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RUDIN-SHAPIRO SEQUENCES FOR ARBITRARY COMPACT GROUPS

For George Szekeres on his sixty-fifth birthday

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Abstract

Let G be a compact group. A sequence $\{f_n\}_{n=1}^{\infty}$ of functions in $L^\infty(G)$ is said to be a *Rudin-Shapiro sequence* (briefly, an RS-sequence) if the following conditions hold:

(1) $$\inf_n \|f_n\|_2 > 0;$$

(2) $$\sup_n \|f_n\|_\infty < \infty;$$

(3) $$\lim_n \|\hat{f}_n\|_\infty = 0;$$

The main purpose here is to prove the following theorem:

THEOREM. *Let G be an infinite compact group. Then G has an RS-sequence consisting of trigonometric polynomials.*

The proof is carried out in section 1 while in section 2 several applications are given concerning set-theoretic relations between certain function spaces in harmonic analysis. The existence of RS-sequences for infinite LCA groups is well-known.

NOTATION. Let G be a compact group. The Banach space of all continuous complex-valued functions on G we denote by $C(G)$, and the Banach space of all complex Radon measures on G by $M(G)$; $L^1(G)$ will be identified in the usual way with the ideal in $M(G)$ of measures that are absolutely continuous with respect to normalized Haar measure λ_G.

The symbol $\hat{G}$ will denote a maximal set of pairwise inequivalent continuous irreducible unitary representations of G. The representation space of $\gamma \in G$ will be denoted by H_γ, and its dimension by d_γ. By $\mathfrak{E}(\hat{G})$ we mean the linear space of all "sections" over $\hat{G}$, i.e. of all those functions

$\Phi: \hat{G} \to \amalg_{\gamma \in \hat{G}} \mathscr{L}(H_\gamma)$ such that $\Phi(\gamma) \in \mathscr{L}(H_\gamma)$ for all $\gamma \in \hat{G}$. Here of course $\mathscr{L}(H_\gamma)$ is the von Neumann algebra of all (bounded) linear operators on H_γ. The Banach spaces $\mathfrak{E}^p(\hat{G})$ $(1 \leqq p \leqq \infty)$ and $\mathfrak{E}_0$, are defined as in (28.34) of Hewitt & Ross (1970). The norms on the $\mathfrak{E}^p(\hat{G})$ are given by

$$\|\Phi\|_\infty = \sup\{\|\Phi(\gamma)\|_{\phi_\infty} : \gamma \in \hat{G}\} \qquad (\Phi \in \mathfrak{E}^\infty(\hat{G}))$$

$$\|\Phi\|_p = \left(\sum_{\gamma \in G} d_\gamma \|\Phi(\gamma)\|_{\phi_p}^p\right)^{1/p} \qquad (\Phi \in \mathfrak{E}^p(\hat{G}))$$

where $\| \quad \|_{\phi_p}$ denotes the pth von Neumann-Schatten norm on $\mathscr{L}(H_\gamma)$. In particular $\|A\|_{\phi_2} = [tr(AA^*)]^{\frac{1}{2}}$, and $\|A\|_{\phi_\infty}$ is the operator norm of A.

The Fourier-Stieltjes transform of $\mu \in M(G)$ we define as an element of $\mathfrak{E}^\infty(\hat{G})$ by

$$\hat{\mu}(\gamma) = \int_G \gamma(x^{-1}) d\mu(x)$$

and its Fourier series is the series (suitably interpreted)

$$\mu \sim \sum_{\gamma \in G} d_\gamma tr(\hat{\mu}(\gamma)\gamma(\cdot)).$$

The closure of the nth derived subgroup of G we denote by $G^{(n)}$.

1. Proof of the theorem

Before commencing the proof we remark that the existence of RS-sequences for infinite LCA groups is well-known; see Gaudry (1970) and (37.19b) of Hewitt & Ross (1970). Also a weaker version is known to exist for infinite compact groups. Specifically, whenever $t \in]2, \infty]$, a sequence $\{f_n\}_{n=1}^\infty$ of functions is said to be a t-RS-sequence if it satisfies conditions (1) and (3) of the definition of an RS-sequence with (2) replaced by

(2)′
$$\sup_n \|f_n\|_t < \infty.$$

In Figà-Talamanca & Price (1972), random Fourier series are used to show that t-RS-sequences with $t < \infty$ exist for all infinite compact groups. Also the existence of such sequences with other useful properties is demonstrated in Figà-Talamanca & Price (1972, 1973). We have not been able to generalise these extra properties to RS-sequences.

Since the definition of an RS-sequence involves only three norms, it is easily verified that any RS-sequence may be replaced by an RS-sequence consisting of trigonometric polynomials. In this section we therefore prove merely the existence of an RS-sequence for any infinite compact group.

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Whenever the supports of the members of an RS-sequence are contained in some open set U, then we say that this sequence is a U-RS-*sequence*.

Our proof begins with two special cases, from which we proceed to deduce the general case.

(1.1) PROPOSITION. (Gaudry (1970), Lemma 2.1). *Let G be an infinite compact abelian group and U a nonvoid open subset of G. Then G has a U-RS-sequence.*

Now, let us say that a compact group G is *tall* if for every positive integer d there are at most finitely many elements of $\hat{G}$ of degree d.

(1.2) PROPOSITION. *Let G be an infinite tall compact group and U a nonvoid open subset of G. Then G has a U-RS-sequence.*

PROOF. The following construction depends on repeated applications of the fact that every measurable subset of G of positive measure has a subset of half its measure.

Let $V \subseteq \bar{V} \subseteq U \subseteq G$ be measurable, $\lambda_G(V) = v > 0$. Let P_1, P_2 be disjoint measurable subsets of V such that $P_1 \cup P_2 = V$, $\lambda_G(P_1) = \lambda_G(P_2)$. Let $\pi_1 = \{P_1, P_2\}$. If π_{n-1} has been defined as a partition of U into 2^{n-1} subsets $P_{n-1,i}$ $(1 \leqq i \leqq 2^{n-1})$ then form π_n by writing $P_{n-1,i}$ as a disjoint union of two measurable subsets $P_{n,2i-1}$, $P_{n,2i}$ of equal measure. Thus π_n is a set $\{P_{n,i} : 1 \leqq i \leqq 2^n\}$ of pairwise disjoint measurable subsets of V of equal measure such that $V = \bigcup_{i=1}^{2^n} P_{n,i}$.

We now define a sequence of Rademacher functions associated with the sequence π_n. Put $r_1 = \chi_{P_{1,1}} - \chi_{P_{1,2}}$, and more generally let $r_n = \sum_{i=1}^{2^n} (-1)^{i+1} \chi_{P_{n,i}}$. Then r_n takes values ± 1 and

$$\int_{P_{n-1,i}} r_n d\lambda_G = \int_{P_{n,2i-1}} r_n d\lambda_G + \int_{P_{n,2i}} r_n d\lambda_G = 0.$$

(This construction is also used in Figà-Talamanca and Gaudry (1970).) Clearly we have $\|r_n\|_p = v^{1/p}$ $(1 \leqq p < \infty)$, $\|r_n\|_\infty = 1$, and $\int_G r_m r_n d\lambda_G = v\delta_{mn}$ for $m, n \in \{1, 2, \cdots\}$. Define $f_n = v^{-\frac{1}{2}} r_n$; then we have

$$\|f_n\|_p = v^{1/p - \frac{1}{2}} (1 \leqq p < \infty),$$

$$\|f_n\|_\infty = v^{-\frac{1}{2}}, \quad \text{and} \tag{4}$$

$$\int_G f_m f_n d\lambda_G = \delta_{mn} \qquad (m, n \in \{1, 2, \cdots\}).$$

We claim that $\{f_n\}_{n=1}^\infty$ is an RS-sequence under the assumption that G is tall.

Indeed, in view of (4) all that is required is to show that $\|\hat{f}_n\|_\infty \to 0$ as $n \to \infty$. Now Parseval's formula for G is

$$\|f\|_2^2 = \sum_{\gamma \in \hat{G}} d_\gamma tr(\hat{f}(\gamma)\hat{f}(\gamma)^*) \qquad (f \in L^2(G))$$

Also, by Hewitt and Ross (1963 and 1970), (D.51) we have, for $A \in \mathscr{L}(H)$, $tr(AA^*) \geqq \|A\|_{\phi_\infty}^2$, and hence

$$\sum_{\gamma \in \hat{G}} d_\gamma \|\hat{f}_n(\gamma)\|_{\phi_\infty}^2 \leqq \|f_n\|_2^2 = 1. \tag{5}$$

This makes it clear for each $n \geqq 1$ and each $\varepsilon > 0$, the set $\{\gamma \in \hat{G} : \|\hat{f}_n(\gamma)\|_{\phi_\infty} > \varepsilon\}$ is finite (this merely reproves the well known fact that $\mathfrak{E}^2 \subseteq \mathfrak{E}_0$). Hence we may conclude that

$$\|\hat{f}_n\|_\infty = \|\hat{f}_n(\gamma_n)\|_{\phi_\infty}$$

for some $\gamma_n \in \hat{G}$. Let $\Delta = \{\gamma_n : n \geqq 1\}$. If Δ is infinite, then $d_{\gamma_n} \to \infty$ as $n \to \infty$, by our assumption about the representations of G. Hence by (5), we have

$$\|\hat{f}_n\|_\infty = \|\hat{f}_n(\gamma_n)\|_{\phi_\infty} \leqq d_{\gamma_n}^{-1/2},$$

showing that $\{f_n\}_{n=1}^\infty$ is an RS-sequence as asserted.

In any case, since $\{f_n\}_{n=1}^\infty$ is orthonormal in $L^2(G)$, it follows that $\|\hat{f}_n(\gamma)\|_{\phi_\infty} \to 0$ for each $\gamma \in \hat{G}$. Thus when Δ is finite, we have

$$\|\hat{f}_n\|_\infty \leqq \sup_{\gamma \in \Delta} \|\hat{f}_n(\gamma)\|_{\phi_\infty} \to 0 \quad \text{as} \quad n \to \infty$$

and again $\{\hat{f}_n\}_{n=1}^\infty$ is an RS-sequence. This completes the proof.

(1.3) LEMMA. *Let Γ be a closed subgroup of the compact group G. Then there is a quasi-invariant normalised measure λ on the coset space G/Γ with the following property: if f is a nonnegative extended real-valued λ_G-integrable function on G, then the set of cosets $x\Gamma$ in G/Γ for which the function $\xi \to f(x\xi)$ $(\xi \in \Gamma)$ is not λ_Γ-integrable is λ-null; the function on G/Γ defined λ-a.e. by $x\Gamma \to \int_\Gamma f(x\xi)d\lambda_\Gamma(\xi)$* is *$\lambda$-integrable, and we have*

$$\int_G f(x)d\lambda_G(x) = \int_{G/\Gamma} \int_\Gamma f(x\xi)d\lambda_\Gamma(\xi)d\lambda(x\Gamma). \tag{6}$$

The reader is referred to Bourbaki (1963), Chapter VII, section 2, discussion following Théorème 2.

(1.4) LEMMA. *Let $\phi : G \to G_1$ be a continuous surjective homomorphism, where G and G_1 are compact groups and G_1 has an RS-sequence. Then G has an RS-sequence.*

This is proved in Edwards & Price (1970), A.3.3.

(1.5) LEMMA. *Let Γ be a closed subgroup of a separable compact group G, and suppose that Γ admits an RS-sequence. Then G does also.*

PROOF. Since G is separable, there is a Borel section B for Γ in G (Mackey (1951)), i.e., a Borel set B in G which meets each coset $x\Gamma$ in exactly one point (we may assume $B \cap \Gamma = \{e\}$). Define a map $b : G/\Gamma \to B$ by setting $b(x\Gamma)$ as the unique member of $B \cap x\Gamma$. Let $h \in C(\Gamma)$. Put

$$h^0(x\xi) = h(\xi) \qquad (x \in B, \xi \in \Gamma).$$

The properties of B ensure that h^0 is well-defined as an element of $L^\infty(G)$. Application of Lemma (1.3) shows immediately that

$$\|h^0\|_2^2 = \int_{G/\Gamma} \|h\|_2^2 \, d\lambda(x) = \|h\|_2^2. \tag{7}$$

It is also easy to see that

$$\|h^0\|_\infty = \|h\|_\infty. \tag{8}$$

Now, let $\sigma \in \hat{G}$ be fixed. Then the restriction $\sigma|_\Gamma$ of σ to Γ admits a decomposition

$$\sigma|_\Gamma = \bigoplus_{\tau \in \Gamma} n_\sigma(\tau) . \tau$$

($n_\sigma(\tau)$ is the multiplicity of τ in $\sigma|_\Gamma$ and $n_\sigma(\tau) = 0$ for all save finitely many τ). This decomposition is given via some unitary intertwining transformation from $\bigoplus_{\tau \in \Gamma} n_\sigma(\tau) H_\tau$ to H_σ which by transport of structure gives rise to a (von Neumann) algebra isomorphism

$$\alpha_\sigma : \bigoplus_{\tau \in \hat{G}} n_\sigma(\tau) \mathscr{L}(H_\tau) \to \mathscr{L}(H_\sigma).$$

Then we have by (1.3)

$$\begin{aligned}
(h^0)^\wedge(\sigma) &= \int_G h^0(x)\sigma(x^{-1}) d\mu(x) = \int_{G/\Gamma} \left(\int_\Gamma h^0(x\xi)\sigma((x\xi)^{-1}) d\lambda_\Gamma(\xi) \right) d\lambda(x\Gamma) \\
&= \int_{G/\Gamma} \left(\int_\Gamma h(\xi)\sigma(\xi^{-1}) d\lambda_\Gamma(\xi) \right) \sigma[(b(x\Gamma))^{-1}] d\lambda(x\Gamma) \\
&= \int_{G/\Gamma} \left(\int_\Gamma h(\xi)\alpha_\sigma\left[\bigoplus_\tau n_\sigma(\tau) \cdot \tau(\xi^{-1}) \right] d\lambda_\Gamma(\xi) \right) \sigma[(b(x\Gamma))^{-1}] \, d\lambda(x\Gamma) \\
&= \alpha_\sigma\left(\bigoplus_\tau n_\sigma(\tau)\hat{h}(\tau) \right) \int_{G/\Gamma} \sigma(b(x\Gamma)^{-1}) d\lambda(x\Gamma).
\end{aligned}$$

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It follows that we have

$$\|(h^0)^\wedge(\sigma)\|_{\Phi_\infty} \leqq \Big\| \alpha_\sigma\Big(\bigoplus_\tau n_\sigma(\tau)\hat{h}(\tau)\Big)\Big\|_{\Phi_\infty} = \Big\| \bigoplus_\tau n_\sigma(\tau)\hat{h}(\tau)\Big\|_{\Phi_\infty}$$

$$\|(h^0)^\wedge(\sigma)\|_{\Phi_\infty} \leqq \Big\| \alpha_\sigma\Big(\bigoplus_\tau n_\sigma(\tau)\hat{h}(\tau)\Big)\Big\|_{\Phi_\infty}$$

$$= \max\{\|\hat{h}(\tau)\|_{\Phi_\infty} : n_\sigma(\tau) \neq 0\} \leqq \|\hat{h}\|_\infty$$

and hence that $\|(h^0)^\wedge\|_\infty \leqq \|\hat{h}\|_\infty$. This combined with equalities (7) and (8) shows that if $\{h_n\}$ is an RS-sequence on Γ (with the h_n restricted to be continuous—see the opening remarks of this section) then $\{h_n^0\}$ is an RS-sequence on G.

(1.6) Lemma. [M. F. Hutchinson, private communication]. *Let G be a prosolvable group (i.e. a projective limit of finite solvable groups) in which each derived factor $G/G^{(n)}$ is finite. Then G is tall.*

Proof. Since G is profinite it is totally disconnected. Let $\gamma \in \hat{G}$. Then $\gamma(G)$ must be finite since it is a totally disconnected compact Lie group. Furthermore, since $\gamma(G)$ is also prosolvable, it must be solvable. Let d be the degree of γ.

By Zassenhaus (1938) there is a number $l > 0$ depending on d only such that the solvable length of $\gamma(G)$ is at most l. Hence, using the fact that $G/\ker \gamma$ and $\gamma(G)$ are isomorphic, $G^{(l)} \leqq \ker \gamma$.

Now let d be a fixed positive integer. The members of $\{\gamma \in \hat{G} : d_\gamma = d\}$ must all satisfy $G^{(l)} \subseteq \ker \gamma$ where $l = l(d)$ and hence this set corresponds under an obvious injective map to a subset of $(G/G^{(l)})^\wedge$. But the latter set is finite by the hypothesis, and the lemma is proved.

(1.7) Conclusion of the Proof. Let G be an infinite compact group. Then G has an infinite separable compact quotient group [Hewitt & Ross (1963), Theorem (8.7)]. Lemma (1.4) indicates that it is enough then to prove the theorem under the assumption that G is separable.

According to McMullen (1974), G either has an infinite abelian subgroup or an infinite closed topologically-2-generator pro-p torsion subgroup (p an odd prime). In either case, let us call the subgroup in question Γ. In the first case, Γ has an RS-sequence by proposition (1.1). In the second, the same conclusion follows from lemma (1.6) and proposition (1.2).

Since G is separable, the theorem now follows from Lemma (1.5).

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2. Applications

Techniques for applying RS-sequences to problems in harmonic analysis are well-known. For example, see Hewitt & Ross (1970), (37.19), Gaudry (1970), Edwards & Price (1970) and Figà-Talamanca & Price (1972).

Here we sketch the details of three applications.

APPLICATION A. (2.1) In the case of compact groups, the Hausdorff-Young theorem states that $\hat{f} \in \mathfrak{E}^{p'}$ whenever $f \in L^p$, $1 \leqq p \leqq 2$, and $1/p + 1/p' = 1$. Thus if $f \in C(G)$, then $\hat{f} \in \mathfrak{E}^q$ for all $q \in [2, \infty]$. When G is infinite and abelian this is known to be best possible in the sense that there exists $f \in C(G)$ such that $\hat{f}$ belongs to no $\mathfrak{E}^q$ for $q \in [1, 2[$ (see Hewitt & Ross (1973), (37.19(c)) where an extension of this result is given for all locally compact abelian groups).

(2.2) PROPOSITION. *Whenever G is an infinite compact group there exists $f \in C(G)$ such that $\hat{f}$ belongs to no $\mathfrak{E}^q$ for $q \in [1, 2[$.*

The proof will use the following lemma, the proof of which follows directly from the definitions of the $\mathfrak{E}^p$ and their respective norms.

LEMMA. *If $\phi \in \mathfrak{E}^p$, where $1 \leqq p < \infty$, then $\phi \in \mathfrak{E}^q$ for all $p \leqq q \leqq \infty$ and moreover*

$$\| \phi \|_q^q \leqq \| \phi \|_p^p \, \| \phi \|_\infty^{q-p}$$

holds for $p \leqq q < \infty$.

PROOF OF (2.2). Suppose that the statement of the proposition is not valid, that is, that $f \to \hat{f}$ defines a map from $C(G)$ into $\cup \{E^q : 1 \leqq q < 2\}$. Let $\{q_n\}$ be a sequence in $[1, 2[$ which approaches 2 monotonically; then $\cup \{\mathfrak{E}^q : 1 \leqq q < 2\} = \cup \{\mathfrak{E}^{q_n} : n \geqq 1\}$. A direct application of Edwards (1965), Theorem 6.5.5, with $E = \mathfrak{E}^2$, $F = C(G)$, u = Fourier transform, $F_n = \mathfrak{E}^{q_n}$ and u_n the identity map on F_n shows that there exists an integer k such that $C(G)^\wedge \subseteq \mathfrak{E}^{q_k}$. It now follows from the closed graph theorem that for some $K > 0$ we have

$$\| \hat{f} \|_{q_k} \leqq K \| f \|_\infty \tag{9}$$

for all $f \in C(G)$. Let f_n be an RS-sequence consisting of continuous functions. Then there exist m, $M > 0$ such that

$$m \leqq \| f_n \|_2 \leqq \| f_n \|_\infty \leqq M$$

for all $n \geqq 1$. From the preceding lemma, we have

$$\| f_n \|_2^2 = \| \hat{f}_n \|_2^2 \leqq \| \hat{f}_n \|_{q_k}^{q_k} \| \hat{f}_n \|_\infty^{2-q_k}$$

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and so we have

$$\|\hat{f}_n\|_{q_k} \geqq \|f_n\|_2^{2/q_k} / \|\hat{f}_n\|_\infty^{(2-q_k)/q_k}$$

$$\geqq m^{2/q_k} / \|\hat{f}_n\|_\infty^{(2-q_k)/q_k} \to \infty \quad \text{as} \quad n \to \infty$$

since $1 \leqq q_k < 2$. But this contradicts (9), in view of the fact that $\|f_n\|_\infty \leqq M$, for $n \geqq 1$.

Application B. (2.3) The Fourier transform $f \to \hat{f}$ carries M into $\mathfrak{E}^\infty$, L^1 into $\mathfrak{E}_0$ and L^p into $\mathfrak{E}^{p'}$ when $1 < p < 2$. It is known that these maps are surjective if and only if G is finite [Hewitt & Ross (1970), (37.4) and (37.19 (a))]; a direct proof of these facts follows from the existence of an RS-sequence. The surjectivity of the maps is trivial when G is finite.

(2.4) Proposition. *Let G be an infinite compact group. The images of $M(G)$, $L^1(G)$ and $L^p(G)$ $(1 < p < 2)$ under the Fourier transform are properly contained in $\mathfrak{E}^\infty$, $\mathfrak{E}_0$, and $\mathfrak{E}^{p'}$ respectively.*

Proof. The proofs are similar in detail to the second part of the proof of the proposition (2.2): one assumes the contrary, establishes an inequality analogous to (9), and obtains a contradiction by substituting therein the members of an RS-sequence.

Application C. (2.5) Given p, $q \in [1, \infty]$, then $\phi \in \mathfrak{E}$ is said to be a (p, q)-multiplier if

$$\sum_{\gamma \in \hat{G}} d_\gamma tr[\phi(\gamma)\hat{f}(\gamma)\gamma(\cdot)]$$

is the Fourier series of a function in L^q whenever $f \in L^p$ (an equivalent definition is available which makes sense for arbitrary locally compact groups). For example, it is well-known that $\hat{\mu}$ is a (p, p)-multiplier for all $p \in [1, \infty]$ whenever $\mu \in M(G)$. On the other hand, when G is an infinite compact abelian group there exist functions in $\mathfrak{E}$ which are (p, q)-multipliers for all $p \in]1, \infty]$ and all $q \in [1, \infty[$, which are *not* Fourier-Stieltjes transforms; see Brainerd & Edwards (1966), Theorem (4.15). The proof is based upon the existence of an infinite Sidon set. In fact, given the existence of certain lacunary subsets of $\hat{G}$, examples of functions in $\mathfrak{E}$ with the preceding properties can easily be produced; c.f. (37.22) of Hewitt & Ross (1970). However, there exist compact groups whose duals possess no reasonable lacunary sets. Suppose that a locally compact group has the property of possessing an RS-sequence of functions whose supports are contained in a fixed compact set. Then Theorem 5.7 of Edwards & Price (1970) shows that for such groups there exists a (p, q)-multiplier for all pairs (p, q) such that

$1 < p \leqq q < \infty$, but which is not a Fourier-Stieltjes transform. Since we have seen that every infinite compact group has an RS-sequence we have:

(2.6) PROPOSITION. *Let G be an infinite compact group. There exists a function in $\mathfrak{E}$ which is a (p,q)-multiplier for all pairs (p,q) such that $1 < p \leqq q < \infty$, but is not the Fourier-Stieltjes transform of any measure.*

(2.7) REMARKS (i). Proposition (2.6) improves Theorem 4.3 of Figà-Talamanca & Price (1972), the proof of which was based on the existence of t-RS-sequences, $t > \infty$, as defined above. As a consequence of the existence of these restricted RS-sequences, roughly all that could be shown was that there exist multipliers of the type under question which are not Fourier transforms of any element in $\cup\{L^r : 1 < r \leqq \infty\}$.

(ii) Further cases of proposition (2.6) for infinite compact groups are accounted for by noting that all functions in $\mathfrak{E}^\infty$ are (p,q)-multipliers when $1 \leqq q \leqq 2 \leqq p \leqq \infty$ (Table (36.20) of Hewitt & Ross (1970)), whereas by proposition (2.4) there exist elements in $\mathfrak{E}^\infty$ which are not Fourier-Stieltjes transforms.

(iii) In the event of the existence of a U-RS-sequence where U is some open subset of G (see propositions (1.1) and (1.2)), application A can be strengthened to show that there exists $f \in C(G)$ with support in $\bar{U}$ such that $\hat{f}$ belong to no $\mathfrak{E}^q$ with $q \in [1, 2[$. Applications B and C can be improved in a like manner.

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