Canad. Math. Bull. Vol. 16 (1), 1973

A FIXED POINT THEOREM FOR WEAKLY UNIFORMLY STRICT CONTRACTIONS(1)

BY
NADIM A. ASSAD

In Meir and Keeler [3], the authors proved a fixed point theorem in a complete metric space (X, d) for a mapping f that satisfies the following condition of weakly uniformly strict contraction:

Given $\varepsilon>0$, there exists $\delta>0$ such that

$$\text{(A)} \qquad \varepsilon \leq d(x, y) < \varepsilon+\delta \quad \text{implies} \quad d(f(x), f(y)) < \varepsilon.$$

Below we prove a new theorem for mappings satisfying (A) in convex metric spaces. As usual for $K \subset X$, ∂K denotes the boundary of K.

THEOREM 1. *Let (X, d) be a complete, metrically convex, metric space and K a nonempty closed subset of X. Suppose that $T:K\to X$ satisfies (A) and $T(x) \in K$ for every $x \in \partial K$. Then T has a unique fixed point in K.*

Proof. We construct a sequence $\{p_n\}$ in K as follows: Let p_0 be an arbitrary point in K. Let $p_1'=T(p_0)$. If $p_1' \in K$, then set $p_1=p_1'$, otherwise we choose $p_1 \in \partial K$ so that $d(p_0, p_1)+d(p_1, p_1')=d(p_0, p_1')$ (cf. [1, p.3]). Suppose that $\{p_i\}$, $\{p_i'\}$, $i=1, \ldots, N$ have been chosen so that

(i) $p_i'=T(p_{i-1})$, $i=1, \ldots, N$;

(ii) either $p_i=p_i' \in K$ or $p_i \in \partial K$ and satisfies the relation:

$$d(p_{i-1}, p_i)+d(p_i, p_i') = d(p_{i-1}, p_i').$$

Now set $p_{N+1}'=T(p_N)$. If $p_{N+1}' \in K$ we put $p_{N+1}=p_{N+1}'$, otherwise we choose $p_{N+1} \in \partial K$ so that

$$d(p_N, p_{N+1}') = d(p_N, p_{N+1})+d(p_{N+1}, p_{N+1}').$$

Thus by induction we are finished.

If there exists $p_j \in \{p_n\}$ such that all of its iterates lie in K, Meir and Keeler [3] showed that this sequence of iterates converges to a fixed point of T. Hence we may assume that there exist infinitely many points $p_i \in \{p_n\}$ for which $p_i\neq p_i'$. Let $\{p_{n_k}\}$ be the subsequence of all such points in $\{p_n\}$, i.e., $p_{n_k}\neq p_{n_k}'$.

Received by the editors June 29, 1971 and, in revised form, September 1, 1971.

(1) This paper is taken from the author's Ph.D. Thesis which was written under the supervision of Professor W. A. Kirk.

We assert that

(B) $$d(p_n, p_{n+1}) \to 0 \quad \text{as} \quad n \to \infty$$

and

(C) $$d(T(p_n), p_n) \to 0 \quad \text{as} \quad n \to \infty.$$

To prove (B) and (C) we first prove that

(G) $$d(p_{n_k-1}, p'_{n_k}) \to 0 \quad \text{as} \quad k \to \infty$$

Here we use the fact that T satisfies (A) implies that T is contractive

$$(d(T(x), T(y)) < d(x, y)).$$

If we put $n_k=r$ and $n_{k+1}=s$, then it follows that

$$\begin{aligned} d(p_{s-1}, p'_s) &< d(p_{s-2}, p_{s-1}) \\ &< \cdots < d(p_r, p_{r+1}) \\ &\leq d(p_r, p'_r)+d(p'_r, p_{r+1}) \\ &< d(p_r, p'_r)+d(p_{r-1}, p_r) \\ &= d(p_{r-1}, p'_r). \end{aligned}$$

Therefore $\{d(p_{n_k-1}, p'_{n_k})\}$ is decreasing. Suppose that $d(p_{n_k-1}, p'_{n_k}) \to \varepsilon > 0$. Then for all $k=1, 2, \ldots, d(p_{n_k-1}, p'_{n_k}) > \varepsilon$. But condition (A) implies there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon+\delta \quad \text{implies} \quad d(T(x), T(y)) < \varepsilon.$$

We know there exists an integer N such that for $k \geq N$, $d(p_{n_k-1}, p'_{n_k}) < \varepsilon+\delta$; so if we let $n_k=r$ and $n_{k+1}=s$, it follows that

$$\begin{aligned} d(p_r, p_{r+1}) &\leq d(p_r, p'_r)+d(p'_r, p_{r+1}) \\ &< d(p_r, p'_r)+d(p_{r-1}, p_r) \\ &= d(p_{r-1}, p'_r) < \varepsilon+\delta. \end{aligned}$$

On the other hand,

$$\begin{aligned} \varepsilon &< d(p_{s-1}, p'_s) \\ &< d(p_r, p_{r+1}). \end{aligned}$$

Therefore,

$$\varepsilon < d(p_r, p_{r+1}) < \varepsilon+\delta.$$

It follows that,

$$\begin{aligned} d(p_{s-1}, p'_s) &\leq \cdots \leq d(p_{r+1}, p'_{r+2}) \\ &= d(T(p_r), T(p_{r+1})) < \varepsilon \end{aligned}$$

and this contradicts the assumption that for all $k=1, 2, \ldots, d(p_{n_k-1}, p'_{n_k}) > \varepsilon$. Therefore we have proved (G). To see how (B) follows from (G), we assume (B) is false. Then there exists $\varepsilon > 0$ such that for every positive integer N, there exists $n \geq N$ such that $d(p_n, p_{n+1}) > \varepsilon$; but by (G), we know that there exists a positive integer M such that for $k \geq M$, $d(p_{n_k-1}, p'_{n_k}) < \varepsilon$. So, let $N=n_k$ for some $k \geq M$.

Clearly, for all $n \geq N$,

$$d(p_n, p_{n+1}) \leq \cdots \leq d(p_{N-1}, p'_N) < \varepsilon,$$

and this is a contradiction. Therefore (B) is true. An identical argument establishes (C).

Now we show that the sequence $\{p_n\}$ is Cauchy. If this sequence is not Cauchy, then there exists $2\varepsilon > 0$ such that $\lim_{m,n\to\infty} \sup d(p_m, p_n) > 2\varepsilon$.

By hypothesis there exists a $\delta > 0$ such that

$$\text{(D)} \qquad \varepsilon \leq d(x, y) < \varepsilon + \delta \quad \text{implies} \quad d(T(x), T(y)) < \varepsilon.$$

Formula (D) remains true with δ replaced by $\delta' = \min(\delta, \varepsilon)$. Also, observe that (B) and (C) imply that there exists an integer M such that for $n \geq M$,

$$d(p_n, p_{n+1}) < \frac{\delta'}{3} \quad \text{and} \quad d(p_n, T(p_n)) < \frac{\delta'}{3}.$$

Now, we choose $m, n > M$ so that $d(p_m, p_n) > 2\varepsilon$. For $j \in [m, n]$,

$$d(p_m, p_j) \leq d(p_m, p_{j+1}) + d(p_j, p_{j+1}).$$

Therefore

$$|d(p_m, p_j) - d(p_m, p_{j+1})| \leq d(p_j, p_{j+1}) < \frac{\delta'}{3};$$

this, together with the fact that

$$d(p_m, p_{m+1}) < \frac{\delta'}{3} < \delta' < \varepsilon,$$

and

$$d(p_m, p_n) > 2\varepsilon = \varepsilon + \varepsilon \geq \varepsilon + \delta',$$

implies that there exists a $j \in [m, n]$ with

$$\text{(E)} \qquad \varepsilon + 2\delta'/3 < d(p_m, p_j) < \varepsilon + \delta'.$$

However, for this m and j,

$$\begin{aligned} d(p_m, p_j) &\leq d(p_m, T(p_m)) + d(T(p_m), T(p_j)) + d(T(p_j), p_j) \\ &< \frac{\delta'}{3} + \varepsilon + \frac{\delta'}{3} \\ &= \varepsilon + 2\delta'/3, \end{aligned}$$

and this contradicts (E). Therefore we may conclude that the sequence $\{p_n\}$ is Cauchy, and it follows that the limit of this sequence is a fixed point of T. The fixed point is unique because, as we observed earlier in the proof, T is a contractive mapping.

REMARK 1. Theorem 1 remains true if, instead of (A), we require T to have the property (*): $d(T(x), T(y)) \leq \psi(d(x, y))$, where $\psi: \bar{S} \to [0, \infty)$ is a function satisfying

https://doi.org/10.4153/CMB-1973-004-3 Published online by Cambridge University Press

$\psi(t)<t$ for all $t \in \bar{S}\backslash\{0\}$. Here $S=\{d(x, y), x, y \in K\}$ and $\bar{S}$ is the closure of S. To see this, it suffices to observe that every mapping T satisfying property (*) is a weakly uniformly strict contraction (cf. [2] and [3]). This remark is a generalization to Theorem 2 in Boyd and Wong [2].

REMARK 2. Observe that a contraction mapping $(d(T(x), T(y)) \leq ad(x, y), 0 \leq a<1)$ is a weakly uniformly strict contraction. Moreover, if X is a compact space, then any contractive mapping $(d(f(x), f(y))<d(x, y))$ $f: X \rightarrow X$ is a weakly uniformly strict contraction (cf. [3, p. 328]).

EXAMPLE 1. This example shows that Theorem 1 fails in an arbitrary complete metric space. Consider the space X that consists of two points $\{a, b\}$, with the discrete metric, i.e., $d(a, b)=1$, $d(a, a)=d(b, b)=0$. Let $K=\{b\}$, a closed subset of X. Define $T: K \rightarrow X$ by $T(b)=a$. Then T satisfies (A), $T(\partial K) \subset K$, but T does not have a fixed point.

EXAMPLE 2. Now we give an example of a space X, a subset K of X, and a mapping T which satisfies (A) and the conditions of Theorem 1 but for which there is some $x \in K$ with $T(x) \notin K$. Let X be the real line with the euclidean metric $(d(a, b)=|a-b|)$, and let $K=\{-\frac{1}{4}\} \cup [0, \frac{1}{2}]$. Define $T: K \rightarrow X$ as follows: $T(-\frac{1}{4})=-\frac{1}{4}$, and for $x \in [0, \frac{1}{2}]$, $T(x)=x^2-\frac{1}{4}$. Then T satisfies (A) because the set K is compact and T is contractive (see Remark 2 above). Also, $\partial K=\{-\frac{1}{4}, 0, \frac{1}{2}\}$ and clearly $T: \partial K \rightarrow K$. Moreover, for all $x \in (0, \frac{1}{2})$, $T(x) \notin K$. We might add that T is not a contraction because for $x \neq y$, $x, y \in [0, \frac{1}{2}]$, $d(T(x), T(y))=|x^2-y^2|= |x-y| \cdot |x+y|=|x+y| \cdot d(x, y)$ which approaches $d(x, y)$ as $x, y \rightarrow \frac{1}{2}$.

REFERENCES

1. N. A. Assad and W. A. Kirk, *Fixed point theorems for set valued mappings of contractive type*, Pacific J. Math. (to appear).

2. D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458–464.

3. E. Keeler and A. Meir, *A theorem on contraction mappings*, J. Math. Anal. Appl. **28** (1969), 326–329.

UNIVERSITY OF IOWA,
IOWA CITY, IOWA

https://doi.org/10.4153/CMB-1973-004-3 Published online by Cambridge University Press