ON A THEOREM OF BAER AND HIGMAN

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1. Introduction

1.1 Baer has shown **(1)** that if the fact that the exponent of a group is m (that is, m is the least common multiple of the periods of the elements) implies a limitation on the class of the group, then m must be a prime. Graham Higman has extended this result by proving **(3)** that for any given integer M there are at most a finite number of prime powers q other than primes, such that the fact that a group has exponent q implies a limitation on the class of the Mth derived subgroup. In fact, given arbitrary positive integers M and N, he produces, by an intricate construction, a finite group G having derived length $M + 2$ and prime-power exponent p^r, such that the class of the Mth derived subgroup of G exceeds N, where

$$p^r - 1 > (p - 1)\,A(M)$$

and $A(M)$ is an integer-valued function:

$$A(0) = 1, \quad A(1) = 3, \quad A(2) = 13, \ldots .$$

The case $M = 0$ is Baer's result.

1.2 In this paper we consider those prime powers p^r for which

$$p^r - 1 = (p - 1)\,A(M),$$

in the special cases $M = 0$ and $M = 1$, that is, $r = 1$ and $p = r = 2$ respectively. We show that no result similar to that of Higman can be obtained in these cases; indeed, an upper bound is given for the class of the Mth derived subgroup in terms of the derived length.

Specifically, the final results are as follows:

THEOREM 1. *If a finitely generated group G has exponent 4 and ϕ-length λ, then the class of $\phi(G)$ is at least $2^{\lambda-2}$ and at most $5^{\lambda-2}$.*

The meaning of $\phi(G)$ and ϕ-length of G is explained in §2.1.

THEOREM 2. *If a finitely generated group G has prime exponent p and derived length d, then the class of G is at least 2^{d-1} and at most p^{d-1}.*

Received July 22, 1955. This paper embodies some of the work carried out by the author for a Ph.D. thesis at the University of Manchester. I wish to express my gratitude to Dr. Graham Higman for his continued interest and advice then, and, in connection with the present paper, for Theorem 5.2 which is essentially due to him both in conception and in proof.

A well-known result due to Hall (2) gives $2^{\lambda-2}$ and 2^{d-1}, respectively, as the lower bounds; we shall be concerned here with the upper bounds. The interest of these results lies, of course, not so much in the bounds given for the class, which are presumably far from best possible if λ and d are large, as in the fact that bounds exist which are independent of the number of generators of the groups in question.

We may mention also an auxiliary theorem which is of interest in itself.

THEOREM 5.2. *If, for a finitely generated group G with prime-power exponent p^r, there exists a positive integer s such that*

$$H_{s+1}(G) \subseteq D_2(G),$$

then

$$H_{qs+1}(G) \subseteq H_{q+1}(D(G)), \qquad q = 1, 2, 3, \ldots .$$

Here $H_i\ (G)$ and $D_2(G)$ are members of the lower central series and the derived series (§2) of G, respectively.

2. Definitions

2.1 The *Frattini subgroup* $\phi(G)$ of a group G is defined to be the intersection of all the maximal subgroups of G; if P is a p-group (by which we mean that the order of P is a power of a prime p) it is known (2) that $\phi(P) = D(P) \cup P^p$ where $D(P)$ is the commutator subgroup and P^p the subgroup generated by the pth powers of all elements in P.

The *Frattini series* is defined inductively:

$$\phi_0(G) = G, \qquad \phi_{i+1}(G) = \phi(\phi_i(G)), \qquad i \geqslant 0.$$

If this series terminates with the identity (as it certainly does for a finite group), so that $\phi_{j-1}(G) \neq \{1\}$ but $\phi_i(G) = \{1\}$ whenever $i \geqslant j$, we shall say that the *ϕ-length of G* is j.

The *derived series* of a group G is defined inductively:

$$D_0(G) = G; \quad D_{i+1}(G) = D(D_i(G)), \qquad i \geqslant 0.$$

If N is a normal subgroup of a p-group P, such that $\phi_i(P) \supseteq N$, it is easily seen that $\phi_i(P/N) = \phi_i(P)/N$.

Since $u^{-1}v^{-1}uv = (u^{-1})^2(uv^{-1})^2v^2$, we see that if P is a 2-group, then $P^2 \supseteq D(P)$ and $\phi(P) = P^2$. If, in addition, P is generated by elements which all have period 2, then $D(P) \supseteq P^2$, and consequently $\phi(P) = D(P) = P^2$.

Again, if P is a 2-group with n independent generators, $P/\phi(P)$ is elementary abelian with order 2^n; thus every factor-group $\phi_i(P)/\phi_{i+1}(P)$ in the ϕ-chain of P is an elementary abelian 2-group (i.e., the direct product of cycles of order 2).

2.2 Square brackets will be used to denote commutation:

$$[x, y] = x^{-1}y^{-1}xy.$$

If $x_1, x_2, x_3, \ldots$ are arbitrary elements in a group, the (complex) commutators in the x_i are defined inductively by the rules

(i) x_i is a commutator;

(ii) if c and d are commutators in these elements, so also is $[c, d]$. In particular, a *left-normed* (or simple) *n-fold commutator* is defined:

$$[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n] \qquad (n > 2).$$

The *weight* of a commutator *in the element* x_i is defined by

(i) the weight of x_j in x_i is 1 if $i = j$, 0 if $i \neq j$.

(ii) the weight of $[c, d]$ in x_i is the sum of the weights of c and d in x_i.

The *weight of a commutator* is the sum of its weights in the components x_i. We recall the commutator identity

$$[x, yz] = [x, z][x, y][x, y, z].$$

The *lower central series* of a group G is defined inductively

$$H_1(G) = G; \quad H_{i+1}(G) = [H_i(G), G], \qquad i \geqslant 1.$$

(That is, H_{i+1} is the subgroup generated by the set of all commutators of the form $[h_i, g]$ with h_i in H_i and g in G.) A discussion of the properties of this series, and its connection with the class of G, may be found in **(2)**.

3. A certain group with exponent 4

3.1 Let G be a 2-group with ϕ-length 3. Let $H = \phi_2(G)$, and let $K \simeq G/H$. It is clear that the exponent of G must be 8 or 4, and the exponent of K is 4. Both H and $\phi(K)$ are non-trivial elementary abelian 2-groups, and $\phi(K) \simeq \phi(G)/H$. We use 1 to represent the unit element of G or K according to context.

In this section we show that the requirement that G have exponent 4 introduces a certain relation into the group ring of K over the field of two elements; and in §3.2 we show that this relation yields an upper bound for the class of $\phi(G)$.

For each element k in K we choose, in the corresponding coset of G/H, a coset representative g_k in G. If x, y are any elements of K, the element $h(x, y)$ in H is determined by the equation

$$g_x g_y = g_z h(x, y) \qquad \text{where } z = xy.$$

The element g_k induces an automorphism of H which depends only on k; if h is any element of H we denote its image by

$$h^k = g_k^{-1} h \, g_k.$$

The automorphisms k belong to the ring Θ of endomorphisms of H, and Θ has characteristic 2. Since $(h^x)^y = h^z$, where x, y and z have the meanings already assigned, the subring of Θ generated by the set $\{k: k \in K\}$ is a homomorphic image of the group ring of K over the field of two elements $(0, 1)$.

Any element g of G can be written uniquely in the form $g = g_k h$ with k in K and h in H. Then

$$g^4 = g_1 h(k^2, k^2)\, h(k, k)^{(1+k)^2}\, h^{(1+k)^3}.$$

We choose $g_1 = 1$; and now in order that G itself may have exponent 4 it is necessary and sufficient that

(i) $h^{(1+k)^3} = 1$ for all choices of h in H and k in K

and

(ii) $h(k^2, k^2)\, h(k, k)^{(1+k)^2} = 1$ for all choices of k in K.

We shall consider condition (i), which is more amenable to treatment than (ii). Since

$$h^{(1+k)} = hg_k^{-1}hg_k = [h, g_k],$$

what relation (i) says, in effect, is that

$$[h, u, v, w] = 1$$

for any element h in H, whenever the elements u, v, w all lie in the same coset of G modulo H.

3.2 Thus we consider the group ring of K over the field of two elements $(0, 1)$, with the relation $(1 + k)^3 = 0$ for all elements k in K.

We use the notation $K_i = 1 + k_i$ for elements k_i in K; throughout what follows we shall *not* use k without a subscript to represent an element of K. To avoid repetition, we make the following convention:

k_1 is an arbitrary element of K

$k_2 = k_1^2$

k_3, k_5, k_7 are arbitrary elements of $\phi(K)$

$k_4 = [k_1, k_3], \quad k_6 = [k_1, k_5], \quad k_8 = [k_1, k_7].$

Thus $K_i^2 = 0$ and $K_i K_j = K_j K_i$ when i and j lie between 2 and 8 inclusive.

The relation in the group ring may now be written

3.21 $$K_1 K_2 = 0.$$

If we replace k_1 here by $k_1 k_3$, then k_2 is replaced by $(k_1 k_3)^2 = k_2 k_4$,

$$K_1 \quad \text{becomes} \quad 1 + k_1 k_3 = 1 + (1 + K_1)(1 + K_3) = K_1 + K_3 + K_1 K_3,$$

and the relation gives

3.22 $$(K_1 + K_3 + K_1 K_3)(K_2 + K_4 + K_2 K_4) = 0.$$

Post-multiplication by K_4 gives $(K_1 + K_3 + K_1 K_3) K_2 K_4 = 0$; then post-multiplication by K_3 gives

3.23 $$K_1 K_3 (K_2 + K_4) = 0.$$

This leaves $(K_1 + K_3)(K_2 + K_4) = 0$, but $K_1 K_2 = 0$, thus finally

3.24 $$K_1 K_4 + K_3 (K_2 + K_4) = 0.$$

Using 3.23, premultiplication by K_1 gives

3.25 $$K_2 K_4 = 0.$$

In 3.24, replace k_3 by k_3k_5, then k_4 is replaced by $[k_1, k_3k_5] = k_4k_6$. Thus

3.26 $$K_3K_6 + K_4K_5 + (K_1 + K_3 + K_5 + K_3K_5)K_4K_6 + K_3K_5(K_2 + K_4 + K_6 + K_4K_6) = 0.$$

If we substitute for k_5 the particular value $k_5 = [k_1, k_\beta]$ where k_β is an arbitrary element in $\phi(K)$, then

$$k_6 = [k_\beta, k_1, k_1] = [k_\beta, k_1^2] = 1$$

and $K_2K_5 = 0$ by 3.25, thus

$$K_4K_5 + K_3K_4K_5 = 0, \quad \text{which implies} \quad K_4K_5 = 0.$$

Consequently, in equation 3.26, where k_5 is again an arbitrary element of $\phi(K)$, we have

3.27 $$K_4K_6 = 0$$

and 3.26 reduces to

3.28 $$K_3K_6 + K_4K_5 + K_2K_3K_5 = 0.$$

Again, in 3.28 replace k_5 by k_5k_7 and k_6 by k_6k_8. Then, since $K_6K_8 = 0$ by 3.27

$$K_3(K_6 + K_8) + K_4(K_5 + K_7 + K_5K_7) + K_2K_3(K_5 + K_7 + K_5K_7) = 0.$$

Using 3.28 this simplifies to

$$K_4K_5K_7 = K_2K_3K_5K_7.$$

On multiplying 3.28 by K_7 and using this, we obtain $K_3K_6K_7 = 0$, which is equivalent to saying $K_4K_5K_7 = 0$. Consequently

3.29 $$K_2K_3K_5K_7 = 0.$$

Now if we take any element k_9 in $\phi(K)$, then

$$K_9 = K_{11}^2 + K_{12}^2 + \ldots + K_{1\xi}^2 + \text{(products of two or more of these squares)}$$

where $k_{11}, k_{12}, \ldots, k_{1\xi}$ are certain elements of K. Thus 3.29 implies that

$$K_3K_5K_7K_9 = 0$$

for all choices of four elements k_3, k_5, k_7, k_9 in $\phi(K)$.

In the group G this means that for arbitrary elements w, x, y, z in $\phi(G)$ and h in H, $[h, w, x, y, z] = 1$. Thus we have proved

THEOREM 3.2. *If G is a finite group with exponent 4 and ϕ-length 3, then the class of $\phi(G)$ is at most 5.*

3.3 Let G be a finitely generated group with exponent 4: such a group is finite **(5)**, hence a 2-group. Thus $\phi(G/\phi_3(G)) = \phi(G)/\phi_3(G)$ and the derived series of $\phi(G)$ coincides with its Frattini series. Thus from Theorem 3.2 we obtain, using the notation previously explained,

COROLLARY 3.3. *If G is a finitely generated group with exponent 4,*

$$H_6(\phi(G)) \subseteq D_2(\phi(G)).$$

4. A similar result

Meier-Wunderli has shown, in **(4)**, that a finitely generated metabelian group with prime exponent p has class at most p. This may be stated as follows:

THEOREM 4.1 (Meier-Wunderli). *If G is a finitely generated group with prime exponent p,*

$$H_{p+1}(G) \subseteq D_2(G).$$

This result bears an obvious resemblance to Corollary 3.3; they will be extended simultaneously by means of the theorems given in the next section.

5. Some theorems on commutators

5.1 LEMMA. *Let $x_1, x_2, \ldots, x_n$ be arbitrary elements of an arbitrary group G; let c be a commutator of positive weight in each of the x_i $(i = 1, 2, \ldots, n)$ and let the equation*

$$c = d_1 d_2 \ldots d_\alpha,$$

where the d_j are also commutators in $x_1, x_2, \ldots, x_n$, be an identity in the group variables x_i. Then there is an equation

$$c = b_1 b_2 \ldots b_\beta$$

also true for all $x_1, x_2, \ldots, x_n$ in G, where the b's are commutators in the elements $d_1, d_2, \ldots, d_\alpha$ such that every b is of positive weight in each of $x_1, x_2, \ldots, x_n$.

Proof. This can be proved by induction on n, being trivially true for $n = 1$. Thus we may suppose that each d_j is of positive weight in each of $x_1, \ldots, x_{n-1}$. We may further suppose that the commutators of zero weight in x_n are those in an initial segment $d_1 d_2 \ldots d_\xi$. For if this is not so, then using the relation $yx = xy[y, x]$ we can bring them to the left of the expression one at a time, by a process which terminates, since the new commutators $[y, x]$ introduced have positive weight in x_n.

When this has been done, let $x_n = 1$. Then every commutator of positive weight in x_n reduces to the identity, while those of zero weight in x_n are not affected. Thus for all $x_1, x_2, \ldots, x_n$ in G

$$1 = d_1 d_2 \ldots d_\xi.$$

Hence also, for all $x_1, x_2, \ldots, x_n$ in G

$$c = d_{\xi+1} d_{\xi+2} \ldots d_\alpha$$

which is the expression required.

5.2 THEOREM. *If, for a finitely generated group G with prime-power exponent p^r, there exists a positive integer s such that*

$$H_{s+1}(G) \subseteq D_2(G),$$

then

$$H_{qs+1}(G) \subseteq H_{q+1}(D(G)), \qquad q = 1, 2, 3, \ldots.$$

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Before proving Theorem 5.2, we make a remark which also has a bearing on Theorem 6.2. A well-known theorem, due to O. Schreier, states that in a finitely generated free group any subgroup of finite index is also finitely generated. Since any finitely generated group is a homomorphic image of a finitely generated free group, the statement remains true when the words "free group" are replaced by "group." In particular, let F be a group with a finite exponent. Then if F is finitely generated, so also is $D(F)$ and every factor $D_i(F)/D_{i+1}(F)$ in the derived series is finite.

Proof. If N is any normal subgroup of G, $H_i(G/N) = \{H_i(G), N\}/N$; hence it is sufficient to prove

$$H_{qs+1}(X) \subseteq H_{q+1}(D(X))$$

for the finite group $X = G/D_\alpha(G)$ where α is chosen large enough to ensure that $D_\alpha(G) \subseteq H_{q+1}(D(G))$.

Thus we consider a p-group X of exponent p^r, generated by a minimal basis $x_1, x_2, \ldots, x_n$. A result due to Hall **(2**; Theorem 2.8.2**)** states that $H_j(X)$ is generated by the set of all left-normed commutators of weight $\geqslant j$ in the components $x_1, x_2, \ldots, x_n$. Thus $D(X) = H_2(X)$ is generated by the simple commutators

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_t}] \qquad t \geqslant 2.$$

But $H_{s+1}(X) \subseteq D_2(X)$; hence the simple commutators with $t \geqslant s+1$ all lie in $D_2(X) \subseteq \phi(D(X))$, and can therefore **(2)** be omitted from any generating set of $D(X)$. Thus $D(X)$ is in fact generated by the simple commutators with $2 \leqslant t \leqslant s$; we denote these by $d_1, d_2, \ldots$.

If we set

$$c = [x_{i_1}, x_{i_2}, \ldots, x_{i_{qs+1}}],$$

then c lies in $D_2(X)$, so

$$c = d_{j_1} d_{j_2} \ldots d_{j_\alpha}.$$

Consequently, by Lemma 5.1,

$$c = b_1 b_2 \ldots b_\beta,$$

where each b_i is a commutator in the d's and therefore also in the x's and is of positive weight in x_{i_ν}, $\qquad \nu = 1, 2, \ldots, qs+1.$

Thus, in particular, each b_i is of weight at least $qs+1$ in the x's. Since no commutator d has weight greater than s in the x's, each b_i must be of weight at least $q+1$ in the d's. The required result follows.

5.3 Corollary. *If, in addition to the assumptions of the previous theorem,*

$$H_{s+1}(D_\beta(G)) \subseteq D_{\beta+2}(G) \textit{ for every positive integer } \beta,$$

then

$$H_{s^d+1}(G) \subseteq D_{d+1}(G) \quad \textit{for any positive integer } d.$$

Proof. $H_{s+1}(G) \subseteq D_2(G)$ gives a basis for induction on d. We assume that the statement is true for an integer d. Then

$$\begin{aligned} H_{s^{1+d}+1}(G) &\subseteq H_{s^d+1}(D(G)) && \text{by Theorem 5.2,} \\ &\subseteq D_{d+1}(D(G)) && \text{by the induction hypothesis,} \end{aligned}$$

since $D(G)$ itself satisfies the conditions stated for G.

6. Final results

All that remains now is to apply 5.3 to the groups considered in 3.3 and 4.1. If G is finitely generated with exponent 4, so also are its successive Frattini subgroups; and

$$D_\beta(\phi(G)) = \phi_\beta(\phi(G)) = \phi(\phi_\beta(G));$$

thus by Corollary 3.3,

$$H_{5+1}(D_\beta(\phi(G)) \subseteq D_{\beta+2}(\phi(G)).$$

Corollary 5.3 now gives

THEOREM 6.1. *If G is a finitely generated group with exponent* 4:

$$H_{5^\lambda+1}(\phi(G)) \subseteq \phi_{\lambda+2}(G).$$

Again, if G is finitely generated with prime exponent p, so also is $D_\beta(G)$, and by Theorem 4.1,

$$H_{p+1}(D_\beta(G)) \subseteq D_{\beta+2}(G).$$

THEOREM 6.2. *If G is a finitely generated group with prime exponent p*:

$$H_{p^d+1}(G) \subseteq D_{d+1}(G).$$

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