SOME SIMPLE PROPERTIES OF SIMPLE NIL RINGS

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An outstanding unsolved problem in the theory of rings is the existence or non-existence of a simple nil ring. Such a ring cannot be locally nilpotent as is well known [5]. Hence, if a simple nil ring were to exist, it would follow that there exists a finitely generated nil ring which is not nilpotent. This seemed an unlikely situation until the appearance of Golod's paper [1] where a finitely generated, non-nilpotent ring is constructed for any $d \geq 2$ generators over any field.

In this paper we prove a few elementary properties of simple nil rings other than non-local-nilpotency.

LEMMA 1. A non-nilpotent ring R is a simple ring if and only if $RxR = R$ for every non-zero x in R.

Proof. Suppose $RxR = R$ for every non-zero x in R. Let A be a non-zero ideal of R. Let x be non-zero in A. Then $R = RxR \subseteq A$. Hence $A = R$ and R is simple.

Conversely, let R be simple. Let $A = \{x \text{ in } R \mid RxR = 0\}$. A is an ideal of R. Hence $A = 0$ or $A = R$. But $A = R$ implies $R^3 = 0$, contrary to hypothesis, so it follows that $A = 0$. This means that for every non-zero x in R we have that RxR is a non-zero ideal of R, which by the simplicity of R, is all of R. Hence the lemma is proved.

THEOREM 1. Let P be a property of rings such that every homomorphic image of a P-ring is also a P-ring. Then, if there exists a simple ring R which contains a non-nilpotent P-subring of the form xRy, then there exists a simple P-ring.

Proof. Let $A = xRy$ be a non-nilpotent P-ring. Then $yx \neq 0$ since A is not nilpotent. Hence, by Lemma 1, we have $(xRy)^2 = x(RyxR)y = xRy$. Define $B = \{z \text{ in } A \mid yzx = 0\}$.

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Then B is a proper ideal of A. For, if u is arbitrary in A, then $u = xu'y$ for some u' in R. Hence for any z in B we have $0 = yzx = yzxu'yx = y(zu)x$ and $0 = yzx = yxu'yzx = y(uz)x$. Hence zu and uz are in B. That B is a proper ideal of A follows from the fact that $B^2 = 0$ and $A^2 = A \neq 0$. We claim that A/B is a non-nilpotent simple P-ring. First, since A is not nilpotent and B is, A/B is not nilpotent. A/B is a P-ring by our choice of P. Finally, our choice of B and Lemma 1 establish that A/B is simple.

COROLLARY 1. If there exists a simple ring R containing a non-nilpotent nil right or left ideal or a non-nilpotent nil subring of the form xRy, then there exists a non-nilpotent, simple nil ring.

Proof. In Theorem 1 take P as the property of being nil and note that a non-nilpotent right or left ideal of R contains a non-nilpotent nil subring of the form xRy.

The above corollary suggests that in order to obtain the result for one-sided ideals, the particular property P employed requires that the property be inherited to sub-rings. However, this is not the case, as Theorem 1 could have been proved in much the same way using directly the assumption that R contain a non-nilpotent left or right P-ideal.

In order to prove Theorem 2 we need the following property of non-nilpotent, simple nil rings.

LEMMA 2. If R is a non-nilpotent, simple nil ring, then there exist elements x and y of R such that neither $xR \subseteq yR$ nor $yR \subseteq xR$.

Proof. Suppose to the contrary that for all s and t in R either $sR \subseteq tR$ or $tR \subseteq sR$. Let s and t be fixed but arbitrary in R. Suppose that $sR \not\subseteq tR$. Then $s \neq 0$. Hence there exists a positive integer k such that $s^{k+1}R \subseteq tR \subseteq s^kR$ because s is nilpotent. Thus $stR \subseteq s^{k+1}R \subseteq tR$. This means that for all s and t in R either $sR \subseteq tR$ or $stR \subseteq tR$. This means tR is an ideal of R for every t in R. Hence $tR = 0$ or $tR = R$. Both possibilities are impossible since R is nil and not nilpotent. Thus the lemma is proved.

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THEOREM 2. If there exists a non-nilpotent, simple nil ring R, then there exists one which is the sum of two proper right (left) ideals.

Proof. By Lemma 2 there exist x and y in R such that $xR \not\subseteq yR$ and $yR \not\subseteq xR$. Let $A = xR + yR$. We prove that an appropriate factor ring of A is the desired ring. Define $B = \{z \text{ in } A \mid zx = zy = 0\}$. Clearly B is an ideal of A and $B^2 = 0$. Further, $A^2 = A$ because xR and yR are idempotent, non-zero right ideals of R. Hence B is a proper ideal of A and A/B is idempotent. We prove A/B is simple. Let $xr_1 + yr_2$ be in A and not in B. Then $(xr_1 + yr_2)x \neq 0$ or $(xr_1 + yr_2)y \neq 0$. Suppose the former. Then

$$A(xr_1 + yr_2)A = (xR + yR)(xr_1 + yr_2)(xR + yR) =$$

$$= xR(xr_1 + yr_2)xR + yR(xr_1 + yr_2)xR + (\ldots)yR =$$

$$= xR + yR + (\ldots)yR = A + (\ldots)yR = A.$$

Similarly, if $(xr_1 + yr_2)y \neq 0$, then $A(xr_1 + yr_2)A = A$. Hence, by Lemma 1, it follows that A/B is simple. It remains to show that xR/B and yR/B are proper right ideals of A/B. It suffices to show that $xR/B \not\subseteq yR/B$ and $yR/B \not\subseteq xR/B$. Suppose $xR/B \subseteq yR/B$. Then for all r in R, there exists an r' in R such that $xr \equiv yr' \pmod{B}$. Hence $(xr - yr')x = (xr - yr')y = 0$. Thus $xRx \subseteq yRx$, and so $xR = xRxR \subseteq yRxR = yR$, contrary to our choice of x and y. Similarly, $yR/B \not\subseteq xR/B$. This completes the proof of the theorem.

Kegel [3] has investigated rings which are sums of two of its subrings. Although he does consider the question of simplicity for such rings, it is not clear how his results could be applied to the ring constructed in Theorem 2.

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