SEMIGROUPS OF OPERATORS IN $C(S)$

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1. Our study in this paper is two-fold: One is that of a semigroup of linear operators on the space $C(S)$ of bounded continuous functions on a locally compact Hausdorff space S, while the other is that of a transition function of measures in the Banach space $M(S)$ of bounded regular Borel measures on S. It will be seen that an informative and essentially non-restrictive theory of the former can be obtained when $C(S)$ is given the strict topology rather than the usual supremum norm topology and that, in this setting, the natural relationship between semigroups and transition functions obtained when S is compact is maintained, essentially because the dual of $C(S)$ with the strict topology is $M(S)$.

More specifically, we determine conditions under which the general theory of semigroups in locally convex spaces applies to our situation and establish a one-to-one relationship between semigroups on $C(S)$ and certain transition functions in $M(S)$. We show, furthermore, that such semigroups are uniquely determined by the infinitesimal generator of the semigroup of adjoint operators in the Banach space $M(S)$.

2. Definitions and notation. Our notation and terminology is that of [**10**] with a few exceptions which we now point out. If T is a topological space, then a mapping $\lambda: T \to M(S)$ will be called a continuous kernel if for each $f \in C(S)$ and $x \in T$ the function $\lambda(f)(x) = \int_S f(y)\lambda(x)\,(dy) = \int_S f(y)\,\lambda(x, dy)$ is bounded and continuous on T. From the uniform boundedness principle, $\|\lambda\| = \sup\{\|\lambda(x)\|: x \in T\} < \infty$. Departing from our terminology in [**10**] we say that the mapping λ is *tight* if for each compact subset Q of T, $\{\lambda(x): x \in Q\}$ is an equicontinuous subset of $M(S)$ as the dual of $C(S)_\beta$ (i.e., $C(S)$ with the strict topology). According to [**2**, Theorem 2.2], this is equivalent to the statement that for each $\epsilon > 0$ there is a compact subset K of S such that $|\lambda(x)|(S\backslash K) = |\lambda|(x, S\backslash K) < \epsilon$ for all $x \in Q$. Furthermore, we note in reference to [**3**; **10**] that it has been shown in [**4**] that the β and β' topologies are equivalent topologies on $C(S)$. We also note from [**1**] that a collection B of operators in $C(S)_\beta$ is equicontinuous if given $\psi \in C_0(S)$ there is a $\phi \in C_0(S)$ such that $\|\phi f\| \leqq 1$ implies $\|\psi Tf\| \leqq 1$ for all $T \in B$.

A semigroup of operators on a locally convex space X is a collection $\{T_t: t > 0\}$ of linear operators on X into itself such that $T_t T_s = T_{t+s}$ for all $t, s > 0$. A transition function on S is a collection $\{\lambda_t: t > 0\}$ of kernels on S

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into $M(S)$ such that $\lambda_{t+s}(x, E) = \int_S \lambda_t(y, E)\lambda_s(x, dy)$ for all $t, s > 0$, $x \in S$, and Borel sets E. These integrals exist according to [**10**]. All terminology related to semigroups is taken from [**7**; **12**].

3. Preliminary results. In this section we restate, for ease of reference, results already contained in [**10**].

THEOREM 3.1. (a) *If A is a continuous operator on $C(S)_\beta$ into $C(T)_\beta$, then there is a unique tight continuous kernel $\lambda: T \to M(S)$ such that $Af = \lambda(f)$ for all $f \in C(S)$, $A^*\nu = \lambda(\nu)$ for $\nu \in M(S)$ (where $\lambda(\nu)(E) = \int_T \lambda(x, E)\nu(dx)$) and $A^{**}f = \lambda(f)$ for any bounded Borel measurable function on S.*

(b) *If $\lambda: T \to M(S)$ is a tight continuous kernel, then the formula $Af = \lambda(f)$ defines a continuous operator on $C(S)_\beta$ into $C(T)_\beta$.*

(c) *If S is a paracompact space, then any continuous kernel $\lambda: T \to M(S)$ is tight.*

We note that the hypothesis in (b) can be weakened. If λ is tight and $\lambda(f) \in C(T)$ for $f \in C_0(S)$, then by [**10**, Theorem 4], λ is a continuous kernel. Finally, Theorem 3.1 implies that $\sup\{||Af||: f \in C(S), ||f|| \leqq 1\} = ||\lambda|| < \infty$. We set $||A|| = ||\lambda||$ so that $||A||$ is the usual operator norm on A on $C(S)$.

Applying Theorem 3.1 to semigroups on $C(S)$, we obtain the following representation and uniqueness theorem.

THEOREM 3.2. (a) *If $\{T_t: t > 0\}$ is a semigroup of continuous operators on $C(S)_\beta$, then there is a unique transition function $\{\lambda_t: t > 0\}$ of tight continuous kernels such that $T_t f = \lambda_t(f)$ for all $f \in C(S)$.*

(b) *Conversely, any transition function $\{\lambda_t: t > 0\}$ of tight continuous kernels gives rise to a semigroup of continuous operators on $C(S)_\beta$ by way of the formula $T_t f = \lambda_t(f)$.*

For a proof, apply Theorem 3.1 to each operator T_t and note that

$$\lambda_{t+s}(x, E) = (T_{t+s}^* \tilde{x})(E) = T_t^*(T_s^* \tilde{x})(E) = \lambda_t(\lambda_s(\tilde{x}))(E) = \int_S \lambda_t(y, E)\, \lambda_s(x, dy),$$

where $\tilde{x}$ is the unit point measure concentrated at x.

We single out for special attention the following result.

THEOREM 3.3. *If S is paracompact and $\{\lambda_t: t > 0\}$ is a transition function of continuous kernels, then $T_t f = \lambda_t(f)$ is a semigroup of continuous linear operators in $C(S)_\beta$.*

In particular then, if S is paracompact and if the semigroup arises from a transition function, as in the study of Markov processes, and leaves $C(S)$ invariant, then it is *a fortiori* a semigroup of continuous operators in $C(S)_\beta$. Hence for most transition functions considered in the literature, a study of the induced semigroup in $C(S)_\beta$ is natural and non-restrictive.

One can also relate semigroups in $C(S)_\beta$ with Markov processes in S. For with the extension of Kolmogorov's work on extending probabilities found in

[**8**, p. 82], it follows that each transition function in $M(S)$, with $||\lambda_t|| = 1$ and $\lambda_t(x)$ non-negative for all t and x, gives rise to a Markov process, since we are dealing with regular measures. Hence a semigroup of positive strictly continuous operators of norm one must arise from a Markov process.

4. Semigroups of class (C_0) in $C(S)_\beta$. In this section we impose the "usual" conditions (that is, conditions analogous to those assumed when S is compact or when one studies semigroups in the Banach space $C_0(S)$) on the semigroup $\{T_t: t > 0\}$ of continuous operators in $C(S)_\beta$ and relate this situation to the general theory found in [**12**; **7**].

More specifically, we will say that the semigroup $\{T_t: t > 0\}$ in $C(S)$ is of class (C) if:

(1) $\lim_{t\to 0} T_t f(x) = f(x)$ for each $x \in S$ and each $f \in C_0(S)$,

(2) there exists a number $a > 0$ such that $\{T_t: t \leqq a\}$ is equicontinuous in $C(S)_\beta$.

We note that Yosida [**12**] defines a semigroup on a sequentially complete locally convex space to be of class (C_0) if $\lim_{t\to t_0} T_t x = T_{t_0} x$ for all $t_0 \geqq 0$ and all $x \in X$ and if $\{T_t: t > 0\}$ is equicontinuous, where T_0 is the identity operator. In the notation of [**7**], a class (C_0) semigroup is referred to as a (τ, τ)-semigroup. Note by Theorem 3.2 (a), that any class (C) semigroup is given by a unique transition function.

It is evident that a class (C) semigroup need not be class (C_0), the translation semigroup $[T_t f](x) = f(t + x)$ in $C[0, \infty)$ being a simple example of this.

We begin with a generalization of a result found in [**6**, p. 36].

THEOREM 4.1. *Let X be a locally convex space and let $\{T_t: t > 0\}$ be a semigroup of linear operators in X such that $\{T_t: t \leqq a\}$ is equicontinuous for some $a > 0$. Let $X_0 = \{x \in X: T_t x \to x$ as $t \to 0\}$, $X_1 = \{x \in X: T_t x \to x$ weakly as $t \to 0\}$. Then, $X_0 = X$ if and only if $X_1 = X$.*

Proof. Clearly we need only show that $X_1 = X$ implies $X_0 = X$.

Fix $x^* \in X^*$ and let $f(t) = \langle T_t x, x^*\rangle$ for a fixed $x \in X_1$. Then f is continuous from the right at each number $t > 0$, and consequently the function $t \to T_t x$ is weakly measurable for each $x \in X = X_1$.

We claim that the values of this function $t \to T_t x$ lie in a separable subspace of X save for all t in a set of Lebesgue measure 0. Let L be the closure of $\{\sum_{i=1}^n a_i T_{r_i} x$: r_i is rational and a_i is a complex number with rational real and imaginary parts$\}$. Hence L is a closed, separable subspace of X. If $T_t x \notin L$ for some $t > 0$, then there is an $x^* \in X^*$ such that $\langle T_t x, x^*\rangle = 1$ while $\langle y, x^*\rangle = 0$ for all $y \in L$, a clear contradiction.

It now follows from [**11**, Remark 1] and the hypothesis of equicontinuity that $t \to T_t x$ is continuous in X at all points $t > 0$. Hence $X_0 \supset \{T_t x: t > 0\}$. Furthermore, X_0 is a closed subspace of X and consequently by an argument similar to the above, $X = X_0$.

Our principal result is the following.

THEOREM 4.2. *Let* $\{T_t\colon t > 0\}$ *be a semigroup of operators in* $C(S)_\beta$ *with transition function* $\{\lambda_t\colon t > 0\}$. *Let* T_0 *denote the identity operator on* $C(S)$ *and let* λ_0 *be its kernel. Finally, for* $a > 0$ *let* $W_a = [0, a] \times S$. *The following are equivalent:*

(a) $\{T_t\colon t > 0\}$ *is of class C;*

(b) *The mapping* $\mu\colon W_a \to M(S)$ *given by* $\mu(t, x) = \lambda_t(x)$ *is tight and* $\lim_{t\to 0} T_t f = f$ *in the strict topology for all* $f \in C(S)$;

(c) μ *is a tight continuous kernel;*

(d) μ *is tight and* $\mu(f) \in C(W_a)$ *for all* $f \in C_0(S)$;

(e) $[Bf](t, x) = [T_t f](x)$ *is a continuous linear operator on* $C(S)_\beta$ *into* $C(W_a)_\beta$.

When S is paracompact, these are equivalent to:

(c′) μ *is a continuous kernel; i.e., the mapping* $(t, x) \to [T_t f](x)$ *is a bounded continuous function on* W_a *for each* $f \in C(S)$.

Proof. Suppose that (a) holds. By the semigroup property, $\{T_t\colon t \leqq a\}$ is equicontinuous for any $a > 0$. If $Q \subset [0, a] \times K$ with K compact in S, let $\phi \in C_0(S)$, $\phi \equiv 1$ on K. Then there is a β-neighbourhood V of zero such that $||\phi T_t f|| \leqq 1$ for all $f \in V$ and $t \leqq a$. Hence if $(t, x) \in Q$, then $|\mu(f)(t, x)| = |\lambda_t(f)(x)| = |\phi(x)[T_t f](x)| \leqq 1$ for all $f \in V$. Consequently, μ is tight. To complete the proof that (b) follows from (a), fix $x \in S$ and let $\nu(t, E) = \mu((t, x), E)$. Then ν is tight and by (a), $\nu(f)$ is continuous at 0 for all $f \in C_0(S)$. It follows from the remark following Theorem 3.1 that $\nu(f)$ is continuous at 0 for all $f \in C(S)$. Hence $[T_t f](x) \to f(x)$ for all $f \in C(S)$ and $x \in S$ as $t \to 0$. Since $\{T_t f\colon t \leqq a\}$ is bounded in the supremum norm, $T_t f \to f$ weakly as $t \to 0$ for each $f \in C(S)_\beta$. Applying Theorem 4.1, we obtain (b).

To see that (b) implies (c), note first that by (b) and the semigroup property, $\lim_{t\to t_0^+} T_t f = T_{t_0} f$ uniformly on compact subsets of S. Let K be a compact subset of S. Then by (b) there is a β-neighbourhood V of 0 such that if $g \in V$, then $|\mu(g)(t, x)| < \epsilon$ for all $(t, x) \in [0, a] \times K$. Furthermore, there is a $\delta > 0$ such that

$$T_{t_0 - t} f - f \in V \quad \text{for } 0 < t_0 - t < \delta.$$

Hence if $x \in K$ and $0 < t_0 - t < \delta$, then

$$|[T_t f](x) - [T_{t_0} f](x)| = |T_t[f - T_{t_0 - t} f](x)| = |\mu(f - T_{t_0 - t} f)(t, x)| < \epsilon.$$

Therefore, $\lim_{t\to t_0^-} T_t f = T_{t_0} f$ uniformly on compact subsets of S. Since S is locally compact, this proves (c).

Clearly (c) implies (d). By Theorem 3.1(b) along with the remark following it, (d) implies (e). Finally, (e) implies (a) since continuity of Bf at $(0, x)$ implies that $T_t f \to f$ pointwise on S while continuity of B implies that $\{T_t\colon t \leqq a\}$ is equicontinuous.

To complete the proof, note that Theorem 3.1(c) yields the equivalence of (c) and (c′).

Before making use of these results, we need the following lemma.

LEMMA 4.3. *Let* $\{T_t\colon t \geqq 0\}$ *be a class* (C) *semigroup of operators in* $C(S)_\beta$. *Then there exist numbers* M, $\gamma > 0$ *such that* $\|T_t\| \leqq Me^{\gamma t}$ *for* $t > 0$.

Proof. We consider the semigroup as a semigroup of bounded operators in the Banach space $C(S)$ with the supremum norm topology. According to (**12**, p. 232), it suffices to show that $\{\log\|T_t\|\colon 0 < t \leqq a\}$ is bounded above for each number $a > 0$. However, since the supremum norm is bounded and the strictly bounded sets in $C(S)$ coincide, according to [**1**], and since $\{T_t\colon 0 < t \leqq a\}$ is equicontinuous in $C(S)_\beta$, it follows that $\sup_{0<t\leqq a}\|T_t\| < \infty$.

We will call the semigroup $\{T_t\colon t > 0\}$ bounded if $\sup_{t>0}\|T_t\| < \infty$. By Lemma 4.3, any class (C) semigroup in $C(S)_\beta$ can be converted into a bounded semigroup upon multiplication by $e^{-\gamma t}$. Our next result generalizes the result of Dorroh [**3**, Theorem 2.5] and shows that if γ is only slightly larger, we obtain a bounded class (C_0) semigroup in $C(S)_\beta$.

THEOREM 4.4. *Let* $\{T_t\colon t > 0\}$ *be a bounded semigroup in* $C(S)_\beta$ *of class* (C) *with transition function* $\{\lambda_t\colon t > 0\}$. *Then* $\{e^{-\alpha t}T_t\colon t > 0\}$ *is a class* (C_0) *semigroup in* $C(S)_\beta$ *for every number* $\alpha > 0$.

Proof. Let $T = [0, \infty) \times S$ and define $\mu\colon T \to M(S)$ by $\mu(t, x) = \lambda_t(x)$. Since the semigroup is of class (C), Theorem 4.2(b) holds for all $a > 0$ and consequently μ is tight. Furthermore, by property (c) and the assumption that $\sup\|T_t\| = \sup_t\|\lambda_t\| < \infty$, we have $\mu(f) \in C(T)$ for each $f \in C(S)$. Hence $Af = \mu(f)$ is a continuous linear operator on $C(S)_\beta$ into $C(T)_\beta$. Consequently, if $\phi \in C_0(S)$, there is a β-neighbourhood V of 0 such that $|e^{-\alpha t}\phi(x)\mu(f)(t, x)| \leqq 1$ for all $(t, x) \in T$ and $f \in V$. Hence, $\|\phi e^{-\alpha t}T_t f\| \leqq 1$ for all $f \in V$ and $t > 0$. That is, $\{e^{-\alpha t}T_t\colon t > 0\}$ is equicontinuous in $C(S)_\beta$. Finally, Theorem 4.2(b) also means that $\lim_{t\to 0} e^{-\alpha t}T_t f = f$ in the strict topology, completing the proof.

This result tells us then that one can always convert the usual kind of semigroup in $C(S)$ into a class (C_0) semigroup in $C(S)_\beta$ to which the general theory in [**7**; **12**] applies.

As a second application of Theorem 4.4, we consider a semigroup of maps in S as defined in [**3**] and obtain certain results therein (see [**3**, Theorems 2.2, 2.4, 2.5, and 2.6]) quite easily.

THEOREM 4.5. *Let* $\{\phi_t\colon t \geqq 0\}$ *be a semigroup of maps in* S. *Let* $[T_t f](x) = f(\phi_t(x))$. *The following are equivalent:*

(a) $\{T_t\colon t > 0\}$ *is of class* (C);

(b) $\{e^{-\alpha t}T_t\colon t > 0\}$ *is of class* (C_0) *for each* $\alpha > 0$;

(c) *The map* $t \to \phi_t(x)$ *is continuous on* $[0, \infty)$ *for each* $x \in S$;

(d) *The map* $(t, x) \to \phi_t(x)$ *is continuous on* $[0, \infty) \times S$;

(e) $\lim_{t\to 0} \phi_t(x) = x$ *for each* $x \in S$, *and for each* $a > 0$ *and each compact set* $K \subset S$, $\bigcup_{t \leqq a} \phi_t(K)$ *is compact;*

(*f*) $\lim_{t\to 0} \phi_t(x) = x$ *and* $\{T_t: t \leqq a\}$ *is equicontinuous in* $C(S)_\beta$ *for some* $a > 0$.

Proof. From Theorem 4.4, with $\lambda_t(x) = \widetilde{\phi_t(x)}$, (a) implies (b). To see that (*b*) implies (c), fix (t_0, x_0), let U be a neighbourhood of $\phi_{t_0}(x_0)$ and let $f \in C_0(S)$ be one at $\phi_{t_0}(x_0)$ and zero on $S \backslash U$. Then there is a $\delta > 0$ such that $|t - t_0| < \delta$ implies $||f \cdot (T_t f - T_{t_0} f)|| < \frac{1}{2}$ and hence that $|f \cdot (\phi_t(x_0) - 1)| < \frac{1}{2}$ and therefore that $\phi_t(x_0) \in U$.

We obtain the implication (c) implies (a) by restricting the semigroup $\{T_t: t > 0\}$ to the invariant subspace $C_0(S)$ and applying Theorem 4.1 under condition (c) to obtain $\lim_{t \to t_0} ||T_t f - T_{t_0} f|| = 0$ for all $f \in C_0(S)$. This, along with an argument similar to the above, yields (d).

Clearly (d) implies (e). To see that (e) implies (*f*), let $\phi \in C_0(S)$, $K_n = \{x: |\phi(x)| \geqq 1/n\}$, $\epsilon_1 = 1/||\phi||$, and $\epsilon_n = n - 1$, for $n \geqq 2$. Let $Q_n = \bigcup_{t \leqq a} \phi_t(K_n)$ and let $V = \{f \in C(S): ||f||_{Q_n} \leqq \epsilon_n\}$. Then V is a β-neighbourhood of 0 and $f \in V$ implies $|\phi(x) f(\phi_t(x))| \leqq 1$ for $t \leqq a$. That is, $f \in V$ implies $||\phi T_t f|| \leqq 1$ for $t \leqq a$ so that $\{T_t: t \leqq a\}$ is equicontinuous in $C(S)_\beta$.

Clearly (*f*) implies (a), completing the proof.

5. The infinitesimal generator of the class (*C*) semigroup. In this section we relate the class (*C*) semigroup to its infinitesimal generator as well as to the infinitesimal generator of the related semigroup of adjoint operators. We first use Theorem 4.4 and the Hille-Yosida theorem to characterize the infinitesimal generator (see [**7**]) of a class (*C*) semigroup.

THEOREM 5.1. *The linear operator* A *defined on a subspace* D_A *of* $C(S)$ *is the infinitesimal generator of a uniquely determined class* (*C*) *semigroup of continuous operators in* $C(S)_\beta$ *if and only if*

(a) D_A *is dense in* $C(S)_\beta$,

(b) A *is sequentially closed in* $C(S)_\beta$, *and*

(c) *for some* $\alpha > 0$, *there is an unbounded sequence* $\lambda_n \geqq 0$ *such that* $(\lambda_n + \alpha - A)^{-1}$ *exists and*

$$\left\{\left(I - \frac{A - \alpha}{\lambda_n}\right)^{-m} : n, m = 1, 2, \ldots\right\}$$

is equicontinuous.

Proof. Suppose that A satisfies (a), (b), and (c). The linear operator $A - \alpha$, defined on D_A, is the infinitesimal generator of a unique class (C_0) semigroup of continuous operators in $C(S)_\beta$, according to [**7**, Propositions 3.9 and 6.1], which we denote by $\{S_t: t \geqq 0\}$. Clearly then $\{e^{\alpha t} S_t: t \geqq 0\}$ is a class (*C*) semigroup in $C(S)_\beta$ with infinitesimal generator $\alpha + (A - \alpha) = A$. Furthermore, suppose that A generates another class (*C*) semigroup $\{T_t: t > 0\}$.

By Lemma 4.3 and Theorem 4.4, $\{e^{-\alpha t}e^{-\gamma t}T_t: t > 0\}$ is a class (C_0) semigroup whose generator is $A - \alpha - \gamma$. However, $\{e^{-\gamma t}S_t: t > 0\}$ is also a class (C_0) semigroup whose generator is $A - \alpha - \gamma$. Hence $T_t = e^{\alpha t}S_t$ for all $t > 0$, proving uniqueness.

Conversely, if A is the generator of a class (C) semigroup, then again appealing to Lemma 4.3 and Theorem 4.4 there is a number $\alpha > 0$ such that $\{e^{-\alpha t}T_t: t > 0\}$ is a class (C_0) semigroup. An appeal to [**7**, Proposition 6.1] completes the proof.

We will now show that the class (C) semigroup is also uniquely determined by the infinitesimal generator of the semigroup of adjoint operators in $M(S)$. This result is indicated by [**6**, Theorem 2.6] and its development.

THEOREM 5.2. *Let $\{T_t: t > 0\}$ be a class (C) semigroup in $C(S)_\beta$ and let $\{U_t: t > 0\}$ be the semigroup of adjoint operators in $M(S)$. The infinitesimal generator of $\{U_t: t > 0\}$ uniquely determines $\{T_t: t > 0\}$.*

Proof. Let A be the infinitesimal generator of $\{U_t: t > 0\}$ and also of some other semigroup $\{V_t: t > 0\}$ of adjoint operators of a class (C) semigroup $\{S_t: t > 0\}$. Referring to [**7**, Proposition 4.1], let τ be the norm topology on $M(S)$ and let σ be the β-weak* topology on $M(S)$; i.e., the weak* topology on $M(S)$ as the dual of $C(S)_\beta$. It follows that

$$X_0 = \{\mu \in M(S): \tau\text{-}\lim_{t\to 0} U_t\mu = \mu\} = \{\mu \in M(S): \tau\text{-}\lim_{t\to 0} V_t\mu = \mu\}$$

is the norm closure of D_A in $M(S)$. The Hille-Yosida theorem then tells us that the operator A uniquely determines a semigroup in the Banach space X_0. Hence $H = \{\mu: U_t\mu = V_t\mu \text{ for all } t > 0\} \supset X_0$.

Let $Y_0 = \{\mu \in M(S): \sigma\text{-}\lim_{t\to 0}((U_t\mu - \mu)/t) \text{ exists}\}$. From the uniform boundedness principle, $Y_0 \subset X_0$. Furthermore, Y_0 is β weak* dense in $M(S)$. To see this, let $\mu \in M(S)$ and note that

$$\frac{U_h\mu_a - \mu_a}{h} = \frac{1}{h}\int_a^{a+h} U_t\mu\, dt - \frac{1}{h}\int_0^h U_t\mu\, dt,$$

where $\mu_a = \int_0^a U_t\mu\, dt$ and where these integrals exist in the sense of Theorem 3.1(a) by virtue of the fact that $\{T_t: t > 0\}$ is of class (C); this follows upon letting $\lambda(t) = U_t\mu$ for $0 < t \leqq a + h$ and $\lambda(0) = \mu$.

It now follows that $\mu_a \in Y_0$ and $\sigma\text{-}\lim_{a\to 0}\mu_a = \mu$. Hence Y_0 is weak* dense in $M(S)$ and since H is weak* closed and contains Y_0, it follows that $H = M(S)$ and finally that $T_t = S_t$ on $C(S)$ for all $t > 0$.

We close this section with an easily obtainable sufficient condition that a class (C) semigroup have a transition function consisting of probability measures.

THEOREM 5.3. *Let $\{T_t: t > 0\}$ be a class (C) semigroup in $C(S)$ with $||T_t|| \leqq 1$ for $t > 0$. If A is the infinitesimal generator of $\{T_t: t > 0\}$ and if the identically*

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one function $1 \in D_A$ *and* $A1 = 0$, *then* $\{T_t: t > 0\}$ *is a semigroup of positive operators in* $C(S)$ *whose transition function consists of probability measures.*

Proof. From [**7**, Proposition 3.10],

$$\lambda_t(x, S) - 1 = T_t 1 - 1 = \int_0^t T_s A(1)\, ds = 0.$$

Hence each measure $\lambda_t(x)$ assigns the measure **1** to S and has norm less than or equal to $||T_t|| \leqq 1$. However, if $\mu \in M(S)$ and $||\mu|| = \mu(S) = 1$, then as is well known, μ is a probability measure.

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