $\langle ab \rangle$ or $\langle ab^2 \rangle$ satisfies the conclusion.

THEOREM 2.4. *Suppose $G$ belongs to $EB(c, p^r)$ and, when $c > 4$, $G/L_c$ has degree of commutativity greater than 0. If $3 \leqq c \leqq p$, then $G/L_c$ and $L_2$ have exponent $p^r$. If $c < p$, the elements of $G$ of order at most $p^r$ form a characteristic subgroup of index at most $p^r$.*

*Proof.* Let $s$ be an element with the property specified in Lemma 2.2 and let $S$ be the centralizer of $s$ in $G$. By modifying Blackburn's proofs [**2**, p. 63] it can be shown [**3**, pp. 20–22] that $S \cap L_2 = L_c$ and that there is a subset $T$ of $S$ with the property that $Z_{c-1} = \langle T, L_2 \rangle$. Then $s^{p^r}$ and for each $t$ in $T$, $t^{p^r}$ are elements of $S \cap L_2$. Hence $s^{p^r}$ and $t^{p^r}$ are elements of $L_c$. By applying Lemma 2.3 to $G/Z_{c-1}$ we see that there is an element $s'$ of $G$ which also has the property specified in Lemma 2.2 and $\langle s \rangle \cap \langle s' \rangle = 1 \pmod{Z_{c-1}}$. Therefore, as before, $s'^{p^r}$ belongs to $L_c$. Since $\langle s \rangle \cap \langle s' \rangle = 1 \pmod{Z_{c-1}}$ and $L_2$ is contained in the Frattini subgroup [**5**, p. 272] we have $G$ is generated by $s$, $s'$ and $T$. Thus $G/L_c$ is generated by a set of elements of order $p^r$. Since $c(G/L_c) < p$, $G/L_c$ is a regular $p$-group and therefore $G/L_c$ is of exponent $p^r$.

If $c < p$, $G$ itself is regular $p$-group. Thus the elements of order at most $p^r$ form a subgroup $E_r$ of $G$. Let $P_r = \langle g^{p^r} : g \in G \rangle$. Then $|G/E_r| = |P_r|$ [**5**, p. 327]. Since $G/L_c$ is of exponent $p^r$ we have $P_r \leqq L_c$ and therefore $|P_r| \leqq p^r$. Hence $|G/E_r| \leqq p^r$. Finally, $P_r \leqq L_c \leqq Z_1$ implies $L_2 \leqq E_r$ [**5**, p. 327] so that $L_2$ is of exponent $p^r$.

If $c = p$ we let $K$ be the subgroup of $G$ defined in Theorem 2.3. If $p > 3$, we have $\exp L_2(K) = p^r$ since $c(K) = p - 1$ satisfies $3 \leqq c(K) < p$ which is a case handled above. If $p = 3$, then $L_3(K) = 1$ and therefore $L_2(K)$ is cyclic of order $p^r$ so has exponent $p^r$. Thus $L_2(K)$ has exponent $p^r$ for any $p$ which satisfies the hypothesis and therefore, by Theorem 2.3, $L_3(G)$ has exponent $p^r$. Let $G$ be generated by $s$, $s_1$ and $C(L_1(G){:}L_3(G))$. Then $L_2(G)$ is generated by $s_2 = [s_1, s]$ and $L_3(G)$ [**5**, p. 258]. Let $x$ belong to $L_2(G)$. Then $x = s_2{}^m u$ where $u$ belongs to $L_3(G) = L_2(K)$. Since $s_2$ and $u$ belong to $K$, $c(K) = p - 1$ and $\exp L_2(K) = p^r$ it follows from regularity that $x^{p^r} = (s_2{}^m)^{p^r}$. Thus in order to show $\exp L_2(G) = p^r$ it is sufficient to show that $s_2{}^{p^r} = 1$. But $s_2 = s_1{}^{-1}s^{-1}s_1 s = (s^{-1})^{s_1}s$ so that $s_2{}^{p^r} = ((s^{-1})^{s_1}s)^{p^r}$. Since $(s^{-1})^{s_1}$ and $s$ belong to $K$ and $\exp L_2(K) = p^r$ we have by regularity

$$((s^{-1})^{s_1}s)^{p^r} = ((s^{-1})^{s_1})^{p^r}s^{p^r} = (s^{-p^r})^{s_1}s^{p^r}.$$

By the first part of the theorem $s^{-p^r}$ is in the center of $G$ so that $s_2{}^{p^r} = s^{-p^r}s^{p^r} = 1$. This proves the theorem.

LEMMA 2.4. *If $G$ belongs to $EB(c, p^r)$, then $G$ has a subgroup $H$ with the properties that $H$ belongs to $MB(c, p^r)$ and $L_i(H) = L_i(G)$ for $i = 2, \ldots, c + 1$.*

*Proof.* Let $y$ and $v$ be defined as in proof of Lemma 2.2 so that $G = \langle y, v, C(L_1{:}L_3)\rangle$. If we let $H = \langle y, v\rangle$ we have $G = HC(L_1{:}L_3)$ and a theorem of Blackburn [**2**, p. 48] shows $L_i(H) = L_i(G)$ for $i = 2, \ldots, c + 1$. Clearly, $H$ has the proper order.

Example 4.1 shows that for every choice of $c$, $p$ and $r$ there are members of $B(c, p^r)$ with the property that $L_i$ is cyclic for all $i \geqq 2$. The next theorem shows that if any one of $L_2, L_3, \ldots, L_{c-1}$ is cyclic then they all are.

THEOREM 2.5. *If $G$ belongs to $B(c, p^r)$ and $L_{c-1}$ is cyclic, then $L_2$ is also cyclic. If $G$ belongs to $EB(c, p^r)$ then $L_{c-1}$ is cyclic if and only if $p = 2$ and $r = 1$.*

*Proof.* If $L_{c-1}$ is cyclic, then by Theorem 2.1, $Z_2 \cap L_2$ is cyclic and we conclude by a result of P. Hall [**5**, p. 306] that $L_2$ is cyclic. If $G$ belongs to $EB(c, p^r)$ Lemma 2.4 shows we may assume $G$ belongs to $MB(c, p^r)$. It follows from the proof of Theorem 2.3 that $|G/Z_{c-1}| = p^{2r}$ so that $Z_{c-1} = L_2$.

Therefore, by Theorem 2.1, $Z_2 = Z_2 \cap L_2 = L_{c-1}$. If $L_{c-1} = Z_2$ is cyclic, then $p = 2$ and $G$ has a cyclic subgroup of index 2 [**5**, p. 305]. It follows from Theorems 4.4 and 4.3 in [**4**, pp. 191-193] that $r = 1$. Finally, if $G$ belongs to $MB(c, 2)$ then Theorems 4.5 and 4.3 in [**4**, pp. 191–194] show that $L_2$ is cyclic.

**3.** THEOREM 3.1 [**1**, p. 533]. *If $G$ belongs to $UC(c, p^r)$, then $[G, Z_i] = Z_{i-1}$ for $i = 1, \ldots, c - 1$.*

THEOREM 3.2. *Let $G$ belong to $UC(c, p^r)$. Then $Z_{c+1-i} \geqq L_i > Z_{c-i}$ for $i = 1, \ldots, c$.*

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*Proof.* This is clearly true when $i = 1$. Suppose $Z_{c+1-i} \geqq L_i > Z_{c-i}$. Then, by Theorem 3.1, $Z_{c-i} \geqq L_{i+1} \geqq Z_{c-i-1}$ and in any nilpotent group $L_{i+1} \neq Z_{c-i-1}$.

THEOREM 3.3. *Let $G$ belong to $B(c, p^r)$. Then $G$ belongs to $MB(c, p^r)$ if and only if the upper central series and the lower central series of $G$ coincide.*

*Proof.* Suppose $G$ belongs to $MB(c, p^r)$. Clearly $Z_i = L_{c+1-i}$ for $i = c$. By the remark made in the proof of Theorem 2.3, $C(L_1:L_3) = Z_{c-1}$ and $|G/C(L_1:L_3)| = p^{2r}$. Thus $Z_i = L_{c+1-i}$ for $i = c - 1$ also. If $i < c - 1$, then $Z_i < Z_{c-1} = L_2$ so that $Z_i = Z_i \cap L_2 = L_{c+1-i}$ by Theorem 2.1. Hence the two series coincide. Now assume that $Z_i = L_{c-1-i}$ for all $i$. In particular, $Z_{c-1} = L_2$. Since $|G/Z_{c-1}| = p^{2r}$ for any member of $B(c, p^r)$, it follows that $G$ belongs to $MB(c, p^r)$.

We remark that Example 4.3 shows there exists a finite $p$-group with the property that the upper central series and the lower central series of $G$ coincide and $|L_i/L_{i+1}| = p^2$ for $i = 2, \ldots, c$ but $L_i/L_{i+1}$ is not cyclic for $i = 2, \ldots, c$.

Theorem 3.3 shows that members of $MB(c, p^r)$ also belong to $UC(c, p^r)$. The next theorem shows that

$$B(c, p^r) \cap UC(c, p^s) = MB(c, p^r).$$

THEOREM 3.4. *A group $G$ belongs to $B(c, p^r)$ and $UC(c, p^s)$ if and only if $G$ belongs to $MB(c, p^r)$.*

*Proof.* If $G$ belongs to $MB(c, p^r)$, then by Theorem 3.3 the lower central series of $G$ and the upper central series of $G$ coincide. Now suppose that $G$ belongs to $B(c, p^r)$ and $UC(c, p^s)$. Then, by Theorem 3.2, $L_2 \geqq Z_1$ and, by Theorem 2.1, $Z_1 \cap L_2 = L_c$. Thus $Z_1 = L_c$. It follows that $r = s$ and $L_i = Z_{c+1-i}$ for $i = 1, \ldots, c + 1$ so that $G$ belongs to $MB(c, p^r)$.

THEOREM 3.5. *If $G$ belongs to $UC(c, p^r)$ and $Z_{c+1-i} = L_i$ for some $i$ with $2 \leqq i \leqq c$, then $G$ belongs to $MB(c, p^r)$.*

*Proof.* By Theorem 3.1, $Z_{c+1-j} = L_j$ for all $j \geqq i$. We show $Z_{c+2-i} = L_{i-1}$. Clearly $|L_{i-1}/L_i| \leqq |Z_{c+2-i}/Z_{c+1-i}|$. Also

$$\begin{aligned} |L_{i-1}/L_i| &\geqq \exp\,(L_{i-1}/L_i) \geqq \exp\,(L_i/L_{i+1}) \\ &= \exp\,(Z_{c+1-i}/Z_{c-i}) \geqq \exp\,(Z_{c+2-i}/Z_{c+1-i}) \\ &= |Z_{c+2-i}/Z_{c+1-i}|. \end{aligned}$$

Here we have used [**5**, Satz 2.13, p. 266]. Thus $Z_{c+2-i} = L_{i-1}$. It follows the lower central series of $G$ and the upper central series of $G$ coincide so that, by Theorem 3.3, $G$ belongs to $MB(c, p^r)$.

THEOREM 3.6. *Suppose $G/Z_{c-1}$ has exponent $p^r$, $|L_i/L_{i+1}| = p^r$ for all $i$ with $2 \leqq i \leqq c$ and $Z_{c+1-j} = L_j$ for some $j$ with $2 \leqq j \leqq c$. Then $G$ belongs to $B(c, p^r)$.*

*Proof.* First we show $Z_{c-j} = L_{j+1}$. Since

$$p^r = \exp (Z_c/Z_{c-1}) \leqq \exp (Z_{c+1-j}/Z_{c-j}) \leqq |Z_{c+1-j}/Z_{c-j}| \leqq |L_j/L_{j+1}| = p^r,$$

it follows that $Z_{c-j} = L_{j+1}$. Repeating this argument the appropriate number of times we obtain $Z_1 = L_c$. Since $p^r \leqq \exp Z_1 = \exp L_c \leqq |L_c| = p^r$, we have $\exp L_c = p^r$. Also $\exp (L_i/L_{i+1}) \geqq p^r = |L_i/L_{i+1}|$. Thus $L_i/L_{i+1}$ is cyclic for all $i \geqq 2$ and therefore $G$ belongs to $B(c, p^r)$.

Example 4.3 also shows that the hypotheses of Theorem 3.6 cannot be weakened by replacing $\exp (G/Z_{c-1}) = p^r$ by $\exp (G/Z_{c-1}) \leqq p^r$ nor by replacing $|L_i/L_{i+1}| = p^r$ by $\exp (L_i/L_{i+1}) = p^r$.

COROLLARY. *Suppose $G/L_2$ has exponent $p^r$ and $|L_i/L_{i+1}| = p^r$ for $i = 2, \ldots, c$. Then $G$ belongs to $MB(c, p^r)$ if and only if $Z_{c-1} = L_2$.*

*Proof.* If $G$ belongs to $MB(c, p^r)$ ,then $Z_{c-1} = L_2$ by Theorem 3.3. If $Z_{c-1} = L_2$, then by Theorem 3.6, $G$ belongs to $B(c, p^r)$ and since for any $G$ in $B(c, p^r)$, $|G/Z_{c-1}| = p^{2r}$ the result follows.

THEOREM 3.7. *Suppose $G/Z_{c-1}$ and $G/L_2$ have exponent $p^r$, $L_i/L_{i+1}$ is cyclic for $i = 2, \ldots, c$ and $Z_{c+1-j} = L_j$ for some $j$ with $2 \leqq j \leqq c$. Then $G$ belongs to $B(c, p^r)$.*

*Proof.* Since $\exp (Z_c/Z_{c-1}) = p^r$ we have $|Z_i/Z_{i-1}| \geqq p^r$ for all $i$ with $1 \leqq i \leqq c$. Hence $|L_j| = |Z_{c+1-j}| \geqq p^{(c+1-j)r}$. Also $p^r = \exp (G/L_2) \geqq \exp (L_i/L_{i+1}) = |L_i/L_{i+1}|$ for $i \geqq 2$. Thus $|L_j| \leqq p^{(c+1-j)r}$. It follows that $|L_j| = p^{(c+1-j)r}$ and $|L_c| = p^r$. Therefore, because $\exp (G/L_2) = \exp L_c = p^r$ we have $\exp (L_i/L_{i+1}) = p^r$ for all $i$ and the result follows.

COROLLARY. *Suppose $G/L_2$ has exponent $p^r$ and $L_i/L_{i+1}$ is cyclic for $i = 2, \ldots, c$. Then $G$ belongs to $MB(c, p^r)$ if and only if $Z_{c-1} = L_2$.*

*Proof.* If $G$ belongs to $MB(c, p^r)$, then $Z_{c-1} = L_2$ by Theorem 3.3. If $Z_{c-1} = L_2$ then, by Theorem 3.7, $G$ belongs to $B(c, p^r)$ and therefore, by the corollary to Theorem 3.6, $G$ belongs to $MB(c, p^r)$.

**4.** In this section we give an example which shows for every choice of $c$, $p$ and $r$ the class $B(c, p^r)$ is not empty and the two examples referred to in the previous sections.

*Example* 4.1. Let $A = \langle a \rangle$ where $|a| = p^{cr}$. The map which sends $a^n$ to $a^{n(1+p^r)}$ defines an automorphism of $A$. Thus there exists a group $G = \langle x, a \rangle$ where $x^{-1}ax = a^{1+p^r}$. Then $L_i(G) = \langle a^{p^{(i-1)r}} \rangle$ for $i = 2, \ldots, c+1$ and therefore $G$ belongs to $B(c, p^r)$.

*Example* 4.2. Let $A = \langle a \rangle \oplus \langle b \rangle$ where $|a| = p^4$ and $|b| = p^6$. The map which sends $a^n b^m$ to $a^{n+m} b^{np^4+m}$ defines an automorphism of $A$. Thus there exists a finite $p$-group $G$ and an element $x$ such that $G = \langle x, A \rangle$ where $x^{-1}ax =$

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$ab^{p^4}$ and $x^{-1}bx = ab$. Then $L_2 = \langle a, b^{p^4}\rangle$, $L_3 = \langle b^{p^4}\rangle$ and $L_4 = 1$. Since $b^{p^4}$ and $a^{p^2}$ belong to $Z_1(G)$ we have $L_2(G) \cap Z_1(G) > L_3(G)$. Thus the remark following Theorem 2.1 is verified.

*Example* 4.3. Let $A = \langle a_1\rangle \oplus \langle a_2\rangle \oplus \langle a_3\rangle \oplus \langle a_4\rangle$ where $|a_1| = |a_2| = p^2$ and $|a_3| = |a_4| = p$ with $p$ an odd prime. The map which sends

$$a_1^{n_1}a_2^{n_2}a_3^{n_3}a_4^{n_4} \quad \text{to} \quad a_1^{n_1+pn_3}a_2^{n_2+pn_4}a_3^{n_1+n_3}a_4^{n_2+n_4}$$

defines an automorphism of $A$. Thus there exists a finite $p$-group $G$ and an element $x$ such that $G = \langle x, A\rangle$ where $|x| = p$,

$$x^{-1}a_1x = a_1a_3, \quad x^{-1}a_2x = a_2a_4, \quad x^{-1}a_3x = a_3a_1^p, \quad x^{-1}a_4x = a_4a_2^p.$$

Then $L_2 = \langle a_1^p, a_2^p, a_3, a_4\rangle$, $L_3 = \langle a_1^p, a_2^p\rangle$, $L_4 = 1$ and $G$ has the following properties:

(i) $\exp(L_i/L_{i+1}) = p$ for $i = 1, 2, \ldots, c$;
(ii) $|L_i/L_{i+1}| = p^2$ for $i = 2, \ldots, c$;
(iii) $L_{c+1-i} = Z_i$ for $i = 0, 1, \ldots, c$;
(iv) $L_i/L_{i+1}$ is not cyclic for $i = 2, \ldots, c$.

Hence the remarks made following Theorems 3.3 and 3.6 are valid.

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