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COMPUTATIONAL COMPLEXITY OF TOPOLOGICAL INVARIANTS

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Abstract We answer the following question posed by Lechuga: given a simply connected space X with both $H_*(X;\mathbb{Q})$ and $\pi_*(X)\otimes\mathbb{Q}$ being finite dimensional, what is the computational complexity of an algorithm computing the cup length and the rational Lusternik–Schnirelmann category of X?

Basically, by a reduction from the decision problem of whether a given graph is k-colourable for $k\geqslant 3$, we show that even stricter versions of the problems above are NP-hard.

Keywords: computational complexity; cup length; Lusternik–Schnirelmann category; pure elliptic space

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1. Introduction

The theory of computational complexity has developed a powerful machinery to describe how 'difficult', i.e. how time-consuming, it is to answer certain posed questions algorithmically. Classically, this requires the following categorization of problems: the complexity class P describes all the problems for which there is a polynomial time-solving algorithm; the class NP is formed by those problems that may at least be verified in polynomial time. Clearly, $\mathrm{P}\subseteq\mathrm{NP}$. However, it is commonly believed that several problems in NP are much harder to solve than the problems in P. Known algorithms typically run at exponential costs.

On the other hand, a problem in NP is said to be *NP-complete* if any other problem in NP can be reduced to it in polynomial time. Finally, in increasing order of difficulty, a problem not necessarily in NP is NP-hard if, again, any problem in NP can be reduced to it in polynomial time. The *graph colouring problem* is a classical example of an NP-complete problem. (For an introduction to the subject see [**7**].)

In the field of algebraic topology it is easy to imagine several problems for which it seems difficult to find efficient solving algorithms. In particular, rational homotopy

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theory has the appeal of providing 'computable problems', which certainly ask for algorithmic treatment. Indeed, rational homotopy theory permits a categorical translation from topology/homotopy theory to algebra at the expense of losing torsion information. Yet, it turns out that the algebraic side now allows for concrete calculations. (As a reference to rational homotopy theory we recommend the textbook [**2**].)

Using this approach, several topological problems were shown to be NP-hard. In [**1**] it is shown that computing the rational homotopy groups $\pi_*(X) \otimes \mathbb{Q}$ of a simply connected CW complex X is NP-hard. So is the problem of whether a simply connected space X with $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$ also has finite-dimensional rational cohomology (see [**6**, Theorem 1, p. 90]). In [**1**] it was also shown that, for formal spaces, i.e. for spaces for which the rational homotopy type can be formally derived from the rational cohomology algebra, the computations of Betti numbers, cup length and the rational Lusternik–Schnirelmann category are NP-hard problems. In [**4**] it is shown that the computation of Betti numbers of a simply connected space with both finite-dimensional rational homotopy and finite-dimensional rational homotopy, i.e. a *rationally elliptic space*, is NP-hard.

However, in [**5**], and explicitly in [**4**], the following question is posed.

Question (Lechuga). Given an elliptic space, what is the computational complexity of computing its rational cup length or its rational Lusternik–Schnirelmann category?

In this paper we shall answer Lechuga's question by revealing these problems as NP-hard. We first do so for the same question posed on the subclass of pure elliptic spaces. For this we specify the following problems.

($\mathcal{P}$) Let X be a (simply connected) rationally elliptic CW complex. What is its cup length?

($\mathcal{Q}$) Let X be a (simply connected) rationally elliptic CW complex. What is its rational Lusternik–Schnirelmann category?

We are interested in the computational complexity of the problems $\mathcal{P}$ and $\mathcal{Q}$ and we prove the following.

Theorem A. *The problems $\mathcal{P}$ and $\mathcal{Q}$ are NP-hard.*

The rational cohomology algebra of a (simply connected) rationally elliptic space satisfies Poincaré duality, which implies that the rational Toomer invariant equals the rational Lusternik–Schnirelmann category. Thus, its computation is also NP-hard.

2. Background and proofs

The codification of a simply connected space X will be given as the data contained in its *minimal Sullivan model* $(\Lambda V_X, \mathrm{d})$, i.e. X will be represented by the degrees of the homogeneous generators $x_1, \ldots, x_l$ of V_X and the coefficients of the polynomials in the x_i that represent the differential.

Recall that a minimal Sullivan model of a simply connected space is a free (graded) commutative graded algebra ΛV over the $\mathbb{Z}$-graded rational vector space $V = V^{\geqslant 2}$

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together with a differential d defined by $\mathrm{d}\colon V^* \to (\Lambda V)^{*+1}$ and extended to ΛV as a derivation. The differential satisfies that its image lies in the subalgebra of elements of word length at least 2 in V, i.e. $\operatorname{im}\mathrm{d} \subseteq \Lambda^{\geqslant 2}V$. One then requires the existence of a quasi-isomorphism $(\Lambda V, \mathrm{d}) \to \mathrm{A}_{\mathrm{PL}}(X)$, i.e. a morphism of differential graded algebras to the polynomial differential forms $\mathrm{A}_{\mathrm{PL}}(X)$ on X inducing an isomorphism on homology. Thus, $(\Lambda V, \mathrm{d})$ encodes the rational homotopy type of X. (See [**2**, Chapters 3, 10 and 12] for the missing definitions.) In particular, the homology algebra $H(\Lambda V, \mathrm{d})$ of the minimal model is the rational cohomology algebra of X.

Thus, the *rational cup length* $c_0(X)$ of X, i.e. the smallest number n (possibly infinite) for which any cup product of length $n+1$ in $H^*(X;\mathbb{Q})$ vanishes, coincides with the cup length $c(\Lambda V, \mathrm{d})$ of $(\Lambda V, \mathrm{d})$. Recall that the *Lusternik–Schnirelmann category* $\operatorname{cat} X$ of a space X is the least number n (possibly infinite) such that X can be covered by $n+1$ open sets that are contractible in X. The *rational Lusternik–Schnirelmann category* (at least on a simply connected CW complex X) agrees with the category of its rationalization $X_{\mathbb{Q}}$. It coincides with the category $\operatorname{cat}(\Lambda V, \mathrm{d})$ (see [**2**, Part V] for a detailed discussion of these invariants).

Thus, problems $\mathcal{P}$ and $\mathcal{Q}$ translate to the problems $\mathcal{P}'$ and $\mathcal{Q}'$ of determining the cup length and the Lusternik–Schnirelmann category, respectively, of a (simply connected) elliptic Sullivan algebra. Moreover, Theorem A is equivalent to the following.

Theorem 2.1. *The problems $\mathcal{P}'$ and $\mathcal{Q}'$ are NP-hard.*

We shall now head towards a proof of Theorem 2.1. To this end, recall that a Sullivan algebra $(\Lambda V, \mathrm{d})$ is *pure* if $V = P \oplus Q$ with $Q = V^{\mathrm{even}}$ and $P = V^{\mathrm{odd}}$ and if the differential d satisfies

$$\mathrm{d}|_Q = 0 \quad \text{and} \quad \mathrm{d}(P) \in \Lambda Q.$$

Classical examples of spaces admitting pure models are biquotients and their particular respective subclass of homogeneous spaces.

Remark 2.2. In order to answer the original question by Lechuga, we shall actually determine the computational complexities of the following stricter problems, i.e. we shall show that they are NP-hard.

> Let $(\Lambda V, \mathrm{d})$ be a (simply connected) pure elliptic Sullivan algebra. What is its cup length? What is its rational Lusternik–Schnirelmann category?

Consider a (simply connected) pure minimal Sullivan algebra $(\Lambda V, \mathrm{d})$ (with finite-dimensional V) and denote by $\{x_1, \dots, x_r\}$ a basis of V^{even}, and by $\{y_1, \dots, y_s\}$ a basis of V^{odd}. If this algebra is elliptic, then

$$N := \sum_{j=1}^{s} \deg y_j - \sum_{i=1}^{r} (\deg x_i - 1) \tag{2.1}$$

equals the largest integer N' for which $H^{N'}(\Lambda V, \mathrm{d}) \neq 0$ (see [**2**, Chapter 32, p. 434]).

Form the graded vector space $Z = \langle z_1, \dots, z_r\rangle$ with $\deg z_i = m \cdot \deg x_i - 1$ with some $m > \frac{1}{2}N + r + 1$ and consider the minimal Sullivan algebra

$$(\Lambda V \otimes \Lambda Z, \mathrm{D}) \tag{2.2}$$

extending $(\Lambda V, \mathrm{d})$ by $\mathrm{D}z_i = x_i^m$.

Proposition 2.3. *The following assertions are equivalent:*

(i) *the algebra* $(\Lambda V, \mathrm{d})$ *is elliptic;*

(ii) *it holds that* $\mathrm{cat}(\Lambda V \otimes \Lambda Z, \mathrm{D}) \leqslant \frac{1}{2}N + r$*;*

(iii) *it holds that* $\mathrm{c}(\Lambda V \otimes \Lambda Z, \mathrm{D}) \leqslant \frac{1}{2}N + r$.

Proof. If $(\Lambda V, \mathrm{d})$ is elliptic, any cohomology class above its formal dimension N vanishes. In particular, we obtain $[x_i]^m = 0$ for $1 \leqslant i \leqslant r$, i.e. $x_i^m = \mathrm{d}\Psi_i$ for some $\Psi_i \in \Lambda V$. Replacing z_i by $z_i - \Psi_i$, the relative algebra splits as

$$(\Lambda V \otimes \Lambda Z, \mathrm{D}) \cong (\Lambda V, \mathrm{d}) \otimes (\Lambda Z, 0).$$

Due to [**2**, Theorem 30.2 (ii)] we compute the category of the tensor product as the sum of the categories of the factors. The category of the second factor is n, as it is a formal space of cup length n (see [**2**, Example 29.4, p. 388]). The category of the first factor is restricted from above by $\frac{1}{2}N$ due to [**2**, Proposition 27.5, p. 354]. Thus, we compute

$$\mathrm{cat}(\Lambda V \otimes \Lambda Z, \mathrm{D}) = \mathrm{cat}(\Lambda V, \mathrm{d}) + \mathrm{cat}(\Lambda Z, 0) \leqslant \tfrac{1}{2}N + r.$$

The implication '(ii) $\Rightarrow$ (iii)' follows directly from [**2**, p. 352].

In the case when $\mathrm{c}(\Lambda V \otimes \Lambda Z, \mathrm{D}) \leqslant \frac{1}{2}N + r$, the ellipticity of $(\Lambda V, \mathrm{d})$ can be deduced as follows: we may assume the x_i (respectively, the z_i) to be sorted by degree, starting with the smallest one. We have that $[x_i]^{m-1} = 0$ in $H(\Lambda V \otimes \Lambda Z, \mathrm{D})$, i.e. $x_i^{m-1} = \mathrm{D}\Psi_i$ for each $1 \leqslant i \leqslant r$. Via an inductive process we shall now show that all the Ψ_i may be chosen from ΛV: by a degree argument we have that $\Psi_1 \in \Lambda V$. Assume now that $x_l^{m-1} = \mathrm{d}\Psi_l$ with $\Psi_l \in \Lambda V$ for $1 \leqslant l \leqslant i-1$. Replacing z_l by $z_l - \Psi_l$, we obtain

$$(\Lambda V \otimes \Lambda Z, \mathrm{D}) \cong (\Lambda V \otimes \Lambda\langle z_i, \dots, z_r\rangle, \mathrm{D}) \otimes (\Lambda\langle z_1, \dots, z_{i-1}\rangle, 0).$$

We then repeat the degree argument to deduce that Φ_i may be chosen from ΛV. It follows that there are only finitely many non-trivial powers of the $[x_i] \in H(\Lambda V, \mathrm{d})$. By [**2**, Proposition 32.1] this is equivalent to $H(\Lambda V, \mathrm{d})$ being finite dimensional, i.e. to $(\Lambda V, \mathrm{d})$ being elliptic. □

Proof of Theorem 2.1. Let $G = (V, E)$ be a (non-directed) finite connected simple graph with vertices $V = \{v_1, \dots, v_n\}$ and edges $E = \{(v_i, v_j) \mid (i, j) \in J\}$ for some index set J. Formulating the graph k-colouring problem, one tends to encode the graph via its vertices and an adjacency matrix. The length of this encoding is bounded polynomially in the number of vertices (see [**6**, p. 92]). Following [**6**, p. 91] we associate to G and a given integer $k \geqslant 2$ a finitely generated simply connected pure Sullivan algebra by

$$V_{G,k}^{\text{even}} = \langle x_1, \dots, x_n \rangle \quad \text{with } \deg x_i = 2 \text{ and } \mathrm{d}x_i = 0$$

for all $1 \leqslant i \leqslant n$, by

$$V_{G,k}^{\text{odd}} = \langle y_{i,j} \rangle_{(i,j) \in J} \quad \text{with } \deg y_{i,j} = 2k - 3$$

and by

$$\mathrm{d}y_{i,j} = \sum_{l=1}^{k} x_r^{k-1} x_s^{l-1} \quad \text{for all } (i,j) \in J.$$

It is minimal for $k \geqslant 3$. Again the length of this codification is polynomial in n, the number of vertices (see [**6**, p. 91]). We form the Sullivan algebra

$$(\Lambda V_{G,k} \otimes \Lambda Z, \mathrm{D})$$

constructed in analogy to (2.2), i.e. with $Z = \langle z_1, \dots, z_n \rangle$ and $\mathrm{d}z_i = x_i^m$ for $m > \frac{1}{2}N + n + 1$, where

$$N = \sum_{(i,j) \in J}^{s} \deg y_{i,j} - \sum_{i=1}^{n} (\deg x_i - 1).$$

Adding this additional data to the codification can equally be done in polynomial time.

The problem of whether a graph G is k-colourable for $k \geqslant 3$ is known to be NP-complete (see, for example, [**3**]). In the proof of [**6**, Corollary 4, p. 92], it is shown that there is a polynomial reduction of this problem to that of whether the Sullivan algebra $(\Lambda V_{G,k}, \mathrm{d})$ from above is elliptic. Indeed, G is k-colourable if and only if $(\Lambda V_{G,k}, \mathrm{d})$ is not elliptic. Due to Proposition 2.3, this is equivalent to the assertion that the Sullivan algebra $(\Lambda V_{G,k} \otimes \Lambda Z, \mathrm{D})$ has both Lusternik–Schnirelmann category and cup length greater than $\frac{1}{2}N + n$. In other words, the problem of deciding whether a simply connected pure algebra, in particular one of the form $(\Lambda V_{G,k} \otimes \Lambda Z, \mathrm{D})$ from above, has Lusternik–Schnirelmann category (respectively, cup length) smaller than a certain number admits a polynomial reduction of the graph k-colouring problem. Therefore, the problems $\mathcal{P}'$ and $\mathcal{Q}$' are NP-hard. □

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