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Mori's Program for $\overline{M}_{0,7}$ with Symmetric Divisors

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Abstract. We complete Mori's program with symmetric divisors for the moduli space of stable seven-pointed rational curves. We describe all birational models in terms of explicit blow-ups and blow-downs. We also give a moduli theoretic description of the first flip, which has not appeared in the literature.

1 Introduction

The aim of this paper is running Mori's program for $\overline{\mathrm{M}}_{0,7}$, the moduli space of stable seven-pointed rational curves, with symmetric divisors. Mori's program, a minimal model program for a given moduli space M, consists of the following:

(1) Compute the cone of effective divisors $\mathrm{Eff}(M)$ for M and the chamber structure on it, the so called stable base locus decomposition.

(2) For an effective divisor D with finitely generated section ring, we can compute a projective model

$$M(D) := \mathrm{Proj} \bigoplus_{m \geq 0} H^0(M, \mathcal{O}(mD))$$

with a rational contraction $M \dashrightarrow M(D)$.

Because any rational contraction is obtained in this way ([29]), by running Mori's program we are able to classify all birational models of M that are simpler than M. Furthermore, since M is a moduli space, we can expect that some of $M(D)$ also have certain good moduli theoretic interpretations.

Since Hassett and Hyeon initiated the study of birational geometry of moduli spaces of stable curves from the viewpoint of Mori's program in [24, 26, 27], there has been much success and progress in this direction. Although the initial motivation, finding the (final log) canonical models of moduli spaces of stable curves $\overline{\mathcal{M}}_g$ succeeded only for a few small genera [10, 12, 24, 30], people have constructed many modular birational models of $\overline{\mathcal{M}}_g$, and these models have been studied in a theoretical framework of Mori's program. Also, the same framework has been applied to many other moduli spaces, for instance, Hilbert schemes of points [3] and the moduli spaces of stable maps [7–9].

We are interested in running Mori's program for $\overline{\mathrm{M}}_{0,n}$, the moduli space of stable n-pointed rational curves. Since $\dim \mathrm{N}^1(\overline{\mathrm{M}}_{0,n})_{\mathbb{Q}}$ grows exponentially, it is almost

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impossible to determine all birational models even for very small n. But if we restrict ourselves to the space $\mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}_{\mathbb{Q}}$ of S_n-invariant divisors (or symmetric divisors), then the dimension grows linearly. Thus, we can try to classify all birational models appearing in Mori's program at least for small n.

The first non-trivial case is $n = 6$, and it was investigated in [43]. In this case, there are two divisorial contractions and no flips. These two contractions are classically well-known varieties, the Segre cubic and Igusa quartic. The next case $n = 7$, which we study in this paper, is interesting because there are two flips of $\overline{\mathrm{M}}_{0,7}$. It seems that in the literature, there has been no description of these spaces.

1.1 The First Main Result: Mori's Program

In the first half of this paper, we classify all projective models appearing in Mori's program. In this case $\dim \mathrm{N}^1(\overline{\mathrm{M}}_{0,7})^{S_7}_{\mathbb{Q}} = 2$ and $\mathrm{Eff}(\overline{\mathrm{M}}_{0,7})^{S_7}$ is generated by two boundary divisors B_2 and B_3. To describe the result in an efficient way, we use the interval notation for divisor classes. For two divisor classes D_1 and D_2, $[D_1, D_2)$ is the set of all divisor classes $aD_1 + bD_2$ where $a \geq 0$ and $b > 0$. Similarly, we can define (D_1, D_2), $(D_1, D_2]$, and $[D_1, D_2]$ as well. All divisor classes below are defined in Section 2. We describe the flipping locus B_2^3 and B_2^2 later in this section.

Theorem 1.1 *Let D be a symmetric effective divisor of $\overline{\mathrm{M}}_{0,7}$.*

(i) *If $D \in (\psi - K_{\overline{\mathrm{M}}_{0,7}}, K_{\overline{\mathrm{M}}_{0,7}} + \frac{1}{3}\psi)$, then $\overline{\mathrm{M}}_{0,7}(D) \cong \overline{\mathrm{M}}_{0,7}$.*

(ii) *If $D \in [K_{\overline{\mathrm{M}}_{0,7}} + \frac{1}{3}\psi, B_3)$, $\overline{\mathrm{M}}_{0,7}(D) \cong \overline{\mathrm{M}}_{0,A}$, the moduli space of weighted pointed stable curves with weight $A = \left(\frac{1}{3}, \dots, \frac{1}{3}\right)$.*

(iii) *If $D = \psi - K_{\overline{\mathrm{M}}_{0,7}}$, then $\overline{\mathrm{M}}_{0,7}(D)$ is isomorphic to the Veronese quotient V_A^3, where $A = \left(\frac{4}{7}, \dots, \frac{4}{7}\right)$.*

(iv) *If $D \in (\psi - 3K_{\overline{\mathrm{M}}_{0,7}}, \psi - K_{\overline{\mathrm{M}}_{0,7}})$, then $\overline{\mathrm{M}}_{0,7}(D) \cong \overline{\mathrm{M}}_{0,7}^3$, which is a flip of $\overline{\mathrm{M}}_{0,7}$ over V_A^3. The flipping locus is B_2^3.*

(v) *If $D = \psi - 3K_{\overline{\mathrm{M}}_{0,7}}$, then $\overline{\mathrm{M}}_{0,7}(D)$ is a small contraction of $\overline{\mathrm{M}}_{0,7}^3$.*

(vi) *If $D \in (\psi - 5K_{\overline{\mathrm{M}}_{0,7}}, \psi - 3K_{\overline{\mathrm{M}}_{0,7}})$, then $\overline{\mathrm{M}}_{0,7}(D) \cong \overline{\mathrm{M}}_{0,7}^2$, which is a flip of $\overline{\mathrm{M}}_{0,7}^3$ over $\overline{\mathrm{M}}_{0,7}(\psi - 3K_{\overline{\mathrm{M}}_{0,7}})$. The flipping locus is the proper transform of B_2^2.*

(vii) *If $D \in (B_2, \psi - 5K_{\overline{\mathrm{M}}_{0,7}}]$, then $\overline{\mathrm{M}}_{0,7}(D) \cong \overline{\mathrm{M}}_{0,7}^1$, which is a divisorial contraction of $\overline{\mathrm{M}}_{0,7}^2$. The contracted divisor is the proper transform of B_2.*

(viii) *If $D = B_2$ or B_3, then $\overline{\mathrm{M}}_{0,7}(D)$ is a point.*

Some of these results are already well known. The birational models in (i)–(iii) are models appearing in [16, 25], and they have certain moduli theoretic meaning. Also, Mori's program for $\overline{\mathrm{M}}_{0,n}$ for a subcone generated by $K_{\overline{\mathrm{M}}_{0,n}}$ and $B = \sum B_i$ has been intensively studied in [1, 11, 33, 48] for arbitrary n. For $n = 7$, this subcone covers (i) and (ii). Thus, the new result is the opposite direction, (iii)–(vii).

Along this direction, the chain of birational maps $\overline{\mathrm{M}}_{0,7} \dashrightarrow \overline{\mathrm{M}}_{0,7}^3 \dashrightarrow \overline{\mathrm{M}}_{0,7}^2 \to \overline{\mathrm{M}}_{0,7}^1$ shows interesting toroidal birational modifications. On $\overline{\mathrm{M}}_{0,7}$, B_2 is a simple normal crossing divisor and at most three irreducible components meet together. Let B_2^i be the union of nonempty intersections of i irreducible components of B_2. For $\overline{M}_{0,7} \dashrightarrow \overline{M}_{0,7}^3$, B_2^3 is the flipping locus and on $\overline{M}_{0,7}^3$ no three irreducible components of B_2 intersect. For $\overline{M}_{0,7}^3 \dashrightarrow \overline{M}_{0,7}^2$, the flipping locus is the proper transform of B_2^2 and on $\overline{M}_{0,7}^2$, the irreducible components of B_2 are disjoint. Finally, on $\overline{M}_{0,7}^2 \to \overline{M}_{0,7}^1$, the modified locus is the proper transform of $B_2^1 = B_2$, the disjoint union of irreducible components, and it is a divisorial contraction.

Very recently, Castravet and Tevelev proved in [6] that $\overline{M}_{0,n}$ is not a Mori dream space if $n \geq 134$. This result was improved to $n \geq 13$ by Gonzalez and Karu [18]. However, since the effective cone of $\overline{M}_{0,n}/S_n$ is simplicial and generated by boundary divisors B_i for $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$, it is believed that $\overline{M}_{0,n}/S_n$ is a Mori dream space. Because Mori's program of $\overline{M}_{0,n}$ with symmetric divisors can be identified with that of $\overline{M}_{0,n}/S_n$ ([43, Lemma 6.1]), we obtain the following result.

Corollary 1.2 *The S_7-quotient $\overline{M}_{0,7}/S_7$ is a Mori dream space.*

In general, we expect that the symmetric cone $\mathrm{Eff}(\overline{M}_{0,n}) \cap \mathrm{N}^1(\overline{M}_{0,n})_{\mathbb{Q}}^{S_n}$ is in the Mori dream region, so while running Mori's program with symmetric divisors, there is no fundamental technical obstruction. In particular, we expect that the answer for the following question, due to Hu and Keel, is affirmative.

Question 1.3 ([29, Implication 3.3]) For each $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$, is there a rational contraction $\overline{M}_{0,n} \dashrightarrow \mathrm{M}(k)$ that contracts all boundary divisors except B_k?

For $n \geq 7$, the only previously known such model was $\mathrm{M}(2)$, which is $(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$ ([33]). The space $\overline{M}_{0,7}^1$ provides $\mathrm{M}(3)$ when $n = 7$.

1.2 The Second Main Result: Modular Interpretation

So far, all modular birational models of $\overline{\mathcal{M}}_{g,n}$ have been constructed in two ways. One way is taking GIT quotients of certain parameter spaces of pointed curves embedded in a projective space by using Chow varieties or Hilbert schemes, and the other way is taking an open proper substack of the stack of all pointed curves. Those two approaches are completely different, but the outcome is essentially moduli spaces of (pointed) curves with worse singularities. For instance, the moduli space $\overline{\mathcal{M}}_g^{ps}$ of pseudostable curves ([47]) can be obtained by allowing cuspidal singularities instead of elliptic tails. By replacing a certain type of subcurves by a cetain type of Gorenstein singularities, we can obtain many other birational models. See [2] for a systematic approach for curves without marked points. Hassett's moduli spaces of weighted stable curves $\overline{\mathcal{M}}_{g,A}$ are also moduli spaces of semi log canonical pairs (see Section 4.1.), so they are moduli spaces of pointed curves with certain types of singularities of pairs as well.

Recently, in [49], Smyth gave a partial classification of possible modular birational models of $\overline{\mathcal{M}}_{g,n}$, which are moduli spaces of curves with certain singularity types. When $g = 0$, his result gives a complete classification. One interesting fact is that all of his birational models are contractions of $\overline{M}_{0,n}$, because there is no positive dimensional moduli of singularities of arithmetic genus zero. Therefore, if one wants to impose a moduli theoretic interpretation of a flip of $\overline{\mathrm{M}}_{0,n}$, then it must *not* be a moduli space of "pointed curves", in the sense of pairs of an abstract curve and a collection of dimension 0 subvarieties.

In the second half of this paper, we give a moduli theoretic meaning to the first flip $\overline{\mathrm{M}}_{0,7}^3$. The main observation is that both $\overline{\mathrm{M}}_{0,7}$ and V_A^3 are constructed as GIT quotients (Remark 4.4), and there is a commutative diagram in Figure 1.

$$\begin{tikzcd}
\overline{\mathrm{M}}_{0,7}(\mathbb{P}^3,3) \arrow[rr, dashed, leftrightarrow] \arrow[dr] \arrow[dd, "//\mathrm{SL}_4"] & & \square \arrow[dl] \arrow[dd, "//\mathrm{SL}_4"] \\
& I \subset \overline{\mathrm{M}}_{0,0}(\mathbb{P}^3,3) \times (\mathbb{P}^3)^7 \arrow[dd, "//\mathrm{SL}_4"] & \\
\overline{\mathrm{M}}_{0,7} \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & \overline{\mathrm{M}}_{0,7}^3 \arrow[dl] \\
& V_A^3 &
\end{tikzcd}$$

Figure 1: SL_4-quotients of incidence varieties

The variety I is the incidence variety in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3,3) \times (\mathbb{P}^3)^7$, where $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3,3)$ is the moduli space of stable maps ([39]). All vertical maps are SL_4-GIT quotients with certain linearizations (thus they are not regular maps.). So we can guess that there is a parameter space X in the node $\square$ such that

(a) there is a functorial morphism $X \to \overline{\mathrm{M}}_{0,0}(\mathbb{P}^3,3) \times (\mathbb{P}^3)^7$;
(b) there is an "incidence variety" $J \subset X$ with SL_4-action;
(c) with an appropriate linearization, $J/\!/\mathrm{SL}_4 \cong \overline{\mathrm{M}}_{0,7}^3$.

Let $\overline{\mathcal{U}}_{0,n}(\mathbb{P}^r,d)$ be the moduli stack of unramified stable maps introduced in [35], and let $\overline{\mathrm{U}}_{0,n}(\mathbb{P}^r,d)$ be the coarse moduli space. By analyzing the difference between $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)$ and $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3,3)$ carefully, we will show that $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3) \times (\mathbb{P}^3)^7$ has the role of X.

Unfortunately, there are just a few known geometric properties of $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)$. For instance, it is not irreducible, and the connectivity and projectivity of the coarse moduli space are unknown. Therefore, the standard GIT approach is unavailable. Instead of that, we introduce a "stable locus" J^s of J and show that J^s/SL_4 is a projective variety that is isomorphic to $\overline{\mathrm{M}}_{0,7}^3$. We will denote J^s/SL_4 by a "formal GIT quotient" $J/\!/\mathrm{SL}_4$, because if we know the projectivity of $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)$, then J^s/SL_4 is indeed isomorphic to $J/\!/\mathrm{SL}_4$ with a standard choice of linearization.

Theorem 1.4 (Theorem 6.8) *The formal GIT quotient $J/\!/\mathrm{SL}_4$ is isomorphic to $\overline{\mathrm{M}}_{0,7}^3$.*

By using this result, we are able to give a modular description of $\overline{\mathrm{M}}_{0,7}^3$. As we mentioned before, it is not a space of pointed curves anymore. It is a parameter space of data $(C, (x_1, x_2, \dots, x_7), C')$, where $(C, x_1, x_2, \dots, x_7)$ is an element of V_A^3, which is an arithmetic genus zero pointed curve with a certain stability condition ([16, Theorem 5.1]), and C' is a *ghost curve*, which is a curve on a non-rigid compactified tangent space $\mathbb{P}(T_x C \oplus \mathbb{C})$ for a non-Gorenstein singularity $x \in C$. For the precise definition, see Sections 5 and 6. The same type of flip appears for Mori's program for all $n \geq 7$ (Remark 6.10). Thus, we believe that to run Mori's program for $\overline{\mathrm{M}}_{0,n}$, it is essential to understand the geometry of $\overline{\mathcal{U}}_{0,n}(\mathbb{P}^d, d)$. We will study geometric properties of this relatively new moduli space in forthcoming papers.

1.3 Structure of the Paper

In Section 2 we recall the definitions of several divisor classes and curve classes on $\overline{\mathrm{M}}_{0,n}$ with their numerical properties. In Section 3, we compute the stable base locus for every symmetric effective divisor on $\overline{\mathrm{M}}_{0,7}$. In Section 4 we prove Theorem 1.1. Section 5 reviews the moduli space of unramified stable maps and its geometric properties. Finally in Section 6, we prove Theorem 1.4. We will work over the complex numbers $\mathbb{C}$.

2 Divisors and Curves on $\overline{\mathrm{M}}_{0,n}$

In this section, we review general facts about divisors and curves on $\overline{\mathrm{M}}_{0,n}$. All material in this section is well known, but we collect the statements we will use for the reader's convenience.

2.1 Divisors on $\overline{\mathrm{M}}_{0,n}$

The moduli space $\overline{\mathrm{M}}_{0,n}$ inherits a natural S_n action permuting the marked points. A divisor D on $\overline{\mathrm{M}}_{0,n}$ is called *symmetric* if it is invariant under the S_n action. The Neron–Severi vector space $\mathrm{N}^1(\overline{\mathrm{M}}_{0,n})_{\mathbb{Q}}$ has dimension $2^{n-1} - \binom{n}{2} - 1$, so the space of divisors on $\overline{\mathrm{M}}_{0,n}$ is quite huge. But the S_n-invariant part

$$\mathrm{N}^1(\overline{\mathrm{M}}_{0,n})_{\mathbb{Q}}^{S_n} \cong \mathrm{N}^1(\overline{\mathrm{M}}_{0,n}/S_n)_{\mathbb{Q}}$$

of $\mathrm{N}^1(\overline{\mathrm{M}}_{0,n})_{\mathbb{Q}}$ is $\lfloor n/2 \rfloor - 1$ dimensional ([32, Theorem 1.3]) so at least for small n, computations on the space are feasible. The following is a list of tautological divisors on $\overline{\mathrm{M}}_{0,n}$.

Definition 2.1 (i) For $I \subset [n] = \{1, 2, \dots, n\}$ with $2 \leq |I| \leq n-2$, let B_I be the closure of the locus of pointed curves $(C, x_1, \dots, x_n)$ with two irreducible components C_1 and C_2 such that C_1 (resp. C_2) contains x_i for $i \in I$ (resp. $i \in I^c$). Then B_I is called a boundary divisor. By definition, $B_I = B_{I^c}$. For $2 \leq i \leq n-2$, let $B_i = \bigcup_{|I|=i} B_I$. Then B_i is a symmetric divisor and $B_i = B_{n-i}$. Finally, let $B = \sum_{i=2}^{\lfloor n/2 \rfloor} B_i$.

(ii) Fix $1 \leq i \leq n$. Let $\mathbb{L}_i$ be the line bundle on $\overline{\mathrm{M}}_{0,n}$ such that over $(C, x_1, \dots, x_n) \in \overline{\mathrm{M}}_{0,n}$, the fiber is Ω_{C,x_i}, the cotangent space of C at x_i. Let $\psi_i = c_1(\mathbb{L}_i)$, the i-th psi class. If we denote $\psi = \sum_{i=1}^{n} \psi_i$, then ψ is a symmetric divisor.

(iii) Let $K_{\overline{\mathrm{M}}_{0,n}}$ be the canonical divisor of $\overline{\mathrm{M}}_{0,n}$. Obviously, it is symmetric.

The symmetric effective cone $\mathrm{Eff}(\overline{\mathrm{M}}_{0,n})^{S_n} \cong \mathrm{Eff}(\overline{\mathrm{M}}_{0,n}/S_n)$, which is $\mathrm{Eff}(\overline{\mathrm{M}}_{0,n}) \cap \mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}_{\mathbb{Q}}$, is generated by symmetric boundary divisors ([32, Theorem 1.3]). Therefore, we can write $K_{\overline{\mathrm{M}}_{0,n}}$ and ψ as nonnegative linear combinations of boundary divisors.

Lemma 2.2 ([46, Proposition 2], [41, Lemma 2.9]) *On $\mathrm{N}^1(\overline{\mathrm{M}}_{0,n})_{\mathbb{Q}}$, the following relations hold:*

(i)
$$K_{\overline{\mathrm{M}}_{0,n}} = \sum_{i=2}^{\lfloor n/2 \rfloor} \Big(\frac{i(n-i)}{n-1} - 2 \Big) B_i,$$

(ii)
$$\psi = K_{\overline{\mathrm{M}}_{0,n}} + 2B.$$

2.2 Curves on $\overline{\mathrm{M}}_{0,n}$

Let $I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4 = [n]$ be a partition. Let F_{I_1,I_2,I_3,I_4} be the F-curve class corresponding to the partition ([32, Section 4]).

Lemma 2.3 ([32]) *Let $F = F_{I_1,I_2,I_3,I_4}$ be an F-curve and let B_J be a boundary divisor.*

(i)
$$F \cdot B_J = \begin{cases} 1, & J = I_i \cup I_j \text{ for some } i \neq j, \\ -1, & J = I_i \text{ for some } i, \\ 0, & \text{otherwise.} \end{cases}$$

(ii)
$$F \cdot \psi_i = \begin{cases} 1, & I_j = \{i\} \text{ for some } j, \\ 0, & \text{otherwise.} \end{cases}$$

If we consider symmetric divisors only, then the intersection numbers do not depend on a specific partition but depend only on the size of the partition. A curve class F_{a_1,a_2,a_3,a_4} is one of any F-curve classes F_{I_1,I_2,I_3,I_4} with $a_i = |I_i|$.

To compute the stable base locus in Section 3, we need to use other curve classes C_j (see [32, Lemma 4.8]). Fix a j-pointed $\mathbb{P}^1$ and let x be an additional moving point on $\mathbb{P}^1$. By gluing a fixed $(n-j+1)$-pointed $\mathbb{P}^1$ whose last marked point is y to the $(j+1)$-pointed $\mathbb{P}^1$ along x and y and stabilizing it, we obtain a one parameter family of n-pointed stable curves over $\mathbb{P}^1$, *i.e.*, a curve $C_j \cong \mathbb{P}^1$ on $\overline{\mathrm{M}}_{0,n}$.

Lemma 2.4 ([32, Lemma 4.8])

$$C_j \cdot B_i = \begin{cases} j, & i = j-1, \\ -(j-2), & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.5 We are able to generalize the idea of this construction. For example, by 1) gluing two 3-pointed $\mathbb{P}^1$ to $(n-2)$-pointed $\mathbb{P}^1$, 2) varying one of two attached points, and 3) stabilizing it, we get a one parameter family of n-pointed stable curves over $\mathbb{P}^1$. Let $A \subset \overline{\mathrm{M}}_{0,7}$ be such a curve class.

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2.3 Numerical Results on $\overline{\mathrm{M}}_{0,7}$

For the reader's convenience, we state a special case of $\overline{\mathrm{M}}_{0,7}$ below. All results are combinations of the lemmas in previous sections.

Corollary 2.6 *The symmetric Neron-Severi space $\mathrm{N}^1(\overline{\mathrm{M}}_{0,7})^{S_7}_{\mathbb{Q}}$ has dimension two. The symmetric effective cone $\mathrm{Eff}(\overline{\mathrm{M}}_{0,7})^{S_7}$ is generated by B_2 and B_3. Moreover,*

(i) $K_{\overline{\mathrm{M}}_{0,7}} = -\frac{1}{3}B_2$,
(ii) $\psi = \frac{5}{3}B_2 + 2B_3$,
(iii) $B_2 = -3K_{\overline{\mathrm{M}}_{0,7}}$,
(iv) $B_3 = \frac{5}{2}K_{\overline{\mathrm{M}}_{0,7}} + \frac{1}{2}\psi$.

We can summarize Corollary 2.6 with Figure 2.

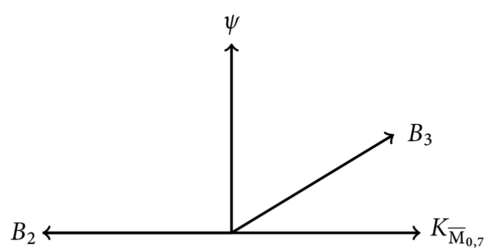

Figure 2: Neron–Severi space of $\overline{\mathrm{M}}_{0,7}$

Corollary 2.7 *On $\overline{\mathrm{M}}_{0,7}$, the intersection of symmetric divisors and curve classes are given by Table 1.*

	ψ	$K_{\overline{\mathrm{M}}_{0,7}}$	B_2	B_3
$F_{1,1,1,4}$	3	-1	3	-1
$F_{1,1,2,3}$	2	0	0	1
$F_{1,2,2,2}$	1	1	-3	3
C_4	4	0	0	2
C_5	5	1	-3	5
C_6	10	-2	6	0
A	3	1	-3	4

Table 1: Intersection numbers on $\overline{\mathrm{M}}_{0,7}$

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3 Stable Base Locus Decomposition

For an effective divisor D, the stable base locus $\mathbf{B}(D)$ is defined as

$$\mathbf{B}(D) = \bigcap_{m\geq 0} \mathrm{Bs}(mD),$$

where $\mathrm{Bs}(D)$ is the set-theoretical base locus of D. As a first step toward Mori's program, we will compute stable base locus decompositions of $\overline{\mathrm{M}}_{0,7}$, which is a first approximation of the chamber decompositions for different birational models.

Definition 3.1 Let B_2^i be the union of intersections of i distinct irreducible components of B_2.

Since B is a simple normal crossing divisor, B_2^i is a union of smooth varieties of codimension i. Moreover, the singular locus of B_2^i is exactly B_2^{i+1}. On $\overline{\mathrm{M}}_{0,7}$, B_2^4 is an emptyset, B_2^3 is the union of all F-curves of type $F_{1,2,2,2}$. Each irreducible component of B_2^2 is isomorphic to $\overline{\mathrm{M}}_{0,5}$. Finally, $B_2^1 = B_2$.

Proposition 3.2 *Let D be a symmetric effective divisor on $\overline{\mathrm{M}}_{0,7}$.*

(i) *If $D \in [\psi - K_{\overline{\mathrm{M}}_{0,7}}, K_{\overline{\mathrm{M}}_{0,7}} + \frac{1}{3}\psi]$, then D is semi-ample.*
(ii) *If $D \in (K_{\overline{\mathrm{M}}_{0,7}} + \frac{1}{3}\psi, B_3]$, then $\mathbf{B}(D) = B_3$.*
(iii) *If $D \in [\psi - 3K_{\overline{\mathrm{M}}_{0,7}}, \psi - K_{\overline{\mathrm{M}}_{0,7}})$, then $\mathbf{B}(D) = B_2^3$.*
(iv) *If $D \in [\psi - 5K_{\overline{\mathrm{M}}_{0,7}}, \psi - 3K_{\overline{\mathrm{M}}_{0,7}})$, then $\mathbf{B}(D) = B_2^2$.*
(v) *If $D \in [B_2, \psi - 5K_{\overline{\mathrm{M}}_{0,7}})$, then $\mathbf{B}(D) = B_2$.*

Proof By [32, Theorem 1.2] and Corollary 2.7, the nef cone of $\overline{\mathrm{M}}_{0,7}$ is generated by $\psi - K_{\overline{\mathrm{M}}_{0,7}}$ and $K_{\overline{\mathrm{M}}_{0,7}} + \frac{1}{3}\psi$. Moreover, $K_{\overline{\mathrm{M}}_{0,7}} + \frac{1}{3}\psi$ is the pull-back of an ample divisor on $\overline{\mathrm{M}}_{0,A}$ where $A = (\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3})$ (See the proof of [41, Theorem 3.1]. In particular, the right-hand side of [41, Equation (7)] is zero.). The opposite extremal ray $\psi - K_{\overline{\mathrm{M}}_{0,7}}$ is also semi-ample. Indeed, by comparing the intersection numbers, it is straightforward that $\psi - K_{\overline{\mathrm{M}}_{0,7}}$ is proportional to the pull-back of the canonical polarization on the Veronese quotient V_A^3 where $A = (\frac{3}{7}, \dots, \frac{3}{7})$ ([17, Theorem 2.1]). Therefore the two endpoints of this interval, and hence all divisors in the interval, are semi-ample divisors. If $D \in (K_{\overline{\mathrm{M}}_{0,7}} + \frac{1}{3}\psi, B_3]$, then $\mathbf{B}(D) \subset B_3$ since $K_{\overline{\mathrm{M}}_{0,7}} + \frac{1}{3}\psi$ is semi-ample and D is an effective linear combination of $K_{\overline{\mathrm{M}}_{0,7}} + \frac{1}{3}\psi$ and B_3. By Corollary 2.7, $F_{1,1,1,4} \cdot D < 0$ so $F_{1,1,1,4} \subset \mathbf{B}(D)$. Since $F_{1,1,1,4}$ covers an open dense subset of B_3, $\mathbf{B}(D) = B_3$. If $D \in [B_2, \psi - K_{\overline{\mathrm{M}}_{0,7}})$, then $\mathbf{B}(D) \subset B_2$ by a similar reason. By Corollary 2.7, $F_{1,2,2,2} \cdot D < 0$ if $D \in [B_2, \psi - K_{\overline{\mathrm{M}}_{0,7}})$, thus $F_{1,2,2,2} \subset \mathbf{B}(D)$. If $D \in [B_2, \psi - 3K_{\overline{\mathrm{M}}_{0,7}})$, $A \cdot D < 0$ and A covers a dense open subset of B_2^2. Thus $B_2^2 \subset \mathbf{B}(D)$. Finally, if $D \in [B_2, \psi - 5K_{\overline{\mathrm{M}}_{0,7}})$, $C_5 \cdot D < 0$. Since C_5 covers an open dense subset of B_2, $B_2 \subset \mathbf{B}(D)$. In particular, we obtain Item (5).

Now it suffices to show that $\mathbf{B}(D) \subset B_2^3$ if $D \in [\psi - 3K_{\overline{\mathrm{M}}_{0,7}}, \psi - K_{\overline{\mathrm{M}}_{0,7}})$ and $\mathbf{B}(D) \subset B_2^2$ if $[\psi - 5K_{\overline{\mathrm{M}}_{0,7}}, \psi - 3K_{\overline{\mathrm{M}}_{0,7}})$. Let B_I be an irreducible component of B_2 and let B_J be an irreducible component of B_3 such that $B_I \cap B_J \neq \varnothing$. For $E = 5B_2 + 3B_3 = \frac{3}{2}(\psi - 5K_{\overline{\mathrm{M}}_{0,7}})$, by using Keel's relations ([31, p. 550]) and a computer algebra system, we can find a divisor $E' \in |E|$ such that E' is a non-negative integral linear combination of boundary divisors such that the coefficients of B_I and B_J are zero. For example, if $I = \{1,2\}$ and $J = \{3,4,5\}$, then

$$\begin{aligned}E \equiv\ & 12B_{\{1,4\}} + 9\big(B_{\{2,5\}} + B_{\{2,6\}} + B_{\{5,6\}}\big)\\ & + 6\big(B_{\{1,3\}} + B_{\{1,7\}} + B_{\{2,3\}} + B_{\{2,7\}} + B_{\{3,4\}} + B_{\{3,7\}} + B_{\{4,7\}}\big)\\ & + 3\big(B_{\{1,5\}} + B_{\{1,6\}} + B_{\{3,5\}} + B_{\{3,6\}} + B_{\{4,5\}} + B_{\{4,6\}} + B_{\{5,7\}} + B_{\{6,7\}}\big)\\ & + 15B_{\{2,5,6\}} + 12\big(B_{\{1,4,7\}} + B_{\{1,3,4\}}\big)\\ & + 6\big(B_{\{1,3,7\}} + B_{\{1,4,5\}} + B_{\{1,4,6\}} + B_{\{2,3,5\}} + B_{\{2,3,6\}}\\ & \quad + B_{\{2,3,7\}} + B_{\{2,5,7\}} + B_{\{2,6,7\}} + B_{\{3,4,7\}}\big)\\ & + 3\big(B_{\{1,5,6\}} + B_{\{3,5,6\}} + B_{\{4,5,6\}} + B_{\{5,6,7\}}\big).\end{aligned}$$

Similarly, if $I = \{1,2\}$ and $J = \{1,2,3\}$, then

$$\begin{aligned}E \equiv\ & 12B_{\{1,4\}} + 9\big(B_{\{2,6\}} + B_{\{2,7\}} + B_{\{6,7\}}\big)\\ & + 6\big(B_{\{1,3\}} + B_{\{1,5\}} + B_{\{2,3\}} + B_{\{2,5\}} + B_{\{3,4\}} + B_{\{3,5\}} + B_{\{4,5\}}\big)\\ & + 3\big(B_{\{1,6\}} + B_{\{1,7\}} + B_{\{3,6\}} + B_{\{3,7\}} + B_{\{4,6\}} + B_{\{4,7\}} + B_{\{5,6\}} + B_{\{5,7\}}\big)\\ & + 15B_{\{2,6,7\}} + 12\big(B_{\{1,3,4\}} + B_{\{1,4,5\}}\big)\\ & + 6\big(B_{\{1,3,5\}} + B_{\{1,4,6\}} + B_{\{1,4,7\}} + B_{\{2,3,5\}} + B_{\{2,3,6\}}\\ & \quad + B_{\{2,3,7\}} + B_{\{2,5,6\}} + B_{\{2,5,7\}} + B_{\{3,4,5\}}\big)\\ & + 3\big(B_{\{1,6,7\}} + B_{\{3,6,7\}} + B_{\{4,6,7\}} + B_{\{5,6,7\}}\big).\end{aligned}$$

These two cases cover all possibilities where $B_I \cap B_J \neq \varnothing$ up to the S_7-action. Thus, the support of E' does not contain a general point of B_I and a general point of $B_I \cap B_J$. Therefore $\mathbf{B}(E)$ must be contained in B_2^2. Since $\psi - K_{\overline{\mathrm{M}}_{0,7}}$ is semi-ample, for all divisors $D \in [\psi - 5K_{\overline{\mathrm{M}}_{0,7}}, \psi - K_{\overline{\mathrm{M}}_{0,7}})$, $\mathbf{B}(D) \subset B_2^2$ and (iv) is proved.

Finally, let B_I, B_K be two irreducible components of B_2 whose intersection is non-empty. For $F = 4B_2 + 3B_3 = \frac{3}{2}(\psi - 3K_{\overline{\mathrm{M}}_{0,7}})$, by using a similar idea, we can find a divisor $F' \in |F|$ such that F' is a non-negative integral linear combination of boundary divisors such that the coefficients of B_I and B_K are zero. Indeed, if $I = \{1,2\}$ and $K = \{3,4\}$, then

$$\begin{aligned}F \equiv\ & 12B_{\{1,3\}} + 9\big(B_{\{2,4\}} + B_{\{2,6\}} + B_{\{4,6\}}\big)\\ & + 6\big(B_{\{1,5\}} + B_{\{1,7\}} + B_{\{3,5\}} + B_{\{3,7\}}\big)\\ & + 3\big(B_{\{2,5\}} + B_{\{2,7\}} + B_{\{4,5\}} + B_{\{4,7\}} + B_{\{5,6\}} + B_{\{5,7\}} + B_{\{6,7\}}\big)\\ & + 18B_{\{2,4,6\}} + 15\big(B_{\{1,3,5\}} + B_{\{1,3,7\}}\big) +\end{aligned}$$

$$+6\big(B_{\{1,5,7\}}+B_{\{2,4,5\}}+B_{\{2,4,7\}}+B_{\{2,5,6\}}+B_{\{2,6,7\}}+B_{\{3,5,7\}}$$
$$+B_{\{4,5,6\}}+B_{\{4,6,7\}}\big)$$
$$+3\big(B_{\{1,2,3\}}+B_{\{1,3,4\}}+B_{\{1,3,6\}}\big).$$

Thus, a general point of B_2^2 is not contained in $\mathbf{B}(F)$. The only remaining locus in B_2 is B_2^3. Hence, $\mathbf{B}(F)\subset B_2^3$ and the same holds for all $D\in[\psi-3K_{\overline{\mathrm{M}}_{0,7}},\psi-K_{\overline{\mathrm{M}}_{0,7}})$. ■

We summarize the above result as Figure 3.

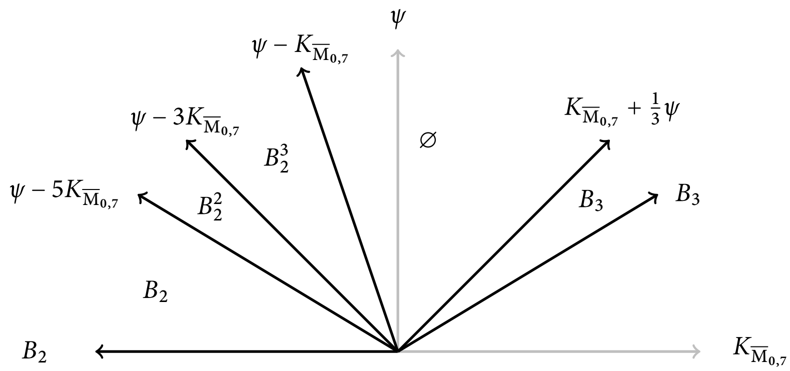

Figure 3: Stable base locus decomposition of $\overline{\mathrm{M}}_{0,7}$

4 Mori's Program for $\overline{\mathrm{M}}_{0,7}$

In this section, we present the first main theorem (Theorem 1.1) of this paper. Before proving it, we describe some moduli spaces appearing in the theorem.

4.1 Moduli of Weighted Pointed Stable Curves

The moduli space $\overline{\mathrm{M}}_{0,A}$ of weighted pointed stable curves, in (ii), is constructed in [25]. For a collection of positive rational numbers (so called weight data) $A=(a_1,a_2,\dots,a_n)$ with $0<a_i\le 1$ and $\sum a_i>2$, there is a fine moduli space of pointed curves $(C,x_1,\dots,x_n)$ such that

- C is a reduced, connected projective curve with $p_a(C)=0$;
- $(C,\sum a_ix_i)$ is a semi-log canonical pair;
- $\omega_C+\sum a_ix_i$ is ample.

In contrast to $\overline{\mathrm{M}}_{0,n}$, for a subset $I\subset[n]$, if $\sum_{i\in I}a_i\le 1$, then $\{x_i\}_{i\in I}$ may collide at a smooth point of C. But because of the last condition, each tail of C has sufficiently many marked points in the sense that their weight sum is greater than one. Also note that $\overline{\mathrm{M}}_{0,n}=\overline{\mathrm{M}}_{0,(1,1,\dots,1)}$.

The moduli space $\overline{\mathrm{M}}_{0,A}$ is smooth and birational to $\overline{\mathrm{M}}_{0,n}$. Furthermore, there is a reduction map $\rho_A\colon\overline{\mathrm{M}}_{0,n}\to\overline{\mathrm{M}}_{0,A}$ for any weight data, which is a divisorial contraction. The map ρ_A sends a pointed curve $(C, x_1, x_2, \dots, x_n)$ to a new curve $(\overline{C}, \overline{x}_1, \overline{x}_2, \dots, \overline{x}_n)$, which is obtained by contracting all tails with weight sums ≤ 1 to the attaching point.

Example 4.1 For the case where $n = 7$ and $A = (\frac{1}{3}, \dots, \frac{1}{3})$, ρ_A is the contraction of B_3. A general point $(C_1 \cup C_2, x_1, x_2, \dots, x_7)$ has a tail with three marked points. Then the sum is precisely one, so the tail is contracted to a point. Note that it forgets the cross ratio of three marked points and a nodal point. Thus, the image of B_3 is a codimension two subvariety of $\overline{\mathrm{M}}_{0,A}$. Figure 4 shows the contraction. The number on a marked point is the multiplicity.

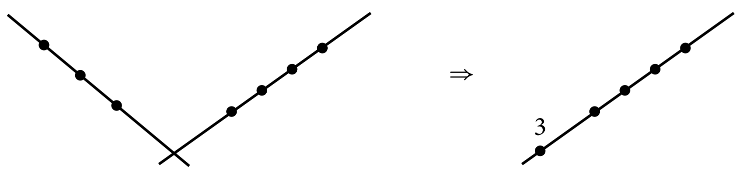

Figure 4: The reduction map $\rho_A \colon \overline{\mathrm{M}}_{0,7} \to \overline{\mathrm{M}}_{0,A}$ where $A = (\frac{1}{3}, \dots, \frac{1}{3})$

4.2 Veronese Quotients

The Veronese quotients, V_A^d in (iii), and their geometric properties have been studied in [15–17]. Originally, they were constructed as GIT quotients of an incidence variety coming from the Chow varieties of curves and points in $\mathbb{P}^d$.

Let $\mathrm{Chow}_{1,d}(\mathbb{P}^d)$ be the irreducible component of the Chow variety that parametrizes rational normal curves and their degenerations. Consider the incidence variety

$$I := \left\{ (C, x_1, \dots, x_n) \in \mathrm{Chow}_{1,d}(\mathbb{P}^d) \times (\mathbb{P}^d)^n \mid x_i \in C \right\}.$$

There is a natural SL_{d+1}-action on I and $\mathrm{Chow}_{1,d}(\mathbb{P}^d) \times (\mathbb{P}^d)^n$. There is also a canonical polarization $\mathcal{O}_{\mathrm{Chow}}(1)$ on $\mathrm{Chow}_{1,d}(\mathbb{P}^d)$. For a sequence of nonnegative rational numbers $(\gamma, a_1, a_2, \dots, a_n)$, define a $\mathbb{Q}$-polarization on I that is the pull-back of

$$L_A := \mathcal{O}_{\mathrm{Chow}}(\gamma) \otimes \mathcal{O}(a_1) \otimes \cdots \otimes \mathcal{O}(a_n)$$

on $\mathrm{Chow}_{1,d}(\mathbb{P}^d) \times (\mathbb{P}^d)^n$. We will normalize the linearization by imposing a numerical condition $(d-1)\gamma + \sum a_i = d + 1$. Thus, γ is determined by $A := (a_1, a_2, \dots, a_n)$ and d. If $0 < a_i < 1$ and $2 < \sum a_i \le d + 1$ (hence $0 \le \gamma < 1$), then the semistable locus I^{ss} is nonempty ([16, Proposition 2.10]), so we obtain a nonempty GIT quotient $V_A^d := I /\!\!/_{L_A} \mathrm{SL}_{d+1}$.

Remark 4.2 A simple observation on the semistability is that every stable curve is non-degenerate. A non-degenerate degree d curve in $\mathbb{P}^d$ has several nice geometric properties: (1) Every connected subcurve of degree e spans $\mathbb{P}^e \subset \mathbb{P}^d$, and (2) all singularities are analytically locally the union of coordinate axes in some $\mathbb{C}^k$ ([16, Corollary 2.4]).

For simplicity, consider general polarizations such that $I^{ss} = I^s$. These quotients have modular interpretations, as moduli spaces of stable polarized pointed curves. For a precise definition and proof, consult [16, Section 5.1].

For any weight data A and $d > 0$, there is a reduction map $\phi\colon \overline{\mathrm{M}}_{0,n} \to V_A^d$ ([16, Theorem 1.1]), which preserves $\mathrm{M}_{0,n}$. For each (possibly reducible) connected tail C' of $(C, x_1, x_2, \dots, x_n) \in \overline{\mathrm{M}}_{0,n}$, we may define a numerical value

$$\sigma(C') := \min\left\{ \max\left\{ \left\lceil \frac{\sum_{x_i \in C'} a_i - 1}{1-\gamma} \right\rceil, 0 \right\}, d \right\}.$$

Because the dual graph of C is a tree, we can define $\sigma(C')$ for every irreducible component C', by setting $\sigma(C') := \sigma(C'' \cup C') - \sigma(C'')$ for any tail C'' such that $C'' \cup C'$ is connected. The reduction map ϕ sends $(C, x_1, x_2, \dots, x_n)$ to a new curve $(\overline{C}, \overline{x}_1, \dots, \overline{x}_n)$, which is obtained by contracting all irreducible components C' with $\sigma(C') = 0$.

Example 4.3 The $n = 7$, $d = 3$, and $A = (\frac{4}{7}, \dots, \frac{4}{7})$ (hence $\gamma = 0$) case. There are only two types of curves in $\overline{\mathrm{M}}_{0,7}$ with contractions.

(i) A chain of curves $C = C_1 \cup C_2 \cup C_3$ such that C_1 has two marked points, C_2 has a marked point, and (possibly reducible) C_3 has four marked points. Then C_2 is contracted to a point.

(ii) A comb of rational curves with three tails C_1, C_2, C_3 with two marked points respectively, and a spine C_4 with a marked point. C_4 is contracted to a triplenodal singularity with a marked point on it.

Note that for the first case, the contracted component has only three special points. Thus, near the point, $\overline{\mathrm{M}}_{0,7}$ and V_A^3 are locally isomorphic. But in the second case, the spine has four special points, so it has a one-dimensional moduli. Thus, the map ϕ contracts the loci of such curves, which are F-curves of type $F_{1,2,2,2}$. So ϕ is a small contraction.

Remark 4.4 An important observation for Example 4.3 is that we can replace the Chow variety by the moduli space of stable maps $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 3)$. There is a cycle map

$$f\colon \overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d) \to \mathrm{Chow}_{1,d}(\mathbb{P}^d).$$

When $d \le 3$, if we take the locus $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)^{\mathrm{nd}}$ parametrizing stable maps with nondegenerate images and if $\mathrm{Chow}_{1,d}(\mathbb{P}^d)^{\mathrm{nd}}$ is the image of it, then the restricted cycle map is an isomorphism, because there is no degree 0 component with positive dimensional moduli. Therefore,

$$\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}} \times (\mathbb{P}^3)^n \longrightarrow \mathrm{Chow}_{1,3}(\mathbb{P}^3)^{\mathrm{nd}} \times (\mathbb{P}^3)^n$$

is an isomorphism and I^s is a subset of $\mathrm{Chow}_{1,3}(\mathbb{P}^3)^{\mathrm{nd}} \times (\mathbb{P}^3)^n$. Therefore, we can replace the Chow variety by $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 3)$.

Furthermore, $\overline{\mathrm{M}}_{0,n} \cong \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) /\!/ \mathrm{SL}_{d+1}$ for an appropriate linearization ([16, Proposition 4.6]). And the morphism $\overline{\mathrm{M}}_{0,7} \to V_A^3$ is obtained by taking the quotient of the map of

$$\overline{\mathrm{M}}_{0,7}(\mathbb{P}^3, 3) \longrightarrow \overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 3) \times (\mathbb{P}^3)^7.$$

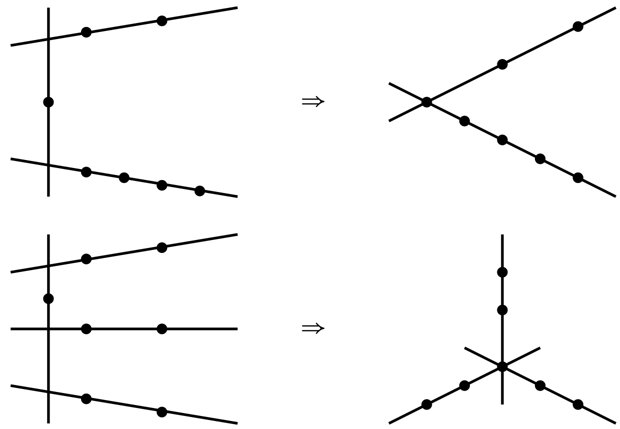

Figure 5: The reduction map $\phi: \overline{\mathrm{M}}_{0,7} \to V_A^3$ where $A = (\frac{4}{7}, \dots, \frac{4}{7})$

The other birational models $\overline{\mathrm{M}}_{0,7}^i$ with $i = 1, 2, 3$ are new spaces that do not appear in the literature. We will describe them concretely using explicit blow-ups and downs.

4.3 Outline of the Proof

The proof of Theorem 1.1 involves explicit but long computations of several birational modifications. So we outline the proof here and prove it in the following sections.

Outline of the Proof of Theorem 1.1 Since the symmetric nef cone is generated by $\psi - K_{\overline{\mathrm{M}}_{0,7}}$ and $K_{\overline{\mathrm{M}}_{0,7}} + \frac{1}{3}\psi$, D in item (i) is an ample divisor. Thus, $\overline{\mathrm{M}}_{0,7}(D) \cong \overline{\mathrm{M}}_{0,7}$.

Item (ii) is established in [41, Theorem 3.1]. If $D = K_{\overline{\mathrm{M}}_{0,7}} + \frac{1}{3}\psi$, then $\overline{\mathrm{M}}_{0,7}(D) \cong \overline{\mathrm{M}}_{0,A}$. Because for D in the range of item (ii) the stable base locus $\mathbf{B}(D)$ is B_3, after removing B_3, we obtain item (ii) in general.

Consider the reduction map $\phi: \overline{\mathrm{M}}_{0,7} \to V_A^3$ in item (iii). By applying [17, Theorem 3.1], we can compute the pull-back D_A of the canonical polarization on V_A^3. With the notation in [17], item (iii) is the case where $\gamma = 0$, $A = (\frac{4}{7}, \frac{4}{7}, \dots, \frac{4}{7})$. So it is straightforward to check that $F_{1,2,2,2} \cdot D_A = 0$. Since $\dim \mathrm{N}^1(\overline{\mathrm{M}}_{0,7})_{\mathbb{Q}}^{S_7} = 2$, this implies that D_A is proportional to $\psi - K_{\overline{\mathrm{M}}_{0,7}}$ by Corollary 2.7. Therefore,

$$\overline{\mathrm{M}}_{0,7}(\psi - K_{\overline{\mathrm{M}}_{0,7}}) \cong \overline{\mathrm{M}}_{0,7}(D_A) \cong V_A^3.$$

Items (iv)–(vii) are obtained by careful computations of flips and contractions. We give a proof of item (iv) in Proposition 4.6. Items (v) and (vi) are proved in Lemma 4.12 and Proposition 4.8, respectively. We prove item (vii) in Proposition 4.15.

Since B_2 and B_3 are rigid, item (viii) follows immediately. ∎

Remark 4.5 The direction toward the canonical divisor have been well understood for all n and all (possibly non-symmetric) weight data. For every n and $A =$

$(a_1, a_2, \ldots, a_n)$,

$$\overline{\mathrm{M}}_{0,n}\Big(K_{\overline{\mathrm{M}}_{0,n}} + \sum a_i\psi_i\Big) \cong \overline{\mathrm{M}}_{0,A}.$$

For a proof, see [41]. Also, for a generalization to $\overline{\mathcal{M}}_{g,n}$ with $g > 0$, consult [42].

4.4 First Flip

In this section, we describe the first flip $\overline{\mathrm{M}}_{0,7} \dashrightarrow \overline{\mathrm{M}}^3_{0,7}$ in terms of blow-ups and downs.

Proposition 4.6 *Let $\widetilde{\mathrm{M}}^3_{0,7}$ be the blow-up of $\overline{\mathrm{M}}_{0,7}$ along B_2^3. A connected component of the exceptional locus is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$. Let $\overline{\mathrm{M}}^3_{0,7}$ be the blow-down of these exceptional loci to the opposite direction. Then $\overline{\mathrm{M}}^3_{0,7}$ is smooth and is the D-flip of $\phi\colon \overline{\mathrm{M}}_{0,7} \to V_A^3$ for $D \in (\psi - 3K_{\overline{\mathrm{M}}_{0,7}}, \psi - K_{\overline{\mathrm{M}}_{0,7}})$ and $\overline{\mathrm{M}}_{0,7}(D) \cong \overline{\mathrm{M}}^3_{0,7}$.*

Proof On $\overline{\mathrm{M}}_{0,7}$, B_2^3 is the disjoint union of 105 F-curves of type $F_{1,2,2,2}$. Take a component F of B_2^3, which is an F-curve $B_I \cap B_J \cap B_K$, where $|I| = |J| = |K| = 2$. The normal bundle $N := N_{F/\overline{\mathrm{M}}_{0,7}}$ is isomorphic to $\mathcal{O}(B_I) \oplus \mathcal{O}(B_J) \oplus \mathcal{O}(B_K)|_F$. By [32, Lemma 4.5], $N \cong \mathcal{O}(-\psi_p) \oplus \mathcal{O}(-\psi_q) \oplus \mathcal{O}(-\psi_r)$ where p, q, r are attaching points of three tails. Since $F \cdot \psi_x = 1$ for any attaching point x, $N \cong \mathcal{O}_{\mathbb{P}^1}(-1)^3$.

Let $\pi_3\colon \widetilde{\mathrm{M}}^3_{0,7} \to \overline{\mathrm{M}}_{0,7}$ be the blow-up. The blown-up space $\widetilde{\mathrm{M}}^3_{0,7}$ is a smooth variety. Also, a connected component E of the exceptional locus is $\mathbb{P}(N) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^3) \cong \mathbb{P}^1 \times \mathbb{P}^2$, and the normal bundle $N_{E/\widetilde{\mathrm{M}}^3_{0,7}}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-1,-1)$. Thus, for a point $y \in \mathbb{P}^2$, the restricted normal bundle to a fiber $\mathbb{P}^1 \times \{y\}$ is $\mathcal{O}_{\mathbb{P}^1}(-1)$. Therefore, there exists a smooth contraction $\overline{\mathrm{M}}^3_{0,7}$, which contracts the $\mathbb{P}^1$-fibration structure of the exceptional divisor. Let $\pi_3'\colon \widetilde{\mathrm{M}}^3_{0,7} \to \overline{\mathrm{M}}^3_{0,7}$ be the contraction. Since the positive dimensional fiber of π_3' is contracted by $\phi \circ \pi_3$, there is a birational map $\phi_3'\colon \overline{\mathrm{M}}^3_{0,7} \to V_A^3$ such that $\phi \circ \pi_3 = \phi_3' \circ \pi_3'$ by the rigidity lemma ([38, Proposition II.5.3]).

$$\begin{tikzcd}
& \widetilde{\mathrm{M}}^3_{0,7} \arrow[dl, "\pi_3"'] \arrow[dr, "\pi_3'"] & \\
\overline{\mathrm{M}}_{0,7} \arrow[dr, "\phi"'] & & \overline{\mathrm{M}}^3_{0,7} \arrow[dl, "\phi_3'"] \\
& V_A^3 &
\end{tikzcd}$$

We claim that $\phi_3'\colon \overline{\mathrm{M}}^3_{0,7} \to V_A^3$ is a D-flip for $D \in (\psi - 3K_{\overline{\mathrm{M}}_{0,7}}, \psi - K_{\overline{\mathrm{M}}_{0,7}})$. The exceptional set of ϕ is exactly $B_2^3 = \cup F_{1,2,2,2}$. From Corollary 2.7, $-D \cdot F_{1,2,2,2} > 0$. Thus, $-D$ is ϕ-ample. Note that a connected component of the positive dimensional exceptional locus of ϕ_3' is isomorphic to $\mathbb{P}^2$. Let $\widetilde{L}$ be a line class of type $(0,1)$ in the exceptional divisor $E \cong \mathbb{P}^1 \times \mathbb{P}^2$ on $\widetilde{\mathrm{M}}^3_{0,7}$, and let $L := \pi_3'(\widetilde{L})$ which is a line on the exceptional locus of ϕ_3'. Note that on ϕ_3'-exceptional $\mathbb{P}^2$, $B_I|_{\mathbb{P}^2}$, $B_J|_{\mathbb{P}^2}$, $B_K|_{\mathbb{P}^2}$ are line classes. So $B_2 \cdot L = 3$. On the other hand, B_3 intersects E three times, and each irreducible component of the intersection is isomorphic to $\{*\} \times \mathbb{P}^2 \subset \mathbb{P}^1 \times \mathbb{P}^2 \cong E$;

the divisor B_3 on $\overline{\mathrm{M}}_{0,7}^3$ vanishes along $\mathbb{P}^2$ with multiplicity three. Hence, $B_3 \cdot L = -3$. Now from $\psi - K_{\overline{\mathrm{M}}_{0,7}} = 2B_2 + 2B_3$, for $D \in (B_2, \psi - K_{\overline{\mathrm{M}}_{0,7}})$, $D \cdot L > 0$, so D is ϕ_3'-ample.

Furthermore, we can see that for $D \in (\psi - 3K_{\overline{\mathrm{M}}_{0,7}}, \psi - K_{\overline{\mathrm{M}}_{0,7}})$, D is ample on $\overline{\mathrm{M}}_{0,7}^3$. If a curve class C is in the image of exceptional $\mathbb{P}^2$, then we already proved that $C \cdot D \geq 0$. If C is not contained in the exceptional locus, from Proposition 3.2, mD is movable for $m \gg 0$ on the outside of B_2^3, thus $C \cdot D \geq 0$ if $D \in [\psi - 3K_{\overline{\mathrm{M}}_{0,7}}, \psi - K_{\overline{\mathrm{M}}_{0,7}}]$. Therefore, the nef cone of $\overline{\mathrm{M}}_{0,7}^3/S_7$ is generated by $\psi - K_{\overline{\mathrm{M}}_{0,7}}$ and $\psi - 3K_{\overline{\mathrm{M}}_{0,7}}$. Since the ample cone is the interior of the nef cone, the desired result follows. ∎

Remark 4.7 After the first flip, the proper transform of B_2^2 becomes a disjoint union of its irreducible components. Each irreducible component is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

4.5 Second Flip

The description of the second flip is more complicated. It is a composition of two smooth blow-ups, a smooth blow-down, and a singular blow-down. In this section, we will describe the second flip. Since the flipping locus is the disjoint union of irreducible components of the proper transform of B_2^2, it is enough to focus on the modification on an irreducible component. We will give an outline of the description first, and after that we give justifications of statements as a collection of lemmas. Figure 6 shows the decomposition of the flip. By abusing notation, we say B_2^2 for the proper transform of B_2^2 on $\overline{\mathrm{M}}_{0,7}^3$.

On $\overline{\mathrm{M}}_{0,7}^3$, let X_0 be an irreducible component of B_2^2. Then X_0 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and its normal bundle $N_{X_0/\overline{\mathrm{M}}_{0,7}^3}$ is isomorphic to $\mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-2)$ (Lemma 4.9). Note that on $\overline{\mathrm{M}}_{0,7}^3$, since we have blown-up B_2^3, X_0 is the intersection of exactly two irreducible components of B_2, and no other irreducible component of B_2 intersects X_0. From the computation of the normal bundle, the direct summands $\mathcal{O}(-2,-1)$ and $\mathcal{O}(-1,-2)$ correspond to the normal bundle to two irreducible components of B_2 containing X_0.

Take the blow-up M_1 of $M_0 := \overline{\mathrm{M}}_{0,7}^3$ along X_0. Then the exceptional divisor X_1 is isomorphic to $\mathbb{P}(\mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-2))$. It has two sections Y_{11} and Y_{12}, which are intersections with the proper transform of irreducible components of B_2. The normal bundle N_{Y_{11}/M_1} is isomorphic to $\mathcal{O}(-2,-1) \oplus \mathcal{O}(1,-1)$ and $N_{Y_{12}/M_1} \cong \mathcal{O}(-1,-2) \oplus \mathcal{O}(-1,1)$ (Lemma 4.10).

Let M_2 be the blow-up of M_1 along $Y_{11} \sqcup Y_{12}$. Let Y_{21} (resp. Y_{22}) be the exceptional divisor over Y_{11} (resp. Y_{12}). Finally, let X_2 be the proper transform of X_1. Since X_2 is a blow-up of two Cartier divisors, $Y_{11}, Y_{12} \subset X_1$, X_2 is isomorphic to X_1. On the other hand, $Y_{21} \cong \mathbb{P}(\mathcal{O}(-2,-1) \oplus \mathcal{O}(1,-1))$ and $Y_{22} \cong \mathbb{P}(\mathcal{O}(-1,-2) \oplus \mathcal{O}(-1,1))$.

If we fix the first coordinate on Y_{11}, then the restriction of N_{Y_{11}/M_1} is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. So its projectivization is $\mathbb{P}^1 \times \mathbb{P}^1$. This implies that Y_{21} has another $\mathbb{P}^1$ fibration structure that does not come from $Y_{21} \to Y_{11}$. Moreover, if we restrict $\mathcal{O}_{Y_{21}}(Y_{21})$ to a fiber, it is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$. Therefore, we can blow-down this $\mathbb{P}^1$ fibration, and the result is smooth. Then Y_{22} can be contracted in the same way. (But note that the direction of fibrations are different.) Let M_3 be the blow-down of Y_{21} and Y_{22}, and let Y_{31} (resp. Y_{32}, X_3) be the image of Y_{21} (resp. Y_{22}, X_2). Then Y_{31}, Y_{32} are isomorphic to $\mathbb{F}_3$ and X_3 is

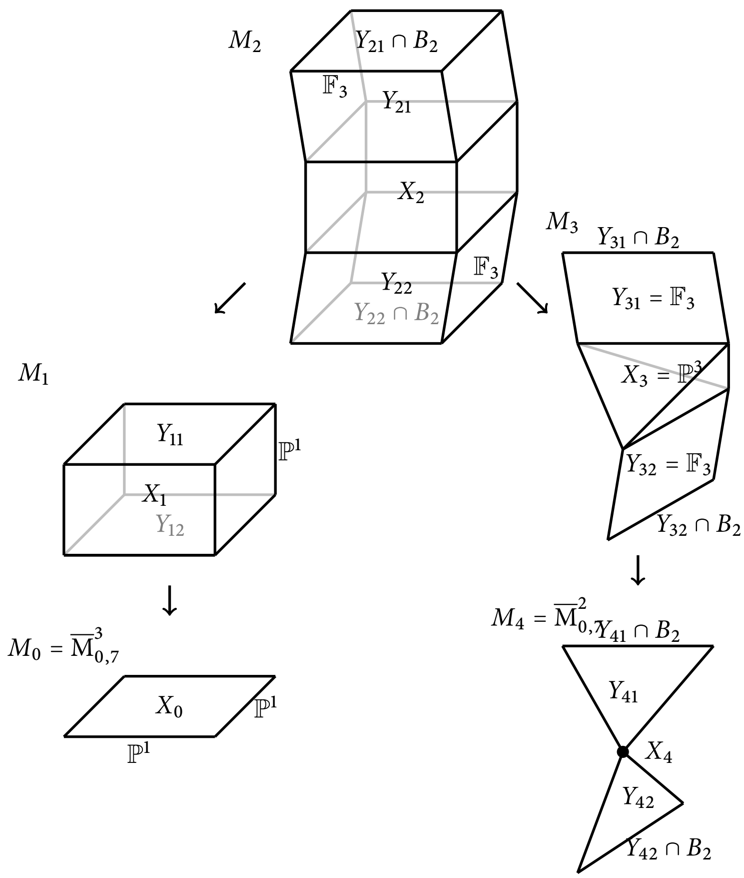

Figure 6: Decomposition of the second flip $\overline{\mathrm{M}}_{0,7}^3 \dashrightarrow \overline{\mathrm{M}}_{0,7}^2$

isomorphic to $\mathbb{P}^3$ and $N_{X_3/M_3} \cong \mathcal{O}(-3)$ (Lemma 4.11). Finally, X_3 can be contracted to a point X_4 in the category of algebraic spaces ([4, Corollary 6.10]). Let M_4 be the contraction. X_4 is a singular point of M_4. The image Y_{41} (resp. Y_{42}) of $Y_{31} \cong \mathbb{F}_3$ (resp. Y_{32}) is the contraction of a (-3) section, hence it is covered by a single family of rational curves passing through the singular point. Let $\overline{\mathrm{M}}_{0,7}^2 := M_4$. We claim that $\overline{\mathrm{M}}_{0,7}^2$ is the second flip. The argument is standard. There is a small contraction $\phi_2 \colon \overline{\mathrm{M}}_{0,7}^3 \to \overline{\mathrm{M}}_{0,7}(\psi - 3K_{\overline{\mathrm{M}}_{0,7}})$ (Lemma 4.12). For two modifications

$$\pi_2 \colon M_2 \longrightarrow \overline{\mathrm{M}}_{0,7}^3 \quad \text{and} \quad \pi_2' \colon \overline{\mathrm{M}}_{0,7}^3 \longrightarrow \overline{\mathrm{M}}_{0,7}^2,$$

by rigidity lemma, there is a morphism $\phi_2' \colon \overline{\mathrm{M}}_{0,7}^2 \to \overline{\mathrm{M}}_{0,7}(\psi - 3K_{\overline{\mathrm{M}}_{0,7}})$ such that $\phi_2 \circ \pi_2 = \phi_2' \circ \pi_2'$. We prove that for $D \in (\psi - 5K_{\overline{\mathrm{M}}_{0,7}}, \psi - 3K_{\overline{\mathrm{M}}_{0,7}})$, D is ample on $\overline{\mathrm{M}}_{0,7}^2$ (Lemma 4.13). Note that this implies the projectivity of $\overline{\mathrm{M}}_{0,7}^2$. In summary, we obtain the following result.

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Proposition 4.8 *The modification $\overline{M}_{0,7}^2$ is the D-flip of $\overline{M}_{0,7}^3$ for*

$$D \in (\psi - 5K_{\overline{M}_{0,7}}, \psi - K_{\overline{M}_{0,7}}).$$

We now state and prove the lemmas mentioned in the outline.

Lemma 4.9 (i) *On $\overline{M}_{0,7}^3$, $X_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$.*
(ii) *The normal bundle $N_{X_0/\overline{M}_{0,7}^3}$ is isomorphic to $\mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-2)$.*

Proof Take an irreducible component of B_2^2 on $\overline{M}_{0,7}$, which is isomorphic to $\overline{M}_{0,5}$. Let p, q be two attaching points. One can also regard $\overline{M}_{0,5}$ as a universal family over $\overline{M}_{0,4} \cong \mathbb{P}^1$ that is also isomorphic to the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ along three diagonal points. Its four sections correspond to 4 marked points for $\overline{M}_{0,5}$. Then there are four sections (say i, j, k, and p) such that three of them are proper transforms of trivial sections, and one of them is the proper transform of the diagonal section. We can assume that p is the diagonal section. The normal bundle $N_{\overline{M}_{0,5}/\overline{M}_{0,7}} \cong \mathcal{O}(-\psi_p) \oplus \mathcal{O}(-\psi_q)$. By intersection number computation, one can show that

$$N_{\overline{M}_{0,5}/\overline{M}_{0,7}} \cong \pi^*\big(\mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-2)\big) \otimes \mathcal{O}(E_i + E_j + E_k),$$

where $\pi: \overline{M}_{0,5} \to \mathbb{P}^1 \times \mathbb{P}^1$ is the blow-up along three intersection points of the diagonal section and E_i, E_j, E_k are three exceptional divisors. On $\overline{M}_{0,7}$, these three exceptional curves are three components of B_2^3. On $\overline{M}_{0,7}^3$, X_0 is the blow-up of $\overline{M}_{0,5}$ along three divisors and contraction along the different direction. Thus, X_0 is the contraction of three exceptional lines E_i, E_j, and E_k and is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. This proves (i).

We denote the proper transform of X_0 in $\widetilde{M}_{0,7}^3$ by $\widetilde{X}$. Let $\pi_1: \widetilde{X} \to \overline{M}_{0,5}$, $\pi_2: \widetilde{X} \to X_0$ be two contractions. (Since $B_2^3 \subset X_0$ is a divisor, π_1 is an isomorphism.) Then by the blow-up formula of normal bundles [13, App. B.6.10.],

$$\begin{aligned} N_{\widetilde{X}/\widetilde{M}_{0,7}^3} &\cong \pi_1^* N_{\overline{M}_{0,5}/\overline{M}_{0,7}} \otimes \mathcal{O}(-E_i - E_j - E_k) \cong \pi_1^*\pi^*\big(\mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-2)\big) \\ &= \pi_2^*\big(\mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-2)\big). \end{aligned}$$

Since the opposite blow-up center is transversal to X,

$$N_{X/\overline{M}_{0,7}^3} \cong \mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-2). \qquad \blacksquare$$

Lemma 4.10 *The normal bundle N_{Y_{11}/M_1} is isomorphic to $\mathcal{O}(-2,-1) \oplus \mathcal{O}(1,-1)$. Similarly, $N_{Y_{12}/M_1} \cong \mathcal{O}(-1,-2) \oplus \mathcal{O}(-1,1)$.*

Proof For a section $Y_{11} = \mathbb{P}(\mathcal{O}(-2,-1)) \subset \mathbb{P}(\mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-2)) = X_1$, the normal bundle is $N_{X_1/M_1}|_{Y_{11}} \cong \mathcal{O}(-2,-1)$ and $N_{Y_{11}/X_1} \cong \mathcal{O}(-1,-2) \otimes \mathcal{O}(-2,-1)^* \cong \mathcal{O}(1,-1)$. From the normal bundle sequence

$$0 \longrightarrow N_{Y_{11}/X_1} \longrightarrow N_{Y_{11}/M_1} \longrightarrow N_{X_1/M_1}|_{Y_{11}} \longrightarrow 0,$$

N_{Y_{11}/M_1} is an extension of $N_{X_1/M_1}|_{Y_{11}}$ by N_{Y_{11}/X_1}. But $\mathrm{Ext}^1(\mathcal{O}(-2,-1), \mathcal{O}(1,-1)) \cong H^1(\mathcal{O}(3,0)) = 0$. Therefore, $N_{Y_{11}/M_1} \cong \mathcal{O}(-2,-1) \oplus \mathcal{O}(1,-1)$. The computation of N_{Y_{12}/M_1} is similar. $\blacksquare$

Lemma 4.11 (i) $Y_{31} \cong Y_{32} \cong \mathbb{F}_3$.
(ii) $X_3 \cong \mathbb{P}^3$.
(iii) $N_{X_3/M_3} \cong \mathcal{O}(-3)$.

Proof Since the restriction of N_{Y_{21}/M_2} to $\mathbb{P}^1 \times \{*\} \subset Y_{11}$ is isomorphic to $\mathcal{O}(-2) \oplus \mathcal{O}(1)$, the restriction of Y_{21} onto the inverse image of $\mathbb{P}^1 \times \{*\}$ is $\mathbb{P}(\mathcal{O}(-2) \oplus \mathcal{O}(1)) \cong \mathbb{F}_3$. Hence, Y_{31} is also isomorphic to the Hirzebruch surface $\mathbb{F}_3$. This proves (i).

The divisor X_2 is isomorphic to $\mathbb{P}(\mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-2))$. Note that two contracted loci $Y_{21} \cap X_2$ and $Y_{22} \cap X_2$ have normal bundles $\mathcal{O}(-1,-2) \otimes \mathcal{O}(-2,-1)^* \cong \mathcal{O}(1,-1)$ and $\mathcal{O}(-1,1)$), respectively. This is isomorphic to the blow-up of $\mathbb{P}^3$ along two lines L_1 and L_2 in general position. Indeed, if we consider the universal (or total) space of all lines intersecting L_1 and L_2, then naturally it is identified with $\mathrm{Bl}_{L_1 \cup L_2}\mathbb{P}^3$. Thus, this blown-up space has a $\mathbb{P}^1$-fibration structure over (both) exceptional divisors isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The normal bundles to these two exceptional divisors are $\mathcal{O}(1,-1)$ and $\mathcal{O}(-1,1)$, respectively. Thus, $X_2 \cong \mathrm{Bl}_{L_1 \cup L_2}\mathbb{P}^3$, and we have $X_3 \cong \mathbb{P}^3$.

For a diagonal embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 = X_0$, if we restrict to $\mathbb{P}(\mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-2)) \to X_0$, we obtain a trivial bundle $\mathbb{P}(\mathcal{O}(-3) \oplus \mathcal{O}(-3)) \to \mathbb{P}^1$. Take a general constant section $s \hookrightarrow \mathbb{P}(\mathcal{O}(-3) \oplus \mathcal{O}(-3))$. Then the restricted normal bundle $N_{X_1/M_1}|_s$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-3)$. We can choose s that does not intersect Y_{ij} during modifications. Thus, $N_{X_1/M_1}|_s = N_{X_3/M_3}|_s$ and s is a line in $X_3 \cong \mathbb{P}^3$. Hence, $N_{X_3/M_3} \cong \mathcal{O}(-3)$. ■

Lemma 4.12 *For $D = \psi - 3K_{\overline{\mathrm{M}}_{0,7}}$, there is a small contraction $\phi_2 : \overline{\mathrm{M}}_{0,7}^3 \to \overline{\mathrm{M}}_{0,7}(D)$ that contracts a connected component of B_2^2 to a point.*

Proof Since X_0 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, it is covered by two rational curve classes $\ell_1 = \mathbb{P}^1 \times \{x\}$ and $\ell_2 = \{y\} \times \mathbb{P}^1$. For a general x, ℓ_1 does not intersect the flipping locus of $\overline{\mathrm{M}}_{0,7} \dashrightarrow \overline{\mathrm{M}}_{0,7}^3$. Moreover, this is a curve class A in Remark 2.5. So by Corollary 2.7, $\ell_1 \cdot D = 0$. For the same reason, $\ell_2 \cdot D = 0$. Since ℓ_1, ℓ_2 generates the cone of curves of $\mathbb{P}^1 \times \mathbb{P}^1$, D is numerically trivial on B_2^2. Because the only numerically trivial divisor on B_2^2 is a trivial divisor, D does not have any base points on B_2^2. By Proposition 3.2, on the outside of B_2^2, there is no base point of mD for $m \gg 0$, too. Thus D is a semi-ample divisor on $\overline{\mathrm{M}}_{0,7}^3$. So there is a regular morphism $\phi_2 : \overline{\mathrm{M}}_{0,7}^3 \to \overline{\mathrm{M}}_{0,7}^3(D) \cong \overline{\mathrm{M}}_{0,7}(D)$ that contracts B_2^2, a codimension two subvariety, to a point. ■

Lemma 4.13 *For $D \in (\psi - 5K_{\overline{\mathrm{M}}_{0,7}}, \psi - 3K_{\overline{\mathrm{M}}_{0,7}})$, D is ample on $\overline{\mathrm{M}}_{0,7}^2$.*

Proof Because it is a contraction of M_3, which is a projective variety, $\overline{\mathrm{M}}_{0,7}^2$ satisfies the assumption of [11, Lemma 4.12]. Thus, we can apply Kleiman's criterion, and we will show that for $D \in [\psi - 5K_{\overline{\mathrm{M}}_{0,7}}, \psi - 3K_{\overline{\mathrm{M}}_{0,7}}]$, D is nef.

Since mD for $m \gg 0$ is base-point-free for all $\overline{\mathrm{M}}_{0,7} - B_2^2 \cong \overline{\mathrm{M}}_{0,7}^2 - Y_{41} \cup Y_{42}$, it is enough to check that for all curve classes on $Y_{41} \cup Y_{42}$, the intersection with D is nonnegative. The curve cone of Y_{41} is generated by a single rational curve ℓ, which is the image of a fiber f in $\mathbb{F}_3$. So it suffices to compute $D \cdot \ell$. The computation of the intersection number of the curve class in Y_{42} is identical.

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It is easy to see that $B_2\cdot\ell = 1$ from the description of M_4. To compute $B_3\cdot\ell$, we need to keep track of the proper transform of B_3. Note that there are seven irreducible components (say $B_{31},\dots,B_{37}$) of B_3 intersecting X_0. If we write $\mathrm{Pic}(X_0) = \langle h_1, h_2\rangle$ where h_1 (resp. h_2) is the curve class of $\mathbb{P}^1\times\{*\}$ (resp. $\{*\}\times\mathbb{P}^1$), three of them (B_{31}, B_{32}, B_{33}) are h_1, another three of them (B_{34}, B_{35}, B_{36}) are h_2, and the last one (B_{37}) is $h_1 + h_2$ class, which is the diagonal set-theoretically. By keeping track of the proper transforms, one can check that on M_3, $Y_{31}\subset B_{3i}$ for $i = 1,2,3,7$, $Y_{31}\cap B_{3j} = \mathbb{P}^1 = f$ for $j = 4,5,6$. Also, $X_3\cap Y_{3k}$ is a plane for $k = 1,2,\dots,6$, but $X_3\cap Y_{37}$ is a quadric containing two skew lines $Y_{31}\cap X_3$, $Y_{32}\cap X_3$.

Analytic locally near X_4, M_4 is isomorphic to a cone over a degree 3 Veronese embedding of $\mathbb{P}^3$ in $\mathbb{P}^{19}$, Y_{41} is a cone over a twisted cubic curve, and M_3 is the blow-up of the conical point. If we take the pull-back of a hyperplane class $H\subset\mathbb{P}^{20}$ containing X_4 for $\pi\colon M_3\to M_4$, then $\pi^*H = \widetilde{H} + X_3$, where $\widetilde{H}$ is the proper transform of H. Note that $\widetilde{H}\cap X_3\subset X_3\cong\mathbb{P}^3$ is a cubic surface. Therefore, $\pi^*\pi_*B_{3i} = B_{3i} + \frac{1}{3}X_3$ for $i = 1,\dots,6$, $\pi^*\pi_*B_{37} = B_{37} + \frac{2}{3}X_3$. Now

$$\begin{aligned} B_3\cdot\ell = \pi^*B_3\cdot f &= \sum_{i=1}^{7} B_{3i}\cdot f + 6\cdot\frac{1}{3}X_3\cdot f + \frac{2}{3}X_3\cdot f \\ &= (B_{31} + B_{32} + B_{33} + B_{37})\cdot f + \frac{8}{3}. \end{aligned}$$

For a 1-dimensional fiber f' of $Y_{21}\to Y_{11}$, f' maps to f by $Y_{21}\to Y_{31}$. By the projection formula for $\rho\colon M_2\to M_3$,

$$B_{3i}\cdot f = \rho^*B_{3i}\cdot f' = \widetilde{B}_{3i}\cdot f' + Y_{21}\cdot f' = Y_{21}\cdot f' = -1$$

if we denote the proper transform of B_{3i} by $\widetilde{B}_{3i}$. Therefore

$$B_3\cdot\ell = -4 + \frac{8}{3} = -\frac{4}{3}.$$

For $D = \psi - aK_{\overline{\mathrm{M}}_{0,7}}$, $D\equiv\frac{5+a}{3}B_2 + 2B_3$ by Corollary 2.6. So $D\cdot\ell = \frac{a-3}{3}$, and it is nonnegative if $a\ge 3$. ∎

4.6 Divisorial Contraction

The last birational model $\overline{\mathrm{M}}^1_{0,7}$ is a divisorial contraction.

Lemma 4.14 *Let $D = \psi - 5K_{\overline{\mathrm{M}}_{0,7}}$. Then D is a semi-ample divisor on $\overline{\mathrm{M}}^2_{0,7}$.*

Proof By Proposition 3.2, the stable base locus is contained in the union of the proper transform of B_2 and $\cup Y_{4i}$. By the proof of Lemma 4.13, D is ample on $\cup Y_{4i}$. So it suffices to show that D is semi-ample on the proper transform of B_2.

Since D is in the closure of the ample cone of $\overline{\mathrm{M}}^2_{0,7}$, D is nef. In particular, if B_I is an irreducible (equivalently on $\overline{\mathrm{M}}^2_{0,7}$, connected) component of B_2, $D|_{B_I}$ is nef. But on $\overline{\mathrm{M}}_{0,7}$, $B_I\cong\overline{\mathrm{M}}_{0,6}$ so it is a Mori dream space ([29, Corollary 2.16], or [5, Theorem 1.4]). Since the proper transform of B_I on $\overline{\mathrm{M}}^2_{0,7}$ is a flip of B_I, it is a Mori dream space as well. Thus, for $m\gg 0$, $mD|_{B_I}$ is base-point-free. Thus, $\mathbf{B}(D) = \varnothing$ on $\overline{\mathrm{M}}^2_{0,7}$, and it is semi-ample. ∎

Let $\overline{\mathrm{M}}_{0,7}^1 = \overline{\mathrm{M}}_{0,7}(\psi - 5K_{\overline{\mathrm{M}}_{0,7}}) = \overline{\mathrm{M}}_{0,7}^2(\psi - 5K_{\overline{\mathrm{M}}_{0,7}})$. Since B_2 is covered by a curve class C_5 such that $C_5 \cdot D = 0$, $\overline{\mathrm{M}}_{0,7}^1$ is a divisorial contraction of $\overline{\mathrm{M}}_{0,7}^2$.

Proposition 4.15 *For $D \in (B_2, \psi - 5K_{\overline{\mathrm{M}}_{0,7}}]$, $\overline{\mathrm{M}}_{0,7}(D) \cong \overline{\mathrm{M}}_{0,7}^1$.*

Proof Note that for $D \in (B_2, \psi - 5K_{\overline{\mathrm{M}}_{0,7}}]$, $D \equiv (\psi - 5K_{\overline{\mathrm{M}}_{0,7}}) + cB_2$ for some $c \geq 0$. Because B_2 is an exceptional divisor for $\phi_1 \colon \overline{\mathrm{M}}_{0,7}^2 \to \overline{\mathrm{M}}_{0,7}^1$,

$$\overline{\mathrm{M}}_{0,7}(D) \cong \overline{\mathrm{M}}_{0,7}^2(D) \cong \overline{\mathrm{M}}_{0,7}^2(\psi - 5K_{\overline{\mathrm{M}}_{0,7}}) \cong \overline{\mathrm{M}}_{0,7}^1. \qquad \blacksquare$$

5 KKO Compactification

In this section, we review of KKO compactification of moduli of curves of genus g in a smooth projective variety X, which will be used to describe a modular interpretation of $\overline{\mathrm{M}}_{0,7}^3$ in the next section. For the details of its construction, consult the original paper of Kim, Kresch, and Oh [35].

5.1 FM Degeneration Spaces

Fix a nonsingular projective variety X. Let $X[n]$ be the Fulton–MacPherson space of n distinct ordered points in X. It is a compactification of the moduli space of n ordered distinct points on X, which is obviously $X^n \smallsetminus \Delta$. See [14] for the construction and its geometric properties. Then $X[n]$ has a universal family $\pi \colon X[n]^+ \to X[n]$ and n disjoint universal sections $\sigma_i \colon X[n] \to X[n]^+$ for $1 \leq i \leq n$.

For a point $p \in X[n]$, the fiber $\pi^{-1}(p)$ is a possibly reducible variety, whose irreducible components are smooth and equidimensional. As an abstract variety, $\pi^{-1}(p)$ can be constructed in the following manner. Set $X_0 := X$. Take a point $x_0 \in X$ and blow-up X_0 along x_0. Let $\widetilde{X}_0 := \mathrm{Bl}_{x_0} X_0$ and let E_1 be the exceptional divisor, which is naturally isomorphic to $\mathbb{P}(T_{x_0} X_0)$. Now consider the compactified tangent space $\mathbb{P}T := \mathbb{P}(T_{x_0} X_0 \oplus \mathbb{C})$, which has a subvariety $\mathbb{P}(T_{x_0} X_0) \cong \mathbb{P}T - T_{x_0} X_0$. Glue $\widetilde{X}_0$ and $\mathbb{P}T$ along $\mathbb{P}(T_{x_0} X_0)$ and let X_1 be the result.

We are able to continue this construction by taking a nonsingular point $x_1 \in X_1$ and constructing X_2 in a same way. If we repeat this procedure several times, we inductively obtain X_k, which is a reducible variety. Thus, $\pi^{-1}(p)$ is isomorphic to X_k for some $k \geq 0$ and some $x_0, x_1, \cdots, x_{k-1}$. Note that there is a natural projection $X_k \to X$. It can be extended to a canonical morphism $\pi_X \colon X[n]^+ \to X$.

Remark 5.1 (i) The singular locus of X_k is isomorphic to a union of disjoint $\mathbb{P}^{r-1}$'s.

(ii) Naturally the dual graph of X_k is a tree with a root. The proper transform of X_0 corresponds to the root. A non-root component is called a *screen*. The *level* of an irreducible component of X_k is defined by the number of edges from the root to the vertex representing the component.

(iii) If an irreducible component Y of X_k does not contain any x_i, then $Y \cong \mathbb{P}^r$. Then Y is called an *end component*.

(iv) If an irreducible component Z of X_k is not the root component and it contains only two singular loci, then $Z \cong \mathrm{Bl}_p \mathbb{P}^r$, which is a ruled variety, and Z is called a *ruled component*.

Definition 5.2 ([35, Definition 2.1.1]) A pair $(\pi_{W/B}: W \to B, \pi_{W/X}: W \to X)$ is called a Fulton–MacPherson degeneration space of X over a scheme B (or an FM degeneration space of X over B) if:

- W is an algebraic space;
- Étale locally it is a pull-back of the universal family $\pi: X[n]^+ \to X[n]$. That is, there is an étale surjective morphism $B' \to B$ from a scheme B', $n > 0$ and a Cartesian diagram

$$\begin{tikzcd} W|_{B'} \arrow[r] \arrow[d] & X[n]^+ \arrow[d] \\ B' \arrow[r] & X[n], \end{tikzcd}$$

where the pull-back of $\pi_{W/X}$ to $W|_{B'}$ is equal to $W|_{B'} \to X[n]^+ \to X$.

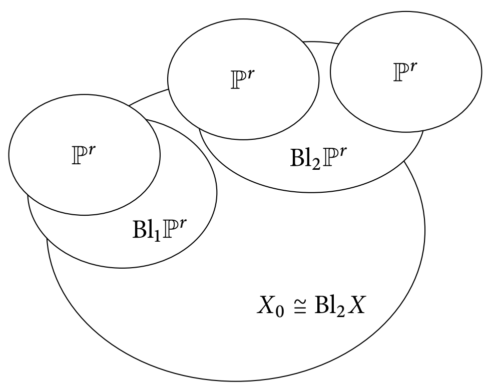

Figure 7: An example of an FM degeneration space

Let W be an FM space over $\mathbb{C}$. An automorphism of W/X is an automorphism $\varphi: W \to W$ fixing the root component, or equivalently, $\pi_{W/X} \circ \varphi = \pi_{W/X}$. If $W \ncong X$, $\mathrm{Aut}(W/X)$ is always positive dimensional. More precisely, for an end component Y of W, the automorphism fixing all W except Y is isomorphic to $\mathbb{C}^r \rtimes \mathbb{C}^*$, the group of homotheties. Also, for a ruled component Z of W, the automorphism fixing W except Z is isomorphic to $\mathbb{C}^*$. The other irreducible components do not contribute to a non-trivial automorphism of W/X.

We state a useful lemma to show several geometric properties of KKO compactifications.

Lemma 5.3 *For $m > n$, there is a commutative diagram*

$$\begin{tikzcd} X[m]^+ \arrow[r] \arrow[d] & X[n]^+ \arrow[d] \\ X[m] \arrow[r] & X[n]. \end{tikzcd}$$

The two vertical maps are universal families, and the horizontal maps are obtained by forgetting $m - n$ marked points and stabilizing.

Proof By induction, it suffices to show for $m = n + 1$ case. Note that $X[n+1]$ is obtained by taking a blow-up of $X[n]^+$ along the image of n sections ([14, p. 195]). On the other hand, $X[n]^+$ is constructed by taking iterated blow-ups of $X[n] \times X$. Hence, we have a commutative diagram

$$\begin{tikzcd} X[n+1]^+ \arrow[rr] \arrow[d] & & X[n]^+ \arrow[d] \\ X[n+1] \times X \arrow[d] & & X[n] \times X \arrow[d] \\ X[n+1] \arrow[rr] \arrow[urruu] & & X[n]. \end{tikzcd}$$

■

5.2 Stable Unramified Maps

Definition 5.4 ([35, Definition 3.1.1]) A collection of data

$$((C, x_1, x_2, \ldots, x_n), \pi_{W/X}: W \to X, f: C \to W)$$

is called an *n-pointed stable unramified map* of type (g, β) to an FM degeneration space W of X if the following hold:

(i) $(C, x_1, x_2, \ldots, x_n)$ is an n-pointed prestable curve with arithmetic genus g;
(ii) $\pi_{W/X}: W \to X$ is an FM degeneration space of X over $\mathbb{C}$;
(iii) $(\pi_{W/X} \circ f)_*[C] = \beta \in A_1(X)$;
(iv) $f^{-1}(W^{sm}) = C^{sm}$, where Y^{sm} is the smooth locus of Y;
(v) $f|_{C^{sm}}$ is unramified everywhere;
(vi) $f(x_i)$ for $1 \le i \le n$ are distinct;
(vii) at each nodal point $p \in C$, there are coordinates

$$\widehat{\mathcal{O}}_p \cong \mathbb{C}[[x, y]]/(x, y) \quad \text{and} \quad \widehat{\mathcal{O}}_{f(p)} \cong \mathbb{C}[[z_1, \ldots, z_{r+1}]]/(z_1 z_2)$$

such that $\widehat{f}^*: \mathbb{C}[[z_1, \ldots, z_{r+1}]]/(z_1 z_2) \to \mathbb{C}[[x, y]]/(xy)$ maps z_1 to x^m and z_2 to y^m for some $m \in \mathbb{N}$;
(viii) there are finitely many automorphisms $\sigma: C \to C$ such that $\sigma(x_i) = x_i$ for $1 \le i \le n$ and $f \circ \sigma = \varphi \circ f$ for some $\varphi \in \mathrm{Aut}(W/X)$.

We can define the *level* of an irreducible component C' of C by the level of the component of W containing $f(C')$. A component C' with a positive level is called a *ghost component*.

Remark 5.5 The last condition about the finiteness of automorphisms can be described by conditions on end components and ruled components in the following way. A map $f: C \to W$ has a finite automorphism group if and only if:

(i) for each end component Y of W, the number of marked points on Y is at least two or there is an irreducible component D of C such that $f(D) \subset Y$ and $\deg f(D) \geq 2$;
(ii) for each ruled component Z of W, there is at least one marked point on Z or there is an irreducible component $D \subset C$ such that $f(D)$ is not contained in a ruling.

Definition 5.6 ([35, Definition 3.2.1]) A collection of data

$$((\pi: \mathcal{C} \to B, \sigma_1, \dots, \sigma_n), (\pi_{W/B}: W \to B, \pi_{W/X}: W \to X), f: \mathcal{C} \to W)$$

is called a *B-family of n-pointed stable unramified maps* of type (g, β) to FM degeneration spaces of X, if:

(i) $(\pi: \mathcal{C} \to B, \sigma_1, \sigma_2, \dots, \sigma_n)$ is a family of n-pointed genus g prestable curves over B;
(ii) $(\pi_{W/B}: W \to B, \pi_{W/X}: W \to X)$ is an FM degeneration space of X over B;
(iii) over each geometric point of B, the data restricted to the fiber is a stable unramified map of type (g, β) to an FM degeneration space of X;
(iv) for every geometric point $b \in B$, if $p \in C_b$ is a nodal point, then there are two identifications
    - $\widehat{\mathcal{O}}_{f(p)} \cong \widehat{\mathcal{O}}_{\pi_{W/B}(p)}[[z_1, z_2, \dots, z_{r+1}]]/(z_1 z_2 - t)$ for some $t \in \widehat{\mathcal{O}}_{\pi_{W/B}(p)}$,
    - $\widehat{\mathcal{O}}_p \cong \widehat{\mathcal{O}}_{\pi(p)}[[x, y]]/(xy - t')$ for some $t' \in \widehat{\mathcal{O}}_{\pi(p)}$ such that $\widehat{f}^*(z_1) = \alpha_1 x^m$, $\widehat{f}^*(z_2) = \alpha_2 y^m$ for some $m \in \mathbb{N}$, $\alpha_1, \alpha_2 \in \widehat{\mathcal{O}}_p^*$, and $\alpha_1 \alpha_2 \in \widehat{\mathcal{O}}_{\pi(p)}$.

Let $\overline{\mathcal{U}}_{g,n}(X, \beta)$ be the fibered category of n-pointed unramified stable maps to FM degeneration spaces of X of type (g, β).

Theorem 5.7 ([35, Corollary 3.3.3]) *The fibered category $\overline{\mathcal{U}}_{g,n}(X, \beta)$ is a proper Deligne–Mumford stack of finite type.*

As in the title of this section, we will call $\overline{\mathcal{U}}_{g,n}(X, \beta)$ the *KKO compactification* of moduli space of embedded curves. By the Keel–Mori theorem, we have a coarse moduli space $\overline{U}_{g,n}(X, \beta)$ in the category of algebraic spaces.

5.3 Some Geometric Properties

In this section, we explain several geometric/functorial properties of $\overline{\mathcal{U}}_{g,n}(X, \beta)$.

As in the case of moduli space of ordinary stable maps, there are several functorial maps. Let $\overline{\mathcal{M}}_{g,n}(X, \beta)$ be the moduli stack of stable maps ([39]).

Proposition 5.8 *There is a functorial morphism*

$$S: \overline{\mathcal{U}}_{g,n}(X, \beta) \longrightarrow \overline{\mathcal{M}}_{g,n}(X, \beta).$$

Proof Let

$$\big((\pi\colon \mathcal{C}\to B,\sigma_1,\dots,\sigma_n),(\pi_{W/B}\colon W\longrightarrow B,\pi_{W/X}\colon W\longrightarrow X),f\colon \mathcal{C}\longrightarrow W\big)$$

be a B-family of n-pointed stable unramified maps of type (g,β) to FM degeneration spaces of X. Then we have $((\pi\colon \mathcal{C}\to B,\sigma_1,\dots,\sigma_n),\pi_{W/X}\circ f\colon \mathcal{C}\to X)$, which is a flat family of maps from n-pointed curves to X. By running relative MMP with respect to $\omega_{\mathcal{C}/B}+\sum\sigma_i$, we can stabilize $\pi_{W/X}\circ f$ and obtain

$$\big((\overline{\pi}\colon\overline{\mathcal{C}}\longrightarrow B,\overline{\sigma}_1,\dots,\overline{\sigma}_n),\overline{f}\colon\overline{\mathcal{C}}\longrightarrow X\big).$$

These two steps are both functorial, we can obtain the desired morphism S. ∎

Proposition 5.9 *There are functorial morphisms*

$$ev_i\colon \overline{\mathcal{U}}_{g,n}(X,\beta)\longrightarrow X$$

for $1\le i\le n$.

Proof Indeed $ev_i = e_i\circ S\colon \overline{\mathcal{U}}_{g,n}(X,\beta)\to\overline{\mathcal{M}}_{g,n}(X,\beta)\to X$ where e_i is the i-th evaluation map for the ordinary moduli space of stable maps. ∎

Proposition 5.10 *For any* $T\subset[n]$*, there is a functorial morphism*

$$F\colon\overline{\mathcal{U}}_{g,n}(X,\beta)\longrightarrow\overline{\mathcal{U}}_{g,T}(X,\beta)$$

obtained by forgetting all marked points with indices in $[n]-T$ *and stabilizing.*

Proof It suffices to show the existence of $F\colon\overline{\mathcal{U}}_{g,n}(X,\beta)\to\overline{\mathcal{U}}_{g,n-1}(X,\beta)$ which forgets the last marked point. For a family

$$\big((\pi\colon \mathcal{C}\longrightarrow B,\sigma_1,\dots,\sigma_n),(\pi_{W/B}\colon W\longrightarrow B,\pi_{W/X}\colon W\longrightarrow X),f\colon \mathcal{C}\longrightarrow W\big)$$

of n-pointed stable unramified maps over B, if we forget the last section σ_n, then the remaining collection of data

$$\big((\pi\colon \mathcal{C}\longrightarrow B,\sigma_1,\dots,\sigma_{n-1}),(\pi_{W/B}\colon W\longrightarrow B,\pi_{W/X}\colon W\longrightarrow X),f\colon \mathcal{C}\longrightarrow W\big)\tag{5.1}$$

is also a family of $(n-1)$-pointed unramified stable maps unless

(a) For a fiber of $b\in B$, there is an end component Y of W_b such that for every component D_i of $\mathcal{C}_b$ mapping to Y, D_i is a rational curve mapping to a line injectively, and there are exactly two marked points $\sigma_n(b)$ and $\sigma_k(b)$ lying on $\cup D_i$, or

(b) for a fiber of $b\in B$, there is a ruled component Z of W_b such that for every component D_j of $\mathcal{C}_b$ mapping to Z, the image of D_j is a ruling and only $\sigma_n(b)$ lies on $\cup D_j$. Note that D_j is a rational curve, because it is a ramified cover of $\mathbb{P}^1$ which has exactly two branch points.

Note that only one of these two cases can happen on a fiber.

We can stabilize the family (5.1) in the following way. Suppose that étale locally, the target space $\pi_{W/B}: W \to B$ comes from the Cartesian diagram

$$\begin{tikzcd} W|_{B'} \arrow[r] \arrow[d] & X[m]^+ \arrow[d] \\ B' \arrow[r] & X[m] \end{tikzcd}$$

for some $m > 0$ and an étale map $B' \to B$. We will modify the family locally, so for simplicity, we can assume that there is a unique connected closed subset $U \subset T$ such that for $b \in U$, the fiber has an end component Y of W_b with property (a). Also, we can assume that there is a unique connected closed subset $V \subset T$ such that for $b \in V$, there is a rule component Z of W_b with property (b). Over U (resp. V), the non-stable end components (resp. ruled components) form a family of irreducible components of $W|_U$ (resp. $W|_V$). Let $\tau_1, \tau_2, \dots, \tau_m: B' \to W|_B$ be the pull-back of universal sections $\sigma_1, \sigma_2, \dots, \sigma_m: X[m] \to X[m]^+$. Let $I \subset [m]$ be the index set of sections such that $i \in I$ if and only if τ_i is on the non-stable end component. Pick any $j \in I$ and let $J := I - \{j\}$. Now we have a forgetting map $X[m] \to X[m - |J|]$ forgetting all sections in J. There is also a contraction map $X[m]^+ \to X[m - |J|]^+$ on the universal family by Lemma 5.3. Take the pull-back of the universal family $X[m - |J|]^+ \to X[m - |J|]$ by $B' \to X[m] \to X[m - |J|]$. Then we have a family $W'|_{B'} \to B'$ of FM degeneration spaces and there is a morphism $W|_{B'} \to W'|_{B'}$.

$$\begin{tikzcd} \mathcal{C}|_{B'} \arrow[r, "f"] \arrow[drr] \arrow[ddr] & W|_{B'} \arrow[rr] \arrow[dr] \arrow[dd] & & X[m]^+ \arrow[dr] \arrow[dd] & \\ & & W'|_{B'} \arrow[rr] \arrow[dd] & & X[m-|J|]^+ \arrow[dd] \\ & B' \arrow[rr] \arrow[dr, equal] & & X[m] \arrow[dr] & \\ & & B' \arrow[rr] & & X[m-|J|] \end{tikzcd}$$

Now there are several irreducible components of $\mathcal{C}_b$ for $b \in V$, which are all tails, such that $f: \mathcal{C}|_{B'} \to W|_{B'} \to W'|_{B'}$ is not finite. By using the standard stabilizing of the domain curve (running the relative MMP over $W|_{B'}$ for $(\mathcal{C}|_{B'}, \omega_{C/B'} + \sum \sigma_i)$), we can contract these irreducible components.

After performing this procedure finitely many times, we can remove all non-stable end components and get a new family of maps $\overline{\mathcal{C}}|_{B'} \to W'|_{B'}$. Note that this procedure does not depend on the choice of m, $B' \to X[m]$ and $J \subset [m]$. We can replace $\mathcal{C}|_{B'}$ by $\overline{\mathcal{C}}|_{B'}$ and $W|_{B'}$ by $W'|_{B'}$ for a notational convenience.

The contraction of a non-stable ruled component in (b) is similar. Take $K \subset [m]$ such that $i \in K$ if and only if τ_i is on the non-stable ruled component. Take the forgetting map $X[m] \to X[m - |K|]$. By taking the pull-back of the universal family $X[m - |K|]^+ \to X[m - |K|]$, we have a family $W''|_{B'} \to B'$, and a B'-morphism

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$W|_{B'} \to W''|_{B'}$. By contracting all non-finite components using standard relative MMP technique, we obtain a family of finite maps $\overline{\mathcal{C}}|_{B'} \to W''|_{B'}$ over B'.

We claim that the result is a family of unramified stable maps. Except Definition 5.4(vii), all other conditions are simple observations of contracting procedures. If we contract a non-stable end component Y of the target, because we contract all irreducible components on the domain whose image lie on Y, there is no relevant singular points on the domain anymore. Furthermore, if we contract a non-stable ruled component Z of the target, then an irreducible component C_i of the domain maps to Z has only two ramification points at two singular points of the domain on C_i. Moreover, since $C_i \cong \mathbb{P}^1$, the ramification indices at two singular points are equal. Thus, after the contraction of the component, the stabilized map has the property (vii). ■

Proposition 5.11 *Let X be a smooth projective variety. Then there is a morphism*

$$T: \overline{\mathcal{U}}_{g,n}(X,\beta) \to \coprod_{\beta' \in A_1(\mathbb{P}(TX),\mathbb{Z})} \overline{\mathcal{M}}_{g,n}\big(\mathbb{P}(TX), \beta'\big),$$

where $\mathbb{P}(TX)$ be the projectivized tangent bundle of X.

Proof This is a direct consequence of [35, Lemma 3.2.4]. For a family

$$\big((\pi: \mathcal{C} \longrightarrow B, \sigma_1, \ldots, \sigma_n), (\pi_{W/B}: W \longrightarrow B, \pi_{W/X}: W \longrightarrow X), f: \mathcal{C} \longrightarrow W\big),$$

we have a family of maps $\widetilde{f}: \mathcal{C} \to \mathbb{P}(TX)$, which is a unique extension of the projectivized tangent map $\mathbb{P}(Tf): \mathcal{C}^{sm} \to \mathbb{P}(TX)$. By stabilizing the domain as usual, we obtain a family of stable maps $\overline{f}: \overline{\mathcal{C}} \to \mathbb{P}(TX)$. ■

Remark 5.12 For a ghost component C' of the domain C, the map $\mathbb{P}(Tf): C' \to \mathbb{P}(TX)$ can be described in the following way. Each screen (after blowing down all higher level screens) is identified with $\mathbb{P}(T_x X \oplus \mathbb{C})$ for some $x \in X$. For a smooth point $p \in C'$, $\mathbb{P}(Tf)(p) = T_p C' \cap \mathbb{P}(T_x X)$, where $\mathbb{P}(T_x X) \subset \mathbb{P}(T_x X \oplus \mathbb{C})$ is the "hyperplane at infinity". Therefore, it is a projection of the tangent variety of C'. If C' is a rational normal curve of degree d in $\mathbb{P}^r$ with $r \geq d$, then $\deg \mathbb{P}(Tf)(C') = 2d-2$ ([22, p. 245]).

Example 5.13 If $X = \mathbb{P}^d$, then the Chow ring of $\mathbb{P}(T\mathbb{P}^d)$ is

$$A^*\big(\mathbb{P}(T\mathbb{P}^r), \mathbb{Z}\big) \cong \mathbb{Z}[H,\zeta]/\Big\langle H^{d+1}, \sum_{i=0}^{d} \binom{d+1}{i} H^i \zeta^{d-i}\Big\rangle,$$

where H is the pull-back of hyperplane class h in $\mathbb{P}^d$ and $\zeta = c_1(\mathcal{O}_{\mathbb{P}(T\mathbb{P}^d)}(1))$.

We claim that for the connected component of $\overline{\mathcal{U}}_{0,n}(\mathbb{P}^d, d)$ containing smooth rational normal curves in $\mathbb{P}^d$, β' in Proposition 5.11 is $dH^{d-1}\zeta^{d-1} + (d+2)(d-1)H^d\zeta^{d-2}$ if $d \geq 2$. First of all, $\deg H^d\zeta^{d-1} = 1$. From the combination of these two relations, we can deduce $H^{d-1}\zeta^d + (d+1)H^d\zeta^{d-1} = 0$, so $\deg H^{d-1}\zeta^d = -(d+1)$. Since $H^{d-1}\zeta^{d-1}$ and $H^d\zeta^{d-2}$ form a basis of $A_1(\mathbb{P}(T\mathbb{P}^d), \mathbb{Z})$, β' is a linear combination of them. For a stable unramified map $f: C \to \mathbb{P}^d$ where $f(C)$ is a smooth rational curve of degree d in $\mathbb{P}^d$, $T(f)(C) = \mathbb{P}(TC) \subset \mathbb{P}(T\mathbb{P}^d)$, thus the restriction of the tautological subbundle to $T(f)(C)$ is $TC \cong \mathcal{O}_{\mathbb{P}^1}(2)$. Hence, $T(f)(C) \cdot \zeta = -2$. On the other hand, from the projection formula, $T(f)(C)\cdot H = f(C)\cdot h = d$. Therefore, from a simple calculation, we obtain $\beta' = dH^{d-1}\zeta^{d-1} + (d+2)(d-1)H^d\zeta^{d-2}$.

In the sequel, we denote $aH^{d-1}\zeta^{d-1} + bH^d\zeta^{d-2}$ by $(\mathbf{a},\mathbf{b})$-*class*.

5.4 Deformation Theory

The dimensions of the deformation and obstruction spaces of $\overline{\mathcal{U}}_{g,n}(X,\beta)$ can by computed indirectly by using Olsson's deformation theory of log schemes [45]. For a family

$$\big((\pi\colon \mathcal{C} \longrightarrow B, \sigma_1,\dots,\sigma_n), (\pi_{W/B}\colon W \longrightarrow B, \pi_{W/X}\colon W \longrightarrow X), f\colon \mathcal{C}\to W\big)$$

of n-pointed stable unramified maps over B, we can introduce natural log structures $M^{\mathcal{C}/B}$ on $\mathcal{C}$, $M^{W/B}$ on W, and $N^{\mathcal{C}/B}$ and $N^{W/B}$ on B such that $(\mathcal{C}, M^{\mathcal{C}/B}) \to (B, N^{\mathcal{C}/B})$ and $(W, M^{W/B}) \to (B, N^{W/B})$ are log smooth morphisms. We obtain a canonical log structure N on B by taking monoid push-out $N^{\mathcal{C}/B} \oplus_{N'} N^{W/B}$, where N' is the submonoid of $N^{\mathcal{C}/B}\oplus N^{W/B}$ generated by $(m\cdot \log t', \log t)$ for each nodal point of $\mathcal{C}$ (for the definition of m, t, t', see Definition 5.6.). We have a stack $\mathcal{B}$ of n-pointed prestable curves, FM degeneration spaces with n distinct smooth points, fine log schemes, and pairs of morphisms of log structures

$$\begin{aligned}\big((\mathcal{C} \longrightarrow B, (\sigma_1,\dots,\sigma_n)), (W \longrightarrow B, (\tau_1,\dots,\tau_n)),\\ (B,N), N^{\mathcal{C}/B} \longrightarrow N, N^{W/B}\to N\big).\end{aligned}$$

The relative tangent/obstruction spaces for $\overline{\mathcal{U}}_{g,n}(X,\beta)\to\mathcal{B}$ are described by cohomology groups. Suppose that $B = \operatorname{Spec} R$ for a Noetherian $\mathbb{C}$-algebra R and $\widetilde{R}$ is a square-zero extension of R by I. Let $\widetilde{B} = \operatorname{Spec}\widetilde{R}$. Also suppose that $\widetilde{\mathcal{C}}$ (resp. $\widetilde{W}$) is an extension of $\mathcal{C}$ (resp. W) over $\widetilde{B}$. Let $\widetilde{N}$ be the extension of N over $\widetilde{B}$ with two extensions $N^{\widetilde{\mathcal{C}}/\widetilde{B}}\to\widetilde{N}$ and $N^{\widetilde{W}/\widetilde{B}}\to\widetilde{N}$. Then the obstruction for a compatible extension of a stable unramified map is an element of $H^1(\mathcal{C}, f^*T_W^\dagger(-\sum\sigma_i)\otimes I)$, and if the obstruction vanishes, the compatible extensions are identified with $H^0(\mathcal{C}, f^*T_W^\dagger(-\sum\sigma_i)\otimes I)$ [35, Proposition 5.1.1]. Here $T_W^\dagger$ means the log tangent sheaf.

On the other hand, there is a log version of moduli space of stable log maps $\overline{\mathcal{U}}^{\log}_{g,n}(X,\beta)$ constructed in [34]. There is a commutative diagram

$$\begin{tikzcd}
\overline{\mathcal{U}}^{\log}_{g,n}(X,\beta) \arrow[d, "\phi"'] \arrow[dr] & \\
\overline{\mathcal{U}}_{g,n}(X,\beta) \arrow[r] & \mathcal{B}
\end{tikzcd}$$

where ϕ is a virtual normalization map [40]. ϕ is finite and degree one.

Let $B^\dagger$ be the log scheme (B,N). Let $\mathcal{C}^\dagger$ be the minimal log curve induced by $N^{\mathcal{C}/B}\to N$ [34, 3.5] and let $W^\dagger$ be the semi-stable log scheme induced by $N^{W/B}\to N$ [34, 4.3]. Let $\operatorname{Aut}_I(\mathcal{C}^\dagger\times_{B^\dagger}W^\dagger)$ be the set of automorphisms of the trivial extensions of $\mathcal{C}^\dagger\times_{B^\dagger}W^\dagger$ over Spec $(\widetilde{R},\widetilde{N})$, whose restriction to $B^\dagger$ is the identity. And let $\operatorname{Def}_I(\mathcal{C}^\dagger\times_{B^\dagger}W^\dagger)$ be the set of isomorphism classes of I-extensions of log schemes

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over $B^\dagger$. There is an R-module exact sequence

$$\begin{aligned}0 \to \mathrm{Aut}_I(\mathcal{C}^\dagger \times_{B^\dagger} W^\dagger) &\to \mathrm{RelDef}(f) = \\ &H^0(\mathcal{C}, f^* T_{W^\dagger/B^\dagger}\Big(-\sum \sigma_i\Big) \otimes_{\mathcal{O}_B} I) \to \mathrm{Def}(f) \\ &\to \mathrm{Def}_I(\mathcal{C}^\dagger \times_{B^\dagger} W^\dagger) \to \mathrm{RelOb}(f) = \\ &H^1(\mathcal{C}, f^* T_{W^\dagger/B^\dagger}\Big(-\sum \sigma_i\Big) \otimes_{\mathcal{O}_B} I) \to \mathrm{Obs}(f) \to 0\end{aligned}$$

[34, Section 7.1].

Now consider the $B = \mathrm{Spec}\,\mathbb{C}$ case. If $H^1(C, f^* T_W^\dagger(-\sum \sigma_i)) = 0$, then ϕ is a local isomorphism, thus $\mathrm{RelOb}(f) = 0$ as well. Also, $\mathrm{Obs}(f) = 0$ hence both $\overline{\mathcal{U}}^{\log}_{g,n}(X, \beta)$ and $\overline{\mathcal{U}}_{g,n}(X, \beta)$ are smooth. Thus, we have the following lemma.

Lemma 5.14 *Let $((C, x_1, x_2, \dots, x_n), \pi_{W/X}: W \to X, f: C \to W)$ be a stable unramified map over* $\mathrm{Spec}\,\mathbb{C}$. *If $H^1(C, f^* T_W^\dagger(-\sum \sigma_i)) = 0$, then $\overline{\mathcal{U}}_{g,n}(X, \beta)$ is smooth at this point.*

6 $\overline{\mathrm{M}}^3_{0,7}$ as a Parameter Space

In this section, we discuss a moduli theoretic interpretation of $\overline{\mathrm{M}}^3_{0,7}$, the first flip of $\overline{\mathrm{M}}_{0,7}$.

In a recent result Smyth [49], described a systematic classification of modular compactifications $\overline{\mathcal{M}}_{g,n}(\mathcal{Z})$ of $\mathcal{M}_{g,n}$, which can be described in term of certain combinatorial data $\mathcal{Z}$. They are moduli spaces of pointed curves with (possibly) worse singularities. In the case where $g = 0$, he obtained a complete classification of such compactifications [49, Theorem 1.21]. When $g = 0$, all such compactifications are obtained by contracting some irreducible components of parameterized curves and obtaining new arithmetic genus 0 singularities there. Because a singularity of arithmetic genus 0 does not have a positive dimensional moduli, all such compactifications are (usually small) contractions of $\overline{\mathrm{M}}_{0,n}$. Therefore, if we want to describe a moduli theoretic meaning of a flip of $\overline{\mathrm{M}}_{0,n}$, then it must not be a moduli of pointed curves with a certain singularity type. In other words, it is not a substack of the stack of all pointed curves ([49, Appendix B]).

From the description of $\overline{\mathrm{M}}^3_{0,7}$, we have several clues on its possible moduli theoretic meaning.

(a) The reduction map $\phi: \overline{\mathrm{M}}_{0,7} \to V_A^3$ contracts F-curves of type $F_{1,2,2,2}$. The image of a contracted F-curve corresponds to a pointed rational curve $(C, x_1, x_2, \dots, x_7)$, which has three irreducible components that meet at a triple nodal singularity. Then ϕ forgets the cross-ratio of four special points on the spine of $F_{1,2,2,2}$.

(b) A connected component of the exceptional fiber of the contraction $\phi_3': \overline{\mathrm{M}}^3_{0,7} \to V_A^3$ is isomorphic to $\mathbb{P}^2$.

Note that the image of $F_{1,2,2,2}$ is exactly the locus of non-nodal (and non-Gorenstein) curves on V_A^3 (See Example 4.3.). From (b), we can guess that $\overline{\mathrm{M}}^3_{0,7}$ is a moduli space of pointed curves parameterized by V_A^3, with some additional structure on non-Gorenstein singularities.

Question 6.1 What kind of infinitesimal structure can we give on non-Gorenstein singularities?

Note that V_A^3 is defined as a GIT quotient of an incidence variety in the product $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3,3)\times(\mathbb{P}^3)^7$. At least as parameter spaces in a weak sense, we are able to construct many new birational models of $\overline{\mathrm{M}}_{0,7}$ by using incidence varieties. For example, if we introduce additional factors such as $\mathbb{G}\mathrm{r}(1,3)^7$ which has the information about a tangent direction at each point, and take the GIT quotient (with an appropriate linearization) of the incidence variety in

$$\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3,3)\times(\mathbb{P}^3)^7\times\mathbb{G}\mathrm{r}(1,3)^7,$$

then we may have a resolution of V_A^3. Also, we can replace a factor by another modular variety. For instance it would be interesting if we consider the Fulton–MacPherson space $\mathbb{P}^3[7]$ instead of $(\mathbb{P}^3)^7$. But in our situation, we need to find a parameter space that fits into the picture of Mori's program for $\overline{\mathrm{M}}_{0,7}$. Thus, a refined question is the following.

Question 6.2 Which parameter space fits into the diagram $\phi_3'\colon\overline{\mathrm{M}}_{0,7}^3\to V_A^3$?

To answer this question, we will use KKO compactification, which we discussed in Section 5.

Let $\overline{\mathcal{U}}_{0,n}(\mathbb{P}^d,d)$ be the KKO compactification of the space of n-pointed rational normal curves in $\mathbb{P}^d$ and let $\overline{\mathrm{U}}_{0,n}(\mathbb{P}^d,d)$ be its coarse moduli space. Similarly, let $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^d,d)$ be the moduli stack of ordinary stable maps and let $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^d,d)$ be its coarse moduli space. We have the following commutative diagram:

$$\begin{tikzcd}
\overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3) \arrow[r, "F'"] \arrow[d, "S"'] & \overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)\times(\mathbb{P}^3)^7 \arrow[d, "S'"] \\
\overline{\mathrm{M}}_{0,7}(\mathbb{P}^3,3) \arrow[r, "F"] & \overline{\mathrm{M}}_{0,0}(\mathbb{P}^3,3)\times(\mathbb{P}^3)^7.
\end{tikzcd}$$

The vertical map S is the stabilization map S in Proposition 5.8, and $S'=S\times\mathrm{id}$. Then F is the product of a forgetful map and evaluation maps for the moduli space of stable maps, and $F'=F\times\prod ev_i$ is that of KKO compactifications (Propositions 5.10 and 5.9).

Let

$$I\subset\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3,3)\times(\mathbb{P}^3)^7$$

be the incidence variety parameterizing $(f\colon C\to\mathbb{P}^3,x_1,\dots,x_7)$ such that $x_i\in\mathrm{im}f$ for all i. It is straightforward to check that $I=\mathrm{im}\phi$. From the description of V_A^3 in Section 4.2, $V_A^3\cong I/\!/_L\mathrm{SL}_4$ with a suitable linearization L, which is a restriction of a linearized ample line bundle on $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3,3)\times(\mathbb{P}^3)^7$. Note that with respect to L, the stability coincides with the semi-stability. Let I^s be the stable locus.

Suppose that we have an incidence variety $J\subset\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)\times(\mathbb{P}^3)^7$. We would like to show that $J/\!/\mathrm{SL}_4\cong\overline{\mathrm{M}}_{0,7}^3$ for an appropriate choice of a linearization. The choice of the linearization is standard. For any G-equivariant projective morphism between two quasi-projective varieties $f: X \to Y$ and a linearization L on Y such that $Y^{ss}(L) = Y^s(L)$, there is a linearization L' on X such that

$$X^{ss}(L') = X^s(L') = f^{-1}\big(Y^s(L)\big)$$

([36, Section 3], [28, Theorem 3.11]). With respect to this linearization, there is a quotient map $\overline{S}: J/\!/_{L'}\mathrm{SL}_4 \to I/\!/_L \mathrm{SL}_4 \cong V_A^3$. Thus, if we carefully analyze the fiber of $\overline{S}$, then we can prove that $J/\!/_{L'}\mathrm{SL}_4 \cong \overline{\mathrm{M}}_{0,7}^3$.

But there are a few technical difficulties with this approach. Because the geometry of $\overline{\mathcal{U}}_{0,n}(\mathbb{P}^r, d)$ is very complicated, there are few results on its geometric properties. For instance, $\overline{\mathcal{U}}_{0,n}(\mathbb{P}^r, d)$ is not irreducible in general, the connectedness is unknown, and we do not know about the projectivity of its coarse moduli space $\overline{\mathrm{U}}_{0,n}(\mathbb{P}^r, d)$ even for $n = 0$ and $r = d = 3$. Furthermore, we do not have a nice modular description nor the deformation theory for the "main component" of $\overline{\mathcal{U}}_{0,n}(\mathbb{P}^r, d)$. So we are unable to apply the above standard approach. Thus, we will use an ad-hoc approach.

Let $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}} \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ be the substack of stable maps with non-degenerate image and let $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}} \subset \overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 3)$ be its coarse moduli space. Since $(f: C \to \mathbb{P}^3) \in \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}}$ has no nontrivial automorphisms, $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}} = \overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}}$ is a smooth open subvariety of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 3)$. Let

$$\overline{\mathcal{U}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}} := S^{-1}\big(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}}\big)$$

for the stabilization map in Proposition 5.8 and let $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}}$ be its coarse moduli space.

Lemma 6.3 *The open subset $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}} \subset \overline{\mathrm{U}}_{0,0}(\mathbb{P}^3, 3)$ is a smooth algebraic space.*

Proof First of all, we will show that $\overline{\mathcal{U}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}}$ is a smooth stack. Because every object $(f: C \to W) \in \overline{\mathcal{U}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}}$ is injective, it has no nontrivial automorphism. Thus, $\overline{\mathcal{U}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}} = \overline{\mathrm{U}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}}$, and the latter one is also smooth as an algebraic space.

Since $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ is a smooth Deligne–Mumford stack, it suffices to check that the smoothness at a map $(f: C \to W) \in \overline{\mathcal{U}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}}$ lying on the locus that

$$S: \overline{\mathcal{U}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}} \longrightarrow \overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$$

is not an isomorphism. If the target space W is $\mathbb{P}^3$, then there is no ghost component, and hence $(f: C \to W = \mathbb{P}^3)$ is already an object in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}}$. Since the image $(\pi \circ f)(C)$ is degenerate in $\mathbb{P}^3$, for any screen (after blowing-down all higher level screens) $Y \cong \mathbb{P}(T_x\mathbb{P}^3 \oplus \mathbb{C})$, $f(C) \cap \mathbb{P}(T_x\mathbb{P}^3)$ is a union of reduced points. If there is an end component $Y \cong \mathbb{P}(T_x\mathbb{P}^3 \oplus \mathbb{C}) \subset W$ of level one such that $\mathbb{P}(T_x\mathbb{P}^3) \cap f(C)$ is a set of two reduced points, then all ghost conics on Y are equivalent to each other and hence there is no non-trivial moduli of them. Hence, $\overline{\mathcal{U}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}}$ is not locally isomorphic to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}}$ along the locus that parametrizes a map $(f: C \to W)$ where the domain has three tails C_1, C_2, C_3, and there is a ghost spine C_4. There are three possibilities. See Figure 8.

(a) The spine C_4 is a level one smooth cubic ghost component.

(b) $C_4 = C_{4,1} \cup C_{4,2} \cup C_{4,3}$ is a chain of rational curves. $C_{4,1}$ has level one and degree two; $C_{4,3}$ has level one and degree one. Finally, $C_{4,2}$ has level two and degree two.
(c) $C_4 = C_{4,1} \cup \cdots \cup C_{4,5}$ is a chain of rational curves; $C_{4,1}, C_{4,3}, C_{4,5}$ are level one linear ghost components and $C_{4,2}, C_{4,4}$ are level two degree two ghost components on two different end components.

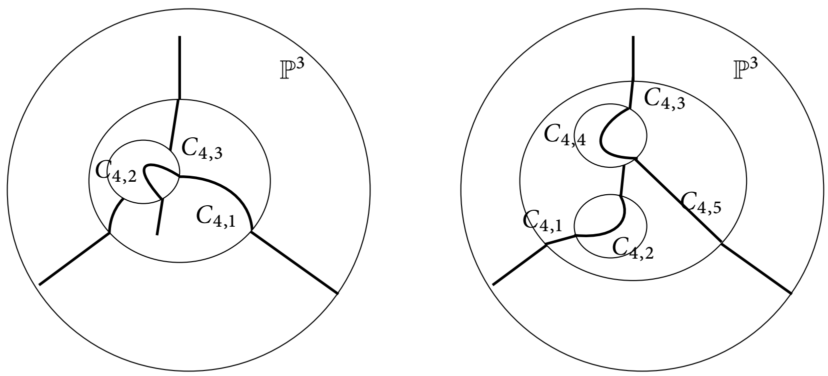

Figure 8: Ghost spines of type (2) and (3)

In each case, we are able to show the smoothness by computing the vanishing of the relative obstruction space (see Section 5.4 and [34, Section 8]). Recall that the relative obstruction is lying on

$$H^1(C, f^* T_W^\dagger),$$

where $T_W^\dagger$ is the logarithmic tangent space of W ([35, Proposition 5.1.1]). If we decompose C into the union of irreducible components $\cup C_j$ and if we denote $f|_{C_j}$ by f_j, then from the short exact sequence

$$0 \longrightarrow f^* T_W^\dagger \longrightarrow \bigoplus_j f_j^* T_W^\dagger \longrightarrow \bigoplus_{\{j \neq k\}} f^* T_W^\dagger|_{C_j \cap C_k} \longrightarrow 0$$

and the derived long exact sequence

$$\bigoplus_j H^0(C_j, f_j^* T_W^\dagger) \longrightarrow \bigoplus_{\{j \neq k\}} f^* T_W^\dagger|_{C_j \cap C_k} \longrightarrow H^1(C, f^* T_W^\dagger) \longrightarrow \bigoplus_j H^1(C_j, f_j^* T_W^\dagger),$$

it suffices to show 1) $H^1(C_j, f_j^* T_W^\dagger) = 0$ and 2) the surjectivity of $\bigoplus_j H^0(C_j, f_j^* T_W^\dagger) \to \bigoplus_{\{j \neq k\}} f^* T_W^\dagger|_{C_j \cap C_k}$.

Each irreducible component C_j is lying on an irreducible component V of W. If V is an end component (which is isomorphic to $\mathbb{P}^3$), then we have an Euler sequence

$$0 \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_V(1)^3 \oplus \mathcal{O}_V \longrightarrow T_W^\dagger|_V \longrightarrow 0,$$

and its pull-back

$$0 \longrightarrow \mathcal{O}_{C_j} \longrightarrow \mathcal{O}_{C_j}(d)^3 \oplus \mathcal{O}_{C_j} \longrightarrow f_j^* T_W^\dagger \longrightarrow 0, \tag{6.1}$$

where $d = \deg C_j$. Since $H^1(\mathbb{P}^1, \mathcal{O}_V(k)) = 0$ for all $k \ge -1$, we have $H^1(C_j, f_j^* T_W^\dagger) = 0$. If V is a root component, then we have

$$(6.2) \qquad 0 \longrightarrow \mathcal{O}_V(-E) \longrightarrow \pi^* \mathcal{O}_{\mathbb{P}^3}(1)(-E)^4 \longrightarrow T_W^\dagger|_V \longrightarrow 0,$$

where E is the exceptional divisor on the root component. Note that for all f above, E is irreducible. Since $f(C_j)$ is a line that intersects E, $H^1(C_j, f_j^*(\pi^*\mathcal{O}_{\mathbb{P}^3}(1)(-E))) = H^1(C_j, \mathcal{O}) = 0$. Finally, if V is a screen which is not an end component, we have

$$0 \longrightarrow \mathcal{O}_V(-E) \xrightarrow{\iota} \pi^*\mathcal{O}_{\mathbb{P}^3}(1)(-E)^3 \oplus \mathcal{O}_V(-E) \longrightarrow T_W^\dagger|_V \longrightarrow 0$$

where E is the union of exceptional divisors on V. In the above cases, the component $f(C_j)$ on V is a conic intersecting an exceptional divisor or a line intersecting one or two exceptional divisors. In any case, $H^1(C_j, f_j^*(\pi^*\mathcal{O}_{\mathbb{P}^3}(1)(-E))) = 0$, thus

$$H^1(C_j, f_j^*\big(\pi^*\mathcal{O}_{\mathbb{P}^3}(1)(-E)^3 \oplus \pi^*\mathcal{O}_V(-E)\big)) \cong H^1\big(C_j, f_j^*(\mathcal{O}_V(-E))\big).$$

Thus, $H^1(\iota)$ is surjective and $H^1(C_j, f_j^*(T_W^\dagger|_V)) = 0$.

For the surjectivity of

$$\bigoplus_j H^0(C_j, f_j^* T_W^\dagger) \longrightarrow \bigoplus_{\{j \ne k\}} f^* T_W^\dagger|_{C_j \cap C_k},$$

we will show a slightly stronger statement: for any level ℓ component C_j with $\ell = 0, 2$,

$$H^0(C_j, f_j^* T_W^\dagger) \longrightarrow \bigoplus_{\{\ell(C_k)=1\}} T_W^\dagger|_{C_j \cap C_k}$$

is surjective. If we denote the intersection point $C_j \cap C_k$ with $\ell(C_k) = 1$ by x_k, then it suffices to show $H^1(C_j, f_j^*(T_W^\dagger(-\sum x_k))) = 0$. For a level zero component, which has a unique x_k, from (6.2) we have

$$0 \longrightarrow \mathcal{O}_{C_j}(-2) \longrightarrow \mathcal{O}_{C_j}(-1)^4 \longrightarrow f_j^* T_W^\dagger|_{C_j}(-x_k) \longrightarrow 0.$$

So $H^1(C_j, f_j^* T_W^\dagger|_{C_j}(-x_k)) = 0$. For a level two component, which has two x_k's, from (6.1), we have

$$0 \longrightarrow \mathcal{O}_{C_j}(-2) \longrightarrow \mathcal{O}_{C_j}^3 \oplus \mathcal{O}_{C_j}(-2) \longrightarrow f_j^* T_W^\dagger\Big(-\sum x_k\Big) \longrightarrow 0.$$

We get the vanishing of $H^1(C_j, f_j^* T_W^\dagger(-\sum x_k))$ in a similar manner. ∎

Let $J^s := S'^{-1}(I^s)$ and J be the closure of J^s in $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3, 3) \times (\mathbb{P}^3)^7$. Then J is the main component of the "incidence subspace" in $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3, 3) \times (\mathbb{P}^3)^7$. Then J and J^s are both SL_4-invariant subspaces.

Lemma 6.4 (i) *The algebraic space J^s is a quasi-projective scheme.*
(ii) *There is a linearization L' on J^s such that for every closed point $x \in J^s$, there is a section $s \in H^0(J^s, L^m)$ such that $s(x) \ne 0$. In other words, $(J^s)^{ss}(L') = J^s$.*

Proof By local computation, we can check that the tangent map in Proposition 5.11

$$T: \overline{\mathrm{U}}_{0,0}(\mathbb{P}^3, 3)^{\mathrm{nd}} \longrightarrow \overline{\mathrm{M}}_{0,0}\big(\mathbb{P}(T\mathbb{P}^3), (3,10)\big)$$

is quasi-finite. Indeed, it might not be injective when $f: C \to W$ has a ghost component of degree 3. Take a rational normal curve N in a non-rigid $\mathbb{P}^3 = \{[x:y:z:w]\}$

passing through three coordinate points on the infinite plane $\{x=0\}$. By using an automorphism of $\mathbb{P}^3$, we can assume that N passes through $p=[1:0:0:0]$. Furthermore, if we fix the image of the tangent map at p, or equivalently, the tangent direction at p, we have a 2-dimensional family of rational normal curves. We can take an explicit 2-dimensional versal family, for instance,

$$f_{a,b}(s:t)=\big[(t-3s)(t-s)(t-2s)s:t(at-s)(t-2s)s:\\ t(t-s)(4t-s)(t-2s):t(bt-2)(2t-s)s\big].$$

By using a computer algebra system, it is straightforward to check that

$$\mathbb{P}(Tf_{a,b})([1:0])=[1:-1:1]$$

is independent of a and b, but for two $(a,b)\neq(a',b')$, the tangent vectors to $\mathbb{P}(Tf_{a,b})(\mathbb{P}^1)$ and $\mathbb{P}(Tf_{a',b'})(\mathbb{P}^1)$ at $[1:-1:1]$ are different. Thus, T is analytic locally injective if f has an irreducible ghost component. The remaining cases are easy to check.

Since the target of T is a scheme, $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)^{\mathrm{nd}}$ is a scheme by [37, Corollary II.6.16]. Furthermore, $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)$ is proper and $\overline{\mathrm{M}}_{0,0}(\mathbb{P}(T\mathbb{P}^3),(3,10))$ is separated. Thus, T is a proper morphism ([23, Corollary II.4.8]). Hence, T (restricted to $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)^{\mathrm{nd}}$) is finite ([21, Theorem 8.11.1]). Thus, T is projective ([19, Corollary 6.1.11]), hence $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)^{\mathrm{nd}}$ is quasi-projective.

Note that $J^s\subset\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)^{\mathrm{nd}}\times(\mathbb{P}^3)^7$. Since J^s is a locally closed subspace of a quasi-projective scheme, it is also quasi-projective. This proves (i).

Note that we have a commutative diagram

$$\begin{tikzcd} J^s \arrow[r] \arrow[d] & \overline{\mathrm{M}}_{0,0}(\mathbb{P}(T\mathbb{P}^3),(3,10))\times(\mathbb{P}^3)^7 \arrow[d, "F"] \\ I^s \arrow[r] & \overline{\mathrm{M}}_{0,0}(\mathbb{P}^3,3)\times(\mathbb{P}^3)^7. \end{tikzcd}$$

Since F is a projective morphism, by [28, Theorem 3.11], there is a linearization L' on $X:=\overline{\mathrm{M}}_{0,0}(\mathbb{P}(T\mathbb{P}^3),(3,10))\times(\mathbb{P}^3)^7$ such that $X^{ss}(L')=X^s(L')=F^{-1}((\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3,3)\times(\mathbb{P}^3)^7)^s(L))$. Since I^s is in the stable locus of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3,3)\times(\mathbb{P}^3)^7$, J^s maps to the stable locus of X. Therefore, the pull-back of L' to J^s is the linearization we want to find. ■

Therefore, by gluing the categorical quotients of affine SL_4-invariant subschemes, we obtain a well-defined quotient scheme J^s/SL_4.

Definition 6.5 The *formal GIT quotient* $J/\!/\mathrm{SL}_4$ is J^s/SL_4.

Remark 6.6 Note that if $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)$ is a projective scheme, then for a standard choice of linearization L' on $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)\times(\mathbb{P}^3)^7$, $J/\!/_{L'}\mathrm{SL}_4\cong J^s/\mathrm{SL}_4$. So far, we do not know the projectivity of $\overline{\mathrm{U}}_{0,n}(\mathbb{P}^r,d)$. We will investigate geometric properties of this moduli space in forthcoming papers.

Lemma 6.7 *The locus J^s is normal.*

Proof Set $J(0) = \overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)^{\mathrm{nd}}$ and for $n \in \mathbb{N}$, let

$$J(n) = \left\{ \left((f\colon C \longrightarrow W), x_1, x_2, \dots, x_n \right) \mid x_i \in \pi \circ f(C) \right\} \subset \overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)^{\mathrm{nd}} \times (\mathbb{P}^3)^n$$

for $\pi\colon W \to \mathbb{P}^3$. We claim that $J(n)$ is normal. Note that $J(0)$ is normal by Lemma 6.3.

Let $p_n\colon J(n) \to J(n-1)$ be the projection map forgetting the last point. Then for any point $((f\colon C \to W), x_1, x_2, \dots, x_{n-1}) \in J(n-1)$, the fiber is isomorphic to $\pi \circ f(C) \subset \mathbb{P}^3$. Since the Hilbert polynomial $P_{\pi\circ f(C)}(m) = 3m+1$ is constant, p_n is flat by [23, Theorem III.9.9].

Note that a general fiber of p_n is smooth because a general element of $J(n-1)$ parametrizes a smooth rational curve. So $J(n)$ is regular in codimension one if $J(n-1)$ is. Also since all fibers are curves, it automatically satisfies Serre's condition S_2. Therefore, $J(n)$ satisfies S_2 by [20, Corollary 6.4.2]. By Serre's criterion, $J(n)$ is normal if $J(n-1)$ is.

Since J^s is an open subset of $J(7)$, we have the desired result. ∎

Now we prove the second main result of this paper.

Theorem 6.8 *The formal GIT quotient $J/\!/\mathrm{SL}_4$ is isomorphic to $\overline{\mathrm{M}}^3_{0,7}$.*

Proof Let

$$\begin{aligned}\overline{\mathrm{M}}_{0,7}(\mathbb{P}^3,3)^s &= F^{-1}(I^s) \subset \overline{\mathrm{M}}_{0,7}(\mathbb{P}^3,3),\\ \overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3)^s &= S^{-1}(\overline{\mathrm{M}}_{0,7}(\mathbb{P}^3,3)^s) \subset \overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3).\end{aligned}$$

We have the following diagram:

$$\begin{tikzcd}
\overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3)^s \arrow[d, "S"'] \arrow[dr, "g"] \arrow[drr, dashed] & & \\
\overline{\mathrm{M}}_{0,7}(\mathbb{P}^3,3)^s \arrow[r, "/\mathrm{SL}_4"'] \arrow[d, "F"'] & \overline{\mathrm{M}}_{0,7} \arrow[d, "\phi"'] & \widetilde{\mathrm{M}}^3_{0,7} \arrow[l, "\pi_3"] \arrow[d, "\pi_3'"] \\
I^s \arrow[r, "/\mathrm{SL}_4"] & V^3_A & \overline{\mathrm{M}}^3_{0,7} \arrow[l, "\phi_3'"']
\end{tikzcd}$$

We first show that there is a morphism $\widetilde{g}\colon \overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3)^s \to \widetilde{\mathrm{M}}^3_{0,7}$. Because π_3 is the blow-up along F-curves of type $F_{1,2,2,2}$, from the universal property of blow-up, it is enough to show that $g^{-1}(F_{1,2,2,2})$ is a Cartier divisor in $\overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3)^s$.

Let $Z^0 \subset \overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)^{\mathrm{nd}}$ be the locally closed subvariety parametrizing $f\colon C \to W$ such that the domain C has three tails C_1, C_2, C_3 of degree one and an irreducible spine C_0 which is a ghost component of level one. Let Z be the closure of Z^0. To obtain $f \in Z^0$, we need to choose three lines C_1, C_2, and C_3 on $\mathbb{P}^3$ meeting at a point, and a cubic rational normal curve C_0 in a *non-rigid* $\mathbb{P}^3$ which passes through three points at rigid $\mathbb{P}^2 \subset \mathbb{P}^3$. Thus the dimension of Z^0 is $3+3\cdot 2+(12-3\cdot 2)-4 = 11$. Hence Z^0 and Z have codimension one in $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)^{\mathrm{nd}}$. Because $\overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)^{\mathrm{nd}}$ is smooth (Lemma 6.3), Z is a Cartier divisor. On the other hand, for $F\colon \overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3) \to \overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)$, $F(\overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3)^s) \subset \overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)^{\mathrm{nd}}$ since $\pi \circ f(C)$ is non-degenerated for all $f\colon C \to W$ in $\overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3)^s$. Finally, for the forgetful map $F\colon \overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3)^s \to \overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3)^{\mathrm{nd}}$, it is straightforward to check that $g^{-1}(F_{1,2,2,2}) = F^{-1}(Z)$. Therefore $g^{-1}(F_{1,2,2,2})$ is a Cartier divisor as well. Thus we have a morphism $\widetilde{g}: \overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3)^s \to \widetilde{\mathrm{M}}_{0,7}^3$. Let $\overline{g} = \pi_3' \circ \widetilde{g}: \overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3)^s \to \overline{\mathrm{M}}_{0,7}^3$.

The forgetful map $F': \overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3)^s \to \overline{\mathrm{U}}_{0,0}(\mathbb{P}^3,3) \times (\mathbb{P}^3)^7$ factors through J^s, because $S' \circ F'(\overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3)^s) = F \circ S(\overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3)^s) = I^s$ and $J^s = S'^{-1}(I^s)$. We have an algebraic fiber space $\overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3)^s \to J^s$ because J^s is normal ([23, Proof of Corollary III.11.4]). The only possible exceptional curve E for $\overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3)^s \to J^s$ is obtained by varying a unique marked point on a ghost component, hence varying the cross-ratio of them. E is contracted by $\overline{g}: \overline{\mathrm{U}}_{0,7}(\mathbb{P}^3,3)^s \to \overline{\mathrm{M}}_{0,7}^3$ because $\overline{g} = \pi_3' \circ \widetilde{g}$ and $\pi_3': \widetilde{\mathrm{M}}_{0,7}^3 \to \overline{\mathrm{M}}_{0,7}^3$ forgets the cross-ratio. Therefore there is a morphism $Q: J^s \to \overline{\mathrm{M}}_{0,7}^3$ ([38, Proposition II.5.3]). Finally, because it is SL_4-equivariant, there is a quotient map $\overline{Q}: J/\!/\mathrm{SL}_4 = J^s/\mathrm{SL}_4 \to \overline{\mathrm{M}}_{0,7}^3$ and a commutative diagram

$$\begin{tikzcd} J/\!/\mathrm{SL}_4 \arrow[r, "\overline{Q}"] \arrow[d] & \overline{\mathrm{M}}_{0,7}^3 \arrow[d, "\phi_3'"] \\ I/\!/_L\mathrm{SL}_4 \arrow[r, "\cong"] & V_A^3. \end{tikzcd}$$

On a point x of the exceptional locus of $\phi_3': \overline{\mathrm{M}}_{0,7}^3 \to V_A^3$, by dimension counting, it is straightforward to check that the inverse image $\overline{Q}^{-1}(x)$ does not have a positive dimensional moduli. Also, on the outside of the exceptional locus, they are isomorphic. Thus, $\overline{Q}$ is a quasi-finite birational morphism to a smooth variety. So it is an isomorphism by [44, Proposition III.9.1]. ∎

Remark 6.9 We can describe an object in $J/\!/\mathrm{SL}_4$ in an intrinsic way. For $(f: C \to W) \in \overline{\mathcal{U}}_{0,0}(\mathbb{P}^3,3)^{\mathrm{nd}}$, suppose that the image of $\pi \circ f: C \to W \to \mathbb{P}^3$ has a non-Gorenstein singularity at $x \in \mathrm{im}\pi \circ f(C)$. Three irreducible components meet at x. The level one component $Y = \mathbb{P}(T_x\mathbb{P}^3 \oplus \mathbb{C})$ of W at x can be regarded as a compactified non-rigid tangent space $\mathbb{P}(T_xC \oplus \mathbb{C})$, because the three irreducible components generate $\mathbb{P}^3$. Hence the infinitesimal structure we can give on the non-Gorenstein singularity $x \in C$, as an answer for Questions 6.1 and 6.2, is a ghost rational cubic curve (and its degeneration) on a compactified non-rigid tangent space of C at x.

Remark 6.10 (i) It would be very interesting if one could define $J/\!/\mathrm{SL}_4$ as a moduli stack directly, instead of describing it as a quotient stack of a certain moduli stack.

(ii) The similar modular flip appears for every $n \geq 7$. For example, if we consider a D-flip for the total boundary divisor B on $\overline{\mathrm{M}}_{0,n}$, then the flipping locus contains the locus covered by $F_{1,i,j,k}$ where $i,j,k \geq 2$. Therefore it is inevitable to study such flips in general, if we would like to study the full symmetric Mori's program for $\overline{\mathrm{M}}_{0,n}$.

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