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# The QCD coupling $\alpha_s$

In this chapter, we shall outline the different determinations of the QCD coupling $\alpha_s$ from $\tau$ decay, $e^+e^- \to I = 1$ hadrons processes and heavy quark systems. This discussion has been already anticipated in Sections 25.5, 25.6 and Part VI and will not be repeated in detail here This chapter will complete these discussions.

## 51.1 $\alpha_s$ from $e^+e^- \to I = 1$ hadrons and $\tau$-decays data

These channels have the great advantage that the spectral functions are measured in a region where pQCD is applicable and therefore the analysis does not suffer from any model dependence in the parametrization of the spectral functions. The $\tau$ decays and $\tau$-like decays in $e^+e^- \to I = 1$ hadron processes (see previous Sections 25.5 and 25.6) are *another prototype of QCD spectral sum rules* like the Weinberg sum rules

- The determinations of the QCD coupling from hadronic $\tau$-decays and $e^+e^- \to I = 1$ hadrons have been already discussed in details in Sections 25.5, 25.6 and are based on the approach proposed by BNP [325] for $\tau$-like decay processes. It relies on the fact that the non-perturbative contributions based on the 'standard SVZ expansion' give a tiny correction at the $\tau$ mass. In addition, recent analysis [161] including the new $D = 2$ dimension tachyonic gluon mass term not considered in the analysis of [325] shows that this effect is small and does not affect in a significant way the determination carried out without this term. It tends to reduce slightly (about 10%) the central value of $\alpha_s$ improving the agreement between the $\tau$-decay prediction and the world average at the $Z^0$ mass. However, this change is marginal as it is of similar size to the theoretical error in the $\alpha_s$ determination from $\tau$-decay.
- The previous original BNP approach [325] has been generalized to higher moments [333] that have been exploited by the different experimental groups [328,33] for extracting $\alpha_s$. The mean value coming from the different structure of pQCD series of the ALEPH/OPAL [33] measurements is:

$$\alpha_s(M_\tau) = 0.323 \pm 0.005(\text{exp}) \pm 0.030(\text{th}) . \tag{51.1}$$

This rather modest accuracy runned until the $Z^0$ mass leads to the precise determinations:

$$\alpha_s(M_{Z^0}) = 0.1181 \pm 0.0007 \,(\text{exp}) \pm 0.0030(\text{th}) , \tag{51.2}$$

which is in excellent agreement with the different determinations summarized in Part VI.

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- In [346,338,391] and [329] (reprinted in Section 52.10) $e^+e^- \to I = 1$ hadrons below 2 GeV and the sum of exclusive $\Delta S = 0$ $\tau$-decay data have been also used to extract the value of $\alpha_s$ as a cross-check of the value obtained from inclusive $\tau$-decay data. This result has been given in Section 25.6. The most recent analysis in [329] gives:

$$\alpha_s(M_\tau) = 0.33 \pm 0.03 , \qquad (51.3)$$

which is in excellent agreement with the $\tau$-decay data result.

- Stability of the previous determinations using lower $\tau$-mass values has been tested using $e^+e^- \to I = 1$ hadrons below 2 GeV [346] and the inclusive distributions of $\tau$-decay [345,33]. This test is reassuring as it indicates that for reasonable values of $M_\tau$ larger than 1 GeV, pQCD still applies, while the OPE parametrized by the few lowest dimension condensates describes the data quite well. It also indicates that the contribution near the real axis where QCD does not apply is negligible due to the phase space double suppression factor $(1 - s/M_\tau^2)^2$. It also indicates that *quark-hadron duality* which is the main idea behind this dispersion relation approach and then behind the QCD spectral sum rule approach is fulfilled by QCD.

## 51.2 $\alpha_s$ from heavy quarkonia mass-splittings

Examining the pioneering SVZ charmonium sum rule [1] (as one can also see in details in [3] and in the next chapter) the QCD side of this sum rule contains three relevant QCD parameters, namely the charm quark mass, the perturbative radiative correction $\alpha_s$ and the gluon condensate $\langle \alpha_s G^2 \rangle$. In order to extract more reliably one of these parameters, one has to find appropriate sum rules or their combinations which can disentangle these parameters.

- In [313] (Section 51.3), it is observed that a double ratio $\mathcal{R}_{P_1^1}/\mathcal{R}_{P_1^3}$ of Laplace sum rules for the $P$ charmonium states can be used for extracting the value of $\alpha_s$:

$$\frac{\mathcal{R}_{P_1^1}}{\mathcal{R}_{P_1^3}} \simeq \frac{M_{P_1^1}^2}{M_{P_1^3}^2} \simeq 1 + \alpha_s \big[ \Delta_\alpha^{13}(\text{exact}) = 0.014^{-0.004}_{+0.008} \big] , \qquad (51.4)$$

where the gluon condensate effect vanishes to leading order. Using the experimental mass, one can deduce at the optimization point $\sigma \simeq (0.6 \pm 0.1)$ GeV$^{-2}$:

$$\alpha_s(\sigma^{-1} = 1.3 \text{ GeV}) = 0.64^{+0.36}_{-0.18} \pm 0.02 \implies \alpha_s(M_Z) = 0.127 \pm 0.009 \pm 0.002 . \qquad (51.5)$$

- The non-relativistic $q^2 = 0$ moments sum rules for the $\Upsilon$ have also been used in [155] in order to extract $\alpha_s$. Recent improvements [156] of the previous analysis including the new $\alpha_s^2$ corrections of the two-point correlator [447] leads to the value:

$$\alpha_s(M_b) = 0.233^{+0.045}_{-0.030} , \qquad (51.6)$$

which runned until $M_Z$ gives:

$$\alpha_s(M_Z) = 0.120^{+0.010}_{-0.008} . \qquad (51.7)$$

These results are less accurate than the determination from $\tau$-decay, $e^+e^-$ and LEP data but are still interesting for an independent determination of $\alpha_s$.

## 51.3 Reprinted paper

# *Heavy quarkonia mass-splittings in QCD: gluon condensate, $\alpha_s$ and $1/m$-expansion*

## S. Narison

## 1. The double ratio of moments

QCD spectral sum rule (QSSR) after SVZ [1] (for a recent review, see, e.g. [2]) has shown since 15 years, its impressive ability for describing the complex phenomena of hadronic physics with the few universal "fundamental" parameters of the QCD Lagrangian (QCD coupling $\alpha_s$, quark masses and vacuum condensates built from the quarks and/or gluon fields), without waiting for a complete understanding of the confinement problem. In the example of the two-point correlator:

$$\Pi_Q(q^2) \equiv i \int d^4 x \, e^{iqx} \langle 0| \mathcal{T} J_Q(x) (J_Q(o))^\dagger |0 \rangle \, , \tag{1}$$

associated to the generic hadronic current: $J_Q(x) \equiv \bar{Q} \Gamma Q(x)$ of the heavy $Q$-quark ($\Gamma$ is a Dirac matrix which specifies the hadron quantum numbers), the SVZ-expansion reads:

$$\Pi_Q(q^2) \simeq \sum_{D=0,2,\dots} \sum_{\dim 0 = D} \frac{C^{(J)}(q^2, M_Q^2, \mu) \langle O(\mu) \rangle}{(M_Q^2 - q^2)^{D/2}} \, , \tag{2}$$

where $\mu$ is an arbitrary scale that separates the long- and short-distance dynamics; $C^{(J)}$ are the Wilson co-efficients calculable in perturbative QCD by means of Feynman diagrams techniques; $\langle O \rangle$ are the non-perturbative condensates of dimension $D$ built from the quarks or/and gluon fields ($D = 0$ corresponds to the case of the naïve perturbative contribution). Owing to gauge invariance, the lowest dimension condensates that can be formed are the $D = 4$ light quark $m_q \langle \bar{\psi} \psi \rangle$ and gluon $\langle \alpha_s G^2 \rangle$ ones, where the former is fixed by the pion PCAC relation, whilst the latter is known to be $(0.07 \pm 0.01) \, \text{GeV}^4$ from more recent analysis of the light [3] quark systems [2]. The validity of the SVZ-expansion has been understood formally, using renormalon techniques (absorption of the IR renormalon ambiguity into the definitions of the condensates and absence of some extra $1/q^2$-terms not included in the OPE) [4,5] and/or by building renormalization-invariant combinations of the condensates (Appendix of [6] and references therein). The SVZ expansion is phenomenologically confirmed from the unexpected accurate determination of the QCD coupling $\alpha_s$ and from a measurement of the condensates from semi-inclusive $\tau$-decays [6–8].

The previous QCD information is transmitted to the data through the spectral function Im $\Pi_Q(t)$ via the Källen–Lehmann dispersion relation (*global duality*) obeyed by the hadronic correlators, which can be improved from the uses of different versions of the sum rules [1,9–11]. In this paper, we shall use the simple duality ansatz parametrization: "*one narrow resonance*"+"*QCD continuum*", from a threshold $t_c$, which gives a good

description of the spectral integral in the sum rule analysis, as has been tested successfully in the light-quark channel from the $e^+e^- \to I = 1$ hadron data and in the heavy-quark ones from the $e^+e^- \to \psi$ or $\Upsilon$ data. We shall work with the relativistic version of the Laplace or exponential sum rules where the QCD expression known to order $\alpha_s$ is given in terms of the pole mass $m(p^2 = m^2)$ [11–14]:[1]

$$\mathcal{L}(\sigma, m^2) \equiv \int\limits_{4m^2}^{\infty} dt \exp(-t\sigma) \frac{1}{\pi} \operatorname{Im} \Pi_Q(t)$$

$$= 4m^2 A_H(\omega) \left[ 1 + \alpha_s a_H(\omega) + \frac{\pi}{36} \frac{\langle \alpha_s G^2 \rangle}{m^4} b_H(\omega) \right] ,$$

$$\mathcal{R}_H(\sigma) \equiv -\frac{d}{d\sigma} \log \mathcal{L}_H(\sigma, m^2) = 4m^2 F_H(\omega)$$

$$\times \left[ 1 + \alpha_s P_H(\omega) + \frac{\pi}{36} \frac{\langle \alpha_s G^2 \rangle}{m^4} Q_H(\omega) \right] , \tag{3}$$

where

$$\omega \equiv 1/x = 4m^2 \sigma \tag{4}$$

is a dimensionless variable, while $\sigma \equiv \tau$ (notation used in the literature) is the exponential Laplace sum rule variable; $F_H$, $P_H$ and $Q_H$ are complete QCD Whittaker functions compiled in [12–14]; $H$ specifies the hadronic channel studied. In principle, the pair $(\sigma, t_c)$ are free external parameters in the analysis, so that the optimal result should be insensitive to their variations. *Stability criteria*, which are equivalent to the variational method, state that the best results should be obtained at the minimas or at the inflexion points in $n$ or $\sigma$, while stability in $t_c$ is useful to control the sensitivity of the result in the changes of $t_c$-values. These stability criteria are satisfied in the heavy quark channels studied here, as the continuum effect is negligible and does not exceed 1% of the ground state contribution [2,12], such that at the minimum in $\sigma$, one expects to a good approximation:

$$\min_{\sigma} \mathcal{R}(\sigma) \simeq M_H^2 . \tag{5}$$

Moreover, one can a posteriori check that, at the stability point, where we have an equilibrium between the continuum and the non-perturbative contributions, which are both small, the OPE is still convergent such that the SVZ-expansion makes sense. The previous approximation can be improved by working with the double ratio of moments[2]:

$$\mathcal{R}_{HH'}(x) \equiv \frac{\mathcal{R}_H}{\mathcal{R}_{H'}} \simeq \frac{M_H^2}{M_{H'}^2}$$

$$= \Delta_0^{HH'} \left[ 1 + \alpha_s \Delta_{\alpha_s}^{HH'} + \frac{4\pi}{9} \langle \alpha_s G^2 \rangle \sigma^2 x^2 \Delta_G^{HH'} \right] , \tag{6}$$

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[1]For consistency, we shall work with the two-loop order $\alpha_s$ expression of the pole mass [15].
[2]This method has also been used in [16] for studying the mass splittings of the heavy-light quark systems.

provided that each ratio of moments stabilizes at about the same value of $\sigma$, as in this case, there is a cancellation of the different leading terms such as the heavy quark mass (and its ambiguous definition used in some previous literatures), the negligible continuum effect (which is already small in the ratio of moments), and each leading QCD corrections. We shall limit ourselves here to the $\alpha_s$-correction for the perturbative contribution and to the leading order one in $\alpha_s$ for the gluon condensate effects. To the order we are working, the gluon condensate is well-defined as the ambiguity only comes from higher order terms in $\alpha_s$, which have, however, a smaller numerical effect than the one from the error of the phenomenological estimate of the condensate.

## 2. Test of the $1/m$-expansion

For this purpose, we use the complete *horrible* results expressed in terms of the pole mass to order $\alpha_s$ given by [12] and checked by various authors [2], which we expand with the help of the Mathematica program. We obtain for different channels the expressions given in Table 1. By comparing the complete and truncated series in $1/m$, one can notice that, at the $c$ and $b$ mass scales, the convergence of the $1/m$-expansion is quite bad due to the increases of the numerical coefficients with the power of $1/m$ and to the alternate signs of the $1/m$ series.

## 3. Balmer-mass formula from the ratio of moments

The Balmer formula derived from a non-relativistic approach $(m \to \infty)$ of the Schrödinger levels reads [17] (see also [18–20]). for the $S_1^3$ vector meson:

$$M_{S_1^3} \simeq 2m \left[ 1 - \frac{2}{9}\alpha_s^2 + 0.23\frac{\pi}{(m\alpha_s)^4}\langle\alpha_s G^2\rangle \right] . \tag{7}$$

It is instructive to compare this result with the mass formula obtained from the ratio of moments within the $1/m$-expansion. Using the different QCD corrections in Table 1, one obtains the mass formula at the minimum in $\sigma$ of $\mathcal{R}$:

$$M_{S_1^3} \simeq \sqrt{\mathcal{R}(\sigma_{\min})} \simeq 2m \left( 1 + \frac{3}{16m^2\sigma} \right)$$
$$\times \left[ 1 - \frac{\sqrt{\pi}}{6m}\frac{\alpha_s(\sigma)}{\sqrt{\sigma}} + \frac{\pi}{12}\sigma^2\langle\alpha_s G^2\rangle \right] . \tag{8}$$

In the case of the $b$-quark, where we expect the *static approximation* to be more reliable, the minimum of this quantity is obtained to leading order at:

$$\sqrt{\sigma_{\text{coul}}} \simeq \frac{9}{4m\alpha_s\sqrt{\pi}} \simeq 0.85 \text{ GeV}^{-1} , \tag{9}$$

Table 1. *Expanded expressions of different QCD corrections in the case of the pole mass $m(p^2 = m^2)$ known to order $\alpha_s$*

| | Vector $S_1^3$ |
|---|---|
| $\pi A_V$ | $\frac{3}{16\sqrt{\pi}}x^{3/2}\left(1 - \frac{3}{4}x + \frac{45}{32}x^2 - \frac{525}{128}x^3 + \cdots\right)$ |
| $a_V$ | $\frac{4}{3\sqrt{\pi}}\left(\frac{\pi}{\sqrt{x}} + 0.040 + 1.952\sqrt{x} - 1.539x - \cdots\right).$ |
| $b_V$ | $-\frac{1}{2x^3} + \frac{3}{2x^2} + \frac{27}{8x} - \frac{21}{8} + \cdots$ |
| $F_V$ | $1 + \frac{3}{2}x - \frac{5}{4}x^2 + 5x^3 - \cdots$ |
| $P_V$ | $-\frac{2}{3}\sqrt{\pi x} + 2.704x^{3/2} - 10.093x^{5/2} + 52.93x^{7/2} - \cdots$ |
| $Q_V$ | $\frac{3}{2x^2} - \frac{1}{4x} + \frac{13}{8} - \frac{41}{4}x + \cdots$ |
| | S-waves splitting |
| $\Delta_0^{VP}$ | $1 - \frac{x^2}{2} + \frac{7}{2}x^3 - \cdots$ |
| $\Delta_\alpha^{VP}$ | $\frac{\sqrt{\pi}}{9}x^{3/2} + 1.539x^2 - 3.0258x^{5/2} - 7.719x^3 + 26.307x^{7/2} + \cdots$ |
| $\Delta_G^{VP}$ | $\frac{5}{x}\left(1 - \frac{4}{5}x + \frac{11}{10}x^2 + \frac{17}{10}x^3 - \cdots\right)$ |
| | P-waves splittings |
| $\Delta_0^{01}$ | $1$ |
| $\Delta_\alpha^{01}$ | $-3.18x^2(1 - 10.17x + 102.1x^2 + \cdots)$ |
| $\Delta_G^{01}$ | $-\frac{2}{x} + \frac{5}{2} - \frac{55}{4}x + \cdots$ |
| $\Delta_\alpha^{13}$ | $1.06x^2(1 - 9.5x + 81.1x^2 - \cdots)$ |
| $\Delta_0^{AT}$ | $1 + x^2 - \frac{23}{2}x^3 + \cdots$ |
| $\Delta_\alpha^{AT}$ | $-0.1576x^{3/2} - 2.545x^2 + 3.95x^{5/2} - \cdots$ |
| $\Delta_G^{AT}$ | $-\frac{6}{x} + \frac{31}{4} - \frac{89}{8}x - \frac{1715}{8}x^2 + \cdots$ |
| | P-versus S-waves splittings |
| $\Delta_0^{VS}$ | $1 - x + 5x^2 - 30x^3 + \cdots$ |
| $\Delta_\alpha^{VS}$ | $-\frac{2}{9}\sqrt{\pi x} - 0.336x^{3/2} + 4.244x^2 + 7.458x^{5/2} - 42.017x^3 - \cdots$ |
| $\Delta_G^{VS}$ | $-\frac{3}{x^2} - \frac{2}{x} - \frac{41}{4} + \frac{389}{4}x - \cdots$ |
| $\Delta_0^{VA}$ | $\Delta_0^{VS}$ |
| $\Delta_\alpha^{VA}$ | $-\frac{2}{9}\sqrt{\pi x} - 0.336x^{3/2} + 1.06x^2 + 7.458x^{5/2} - 9.655x^3 \cdots$ |
| $\Delta_G^{VA}$ | $-\frac{3}{x^2} - \frac{4}{x} - \frac{31}{4} + \frac{167}{2}x - \cdots$ |
| $\Delta_0^{VT}$ | $1 - x + 6x^2 - \frac{85}{2}x^3 + \cdots$ |
| $\Delta_\alpha^{VT}$ | $-\frac{2}{9}\sqrt{\pi x} - 0.493x^{3/2} - 1.484x^2 + 11.409x^{5/2} + 18.248x^3 - \cdots$ |
| $\Delta_G^{VT}$ | $-\frac{3}{x^2} - \frac{10}{x} + \frac{579}{8}x - \frac{16719}{16}x^2 + \cdots$ |

where we have used for 5 flavours[3]: $\alpha_s(\sigma) \simeq 0.32 \pm 0.06$ after evolving the value $\alpha_s(M_z) = 0.118 \pm 0.006$ from LEP [21] and $\tau$-decay data [6–8]. The inclusion of the gluon

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[3]In the approximation we are working, the effect of the number of flavours enters only through the $\beta$-function and therefore is not significant.

condensate correction shifts the value of $\sigma_{\min}$ to:

$$\sqrt{\sigma_{\min}} \simeq 0.74 \sqrt{\sigma_{\text{coul}}} \,. \tag{10}$$

These previous values of $\sigma$ confirm the more involved numerical analysis in [12] and indicate the relevance of the gluon condensate in the analysis of the spectrum. By introducing the previous leading order expression of $\sigma_{\text{coul}}$ into the sum rule, one obtains:

$$M_\Upsilon \simeq 2m_b \left(1 + \frac{\pi}{27}\alpha_s^2\right)\left[1 - \frac{2}{9}\left(\frac{\pi}{3}\right)\alpha_s^2 + \left(\frac{27}{128}\right)\left(\frac{3}{\pi}\right)^2 \frac{\pi}{(m_b\alpha_s)^4}\langle\alpha_s G^2\rangle\right] , \tag{11}$$

where one can deduce by identification in the *static limit* ($m_b \to \infty$) that the Coulombic effect is exactly the same in the two approaches. The apparent factor $\pi/3$ is due to the fact that we use here the approximate Schwinger interpolating formula for the two-point correlator. The gluon condensate coefficient is also about the same in the two approaches. This agreement indicates the consistency of the potential model and sum rule approach in the static limit, though a new extra $\alpha_s^2$ correction due to the $v^2$ (finite mass) terms in the free part appears here (for some derivations of the relativistic correction in the potential approach see [22,23]), and tends to reduce the Coulombic interactions. On the other hand, at the $b$-quark mass scale, the dominance of the gluon condensate contribution indicates that the $b$-quark is not enough heavy for this system to be Coulombic rendering the non-relativistic potential approach to be a crude approximation at this scale.

## 4. $S_1^3 - S_0^1$ hyperfine and $P - S$-wave splittings

In the non-relativistic approach used in [20], the hyperfine and $S-P$ wave splittings are given to leading order by:

$$M\left(S_1^3\right) - M\left(S_0^1\right) \simeq 2m\frac{(C_F\alpha_s)^4}{6}\left[1 + 3.255\frac{\pi}{2m^4\alpha_s^6}\langle\alpha_s G^2\rangle\right] ,$$

$$M\left(P_1^3\right) - M\left(S_3^1\right) \simeq 2m\left[\frac{3(C_F\alpha_s)^2}{32} + \frac{2^5\pi}{(C_F m\alpha_s)^4}\langle\alpha_s G^2\rangle\right] , \tag{12}$$

where $C_F = 4/3$. Using the double ratio of moments and the QCD corrections given in Table 1, one obtains at $\sigma_{\text{coul}}$:

$$\frac{M\left(S_1^3\right) - M\left(S_0^1\right)}{M\left(S_1^3\right)} \approx -4\pi^2\left(\frac{\alpha_s}{9}\right)^4 + \frac{8}{9}\left(\frac{\sqrt{\pi}\alpha_s}{9}\right)^3\alpha_s + \frac{45}{32m^4\alpha_s^2}\langle\alpha_s G^2\rangle + \cdots ,$$

$$\frac{M\left(P_1^3\right) - M\left(S_1^3\right)}{M\left(S_1^3\right)} \approx \frac{4\pi}{81}\alpha_s^2 + \frac{2\pi}{81}\alpha_s^2 + \frac{27}{8m^4\alpha_s^2}\langle\alpha_s G^2\rangle + \cdots , \tag{13}$$

where the corrections are, respectively, relativistic, Coulombic and non-perturbative. By comparing the sum rules in Eqs. (8) and (13), one can realize that the leading $x$ or $1/\sigma$-terms cancel in the hyperfine splitting, while the $x$-expansion is slowly convergent for the

$\alpha_s$-term at the $b$-mass. Comparing now this result with the one from the non-relativistic approach, it is interesting to notice that both approaches lead to the same $\alpha_s$-behaviour of the Coulombic and gluon condensate contributions. A one to one correspondence of each QCD corrections is not very conclusive, and needs an evaluation of the correlator at the next-next-to-leading order for a better control of the $\alpha_s^2 x$ terms. However, as the Coulombic potential is a fundamental aspect of QCD, we shall, however, expect that, after the resummation of the higher order terms in $\alpha_s$, the coefficient of the $\alpha_s^4$-term in the hyperfine splitting will be the same in the two alternative approaches. In the case of the $S - P$ wave splitting, the sum of the $\alpha_s^2$ corrections agrees from the two methods, though one can also notice that the relativistic correction is larger than the Coulombic one. The discrepancy for the coefficients of the gluon condensate in the two approaches is more subtle and may reflect the difficulty of Bell-Bertlmann [19] to find a bridge between the field theory after SVZ (flavour-dependent confining potential) and the potential models (flavour-independence). Resolving the different puzzles encountered during this comparison is outside the scope of the present paper.

## 5. Leptonic width and quarkonia wave function

Using the sum rule $\mathcal{L}_H$ and saturating it by the vector $S_1^3$ state, we obtain, to a good approximation, the sum rule:

$$M_V \Gamma_{V \to e^+ e^-} \simeq (\alpha e_Q)^2 \frac{e^{2\delta m M_V \sigma}}{72\sqrt{\pi}} \frac{\sigma^{-3/2}}{m} \times \left[ 1 + \frac{8}{3}\sqrt{\pi\sigma}m\alpha_s - \frac{4\pi}{9}\langle \alpha_s G^2 \rangle m \sigma^{5/2} \right] ,$$
(14)

where $e_Q$ is the quark charge in units of e; $\delta m \equiv M_V - 2m$ is the meson-quark mass gap. In the case of the $b$-quark, we use [15] $\delta m \simeq 0.26$ GeV, and the value of $\sigma_{\min}$ given in Eq. (10). Then:

$$\Gamma_{\Upsilon(S_1^3) \to e^+ e^-} \simeq 1.2 \text{ keV} ,$$
(15)

in agreement with the value found from the data, 1.3 keV. However, one should remark from Eq. (14) that the $\alpha_s$ correction is huge and needs an evaluation of the higher order terms (the gluon condensate effect is negligible), while the exponential factor effect is large, such that one can *reciprocally* use the data on the width to fix either $\alpha_s$ or/and the quark mass. Larger value of the heavy quark mass at the two-loop level (see, e.g. [26]) corresponding to a negative value of $\delta m$, would imply a smaller value of the leptonic width in disagreement with the data.

In the non-relativistic approach, one can express the quarkonia leptonic width in terms of its wave function $\Psi(0)_Q$:

$$\Gamma_{V \to e^+ e^-} = \frac{16\pi\alpha^2}{M_V^2} e_Q^2 |\Psi(0)|_Q^2 \left( 1 - 4C_F \frac{\alpha_s}{\pi} \right) ,$$
(16)

where (see, e.g. [20]):

$$16\pi |\Psi(0)|_Q^2 \left( 1 - 4C_F \frac{\alpha_s}{\pi} \right) \simeq 2(m C_F \alpha_s)^3 \approx 15 \text{ GeV}^3 .$$
(17)

In our approach, one can deduce:

$$16\pi\,|\Psi(0)|_Q^2\left(1 - 4C_F\frac{\alpha_s}{\pi}\right) \simeq \frac{1}{72\sqrt{\pi}}\,e^{2\delta m M_V \sigma}\sigma^{-3/2}\frac{M_V}{m}$$
$$\times\left[1 + \frac{8}{3}\sqrt{\pi\sigma}\,m\alpha_s - \frac{4\pi}{9}\langle\alpha_s G^2\rangle m\sigma^{5/2}\right] \simeq 18.3\ \text{GeV}^3\ . \tag{18}$$

Using the expression of $\sigma_{\text{coul}}$, one can find that, to leading order, the two approaches give a similar behaviour for $\Psi(0)_Q$ in $\alpha_s$ and in $m$ and about the same value of this quantity, though, one should notice that in the present approach, the QCD coupling $\alpha_s$ is evaluated at the scale $\sigma$ as dictated by the renormalization group equation obeyed by the Laplace sum rule [27] but *not* at the resonance mass!

## 6. Gluon condensate from $M_{\psi(S_1^3)} - M_{\eta_c(S_0^1)}$

The value of $\sigma$, at which, the $S$-wave charmonium ratio of sum rules stabilize is [12]:

$$\sigma \simeq (0.9 \pm 0.1)\ \text{GeV}^{-2}\ . \tag{19}$$

Using the range of the charm quark pole mass to order $\alpha_s$ accuracy [15][4]: $m_c \simeq 1.2$–$1.5\ \text{GeV}$ one can deduce the conservative value of $x$:

$$\omega \equiv 1/x \simeq 6.6 \pm 1.8\ . \tag{20}$$

The ratio of the mass squared of the vector $V(S_1^3)$ and the pseudoscalar $P(S_0^1)$ is controlled by the double ratio of moments given generically in Eq. (6), where the exact expressions of the corrections read:

$$\Delta_0^{VP} \simeq 0.995^{+0.001}_{-0.004}\ , \qquad \Delta_\alpha^{VP} \simeq 0.0233^{-0.009}_{+0.011}\ ,$$
$$\Delta_G^{VP} \simeq 29.77^{+8.86}_{-10.23}\ , \tag{21}$$

where each terms lead to be about 0.5, 2 and 7% of the leading order one. One can understand from the approximate expressions in Table 1 that the leading $x$-corrections appearing in the ratio of moments cancel in the double ratio, while the remaining corrections are relatively small. However, the $x$-expansion is not convergent for the $\alpha_s$-term at the charm mass, which invalidates the use of the $1/m$-expansion done in [28] in this channel. Using for 4 flavours [15]: $\alpha_s(\sigma) \simeq 0.48^{+0.17}_{-0.10}$, and the experimental data [25]: $\mathcal{R}_{VP}^{\text{exp}} = 1.082$, one can deduce the value of the gluon condensate:

$$\langle\alpha_s G^2\rangle \simeq (0.10 \pm 0.04)\ \text{GeV}^4\ . \tag{22}$$

We have estimated the error due to higher order effects by replacing the coefficient of $\alpha_s$ with the one obtained from the effective Coulombic potential, which tends to reduce the

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[4]For a recent review on the heavy quark masses, see e.g. [24,25].

estimate to $0.07 \, \text{GeV}^4$. We have tested the convergence of the QCD series in $\sigma$, by using the numerical estimate of the dimension-six gluon condensate $g\langle f_{abc} G^a G^b G^c \rangle$ contributions given in [14]. This effect is about 0.1% of the zeroth order term and does not influence the previous estimate in Eq. (22), which also indicates the good convergence of the ratio of exponential moments already at the charm mass scale in contrast with the $q^2 = 0$ moments studied in [1,29]. We also expect that in the double ratio of moments used here, the radiative corrections to the gluon condensate effects (their expression for the two-point correlator is however available in the literature [30]) are much smaller than in the individual moments, such that they will give a negligible effect in the estimate of the gluon condensate. This value obtained at the same level of $\alpha_s$-accuracy as previous sum rule results, confirm the ones of Bell-Bertlmann [11,12,14,30,2] from the ratio of exponential moments and from FESR-like sum rule for quarkonia [31,2] claiming that the SVZ value [1] has been under-estimated by about a factor 2 (see also [29,32]). Our value is also in agreement with the more recent estimate $(0.07 \pm 0.01) \, \text{GeV}^4$ from the $\tau$-like sum rules [3], and FESR [34] in $e^+ e^- \to I = 1$ hadrons. A more complete comparison of different determinations is done in Table 2.

## 7. Charmonium $P$-wave splittings

The analysis of the different ratios of moments for the $P$-wave charmonium shows [11–14] that they are optimized for:

$$\sigma \simeq (0.6 \pm 0.1) \, \text{GeV}^{-2} \,, \qquad \Longrightarrow \alpha_s(\sigma) \simeq 0.41^{+0.11}_{-0.07} \,, \qquad 1/x = 4.5 \pm 1.5 \,. \tag{23}$$

In the case of the *Scalar $P_0^3$-axial $P_1^3$ mass splitting*, the different exact QCD coefficient corrections of the corresponding double ratio of moments read:

$$\Delta_0^{01} = 1 \,, \qquad \Delta_\alpha^{01} \simeq -\left(0.045^{-0.014}_{+0.028}\right) \,, \qquad \Delta_G^{01} \simeq -\left(7.75^{+2.84}_{-2.77}\right) \,. \tag{24}$$

Using the correlated values of the different parameters, one obtains the mass-splitting $\Delta M_{10}^3 \equiv M_{P_1^3} - M_{P_0^3} \simeq (60^{-16}_{+35}) \, \text{MeV}$, where we have used the experimental value $M_{P_1^3} = 3.51 \, \text{GeV}$. Adding the $\langle g G^3 \rangle$ dimension-six condensate effect, which is about $-1.6\%$ of the leading term in $\mathcal{R}_{01}$, one can finally deduce the prediction in Table 2, which is in excellent agreement with the data. One should remark that the previous predictions indicate that, for the method to reproduce correctly the mass-splittings of the $S$ and $P$-wave charmonium states, one needs *both* larger values of $\alpha_s$ and $\langle \alpha_s G^2 \rangle$ than the ones favoured in the early days of the sum rules.

In the case of the *Tensor $P_2^3$-axial $P_1^3$ mass splitting*, the different exact QCD corrections for the double ratio of the tensor over the axial meson moments read:

$$\Delta_0^{TA} = \left(0.989^{+0.003}_{-0.006}\right) \,, \qquad \Delta_\alpha^{TA} = \left(0.029^{-0.004}_{+0.013}\right) \,, \qquad \Delta_G^{TA} = \left(22.1^{+8.5}_{-8.2}\right) \,, \tag{25}$$

from which, one can deduce the prediction in Table 2, which is slightly higher than the

Table 2. *Predictions for the gluon condensate, for the different mass-splittings (in units of MeV) and for the leptonic widths (in units of keV). We use $\alpha_s(M_Z) = 0.118 \pm 0.006$ from LEP and $\tau$-decay*

| Observables | Input | Predictions | Data/comments |
|---|---|---|---|
| $\langle \alpha_s G^2 \rangle [\text{GeV}]^4 \times 10^2$ | $M_\Psi - M_{\eta c} = 108$ | $10 \pm 4$ | This work |
| | $M_{\chi b}^{\text{c.o.m.}} - M_\Upsilon = 440$ | $6.5 \pm 2.5$ | – |
| | Average | $7.5 \pm 2.5$ | Mass splittings |
| | Charmonium masses | $\approx 4$ | SVZ-value [1] |
| | | | $q^2 = 0$-mom. |
| | – | $5.3 \pm 1.2$ | $q^2$-mom. [9] |
| | – | $10 \pm 2$ | exp. mom. [11,12] |
| | – | $9.2 \pm 3.4$ | mom. [31] |
| | $e^+ e^- \to I = 1$ hadrons | $4 \pm 1$ | ratio of mom. [33] |
| | – | $13^{+5}_{-7}$ | FESR [34] |
| | – | $7 \pm 1$ | $\tau$-like decay [3] |
| | $\tau$-decay (axial) | $6.9 \pm 2.6$ | [35] |
| | $\tau$-decay | | data |
| | ALEPH | | $7.5 \pm 3.1$ [8] |
| | CLEO | | $2.0 \pm 3.8$ [8] |
| $\alpha_s(1.3 \text{ GeV})$ | $M_{\chi c}(P_1^1) - M_{\chi c}(P_1^3)$ | $0.64^{+0.36}_{-0.18} \pm 0.02$ | $\alpha_s(M_Z) \simeq 0.127 \pm 0.011$ |
| $M_{\chi c}(P_1^1) - M_{\chi c}(P_1^3)$ | $\alpha_s$ from LEP/$\tau$-decay | $10.1^{-4.1}_{+9.9}$ | 11.1 (c.o.m.) |
| | | | 15.6 (data) |
| $M_{\chi c}(P_1^3) - M_{\chi c}(P_0^3)$ | $\langle \alpha_s G^2 \rangle$average | $89^{-16}_{+35}$ | 95 |
| $M_{\chi c}(P_2^3) - M_{\chi c}(P_1^3)$ | – | $77^{+26}_{-11}$ | 50 |
| $M_\gamma - M_{\eta b}$ | $\langle \alpha_s G^2 \rangle$average | $13^{-7}_{+10}$ | order $\alpha_s$ |
| | | $63^{-29}_{+51}$ | coeff.$\alpha_s$: Coul. pot. |
| $M_{\chi b}(P_0^3) - M_\Upsilon$ | – | $475^{+75}_{-64}$ | 400 |
| $M_{\chi b}(P_1^3) - M_\Upsilon$ | – | $485^{+25}_{-68}$ | 432 |
| $M_{\chi b}(P_2^3) - M_\Upsilon$ | – | $500 \pm 71$ | 453 |
| $M_{\chi b}(P_1^1) - M_\Upsilon$ | c.o.m. | $492^{+56}_{-69}$ | 440 |
| $M_T - 2m_t$ | $\langle \alpha_s G^2 \rangle$average | $-906$ | two-loop pole mass |
| $M_T - M_{\eta t}$ | – | 1.8 | order $\alpha_s$ |
| | | 93 | coeff.$\alpha_s$: Coul. pot. |
| $M_{\chi t} - M_T$ | – | 1800 | – |
| $\Gamma_{\gamma \to e^+ e^-}$ | – | 1.2 | 1.32 |
| $\Gamma_{T \to e^+ e^-}$ | – | 0.16 | – |

data of 50 MeV. This small discrepancy may be attributed to the unaccounted effects of the dimension-six condensate or/and to the (usual) increasing role of the continuum for state with higher spins. However, the prediction is quite satisfactory within our approximation.

## 8. $\alpha_s$ from the $P_1^1$-$P_1^3$ axial mass splitting

The corresponding double ratio of moments has the nice feature to be independent of the gluon condensate to leading order in $\alpha_s$ and reads:

$$\frac{M_{P_1^1}^2}{M_{P_1^3}^2} \simeq 1 + \alpha_s \left( \Delta_\alpha^{13}(\text{exact}) = 0.014_{+0.008}^{-0.004} \right) . \tag{26}$$

The recent experimental value $M_{P_1^1} = 3526.1$ MeV of the $P_1^1$ state denoted by $h_c(1P)$ in PDG [25] almost coincides with the one of the center of mass energy:

$$M_P^{\text{c.o.m}} = \frac{1}{9} \left[ 5M_{P_2^3} + 3M_{P_1^3} + M_{P_0^3} \right] \simeq 3521.6 \text{ MeV} \tag{27}$$

as expected from the short range nature of the spin-spin force [36]. Using this experimental value, one can deduce:

$$\alpha_s(\sigma^{-1} \simeq 1.3 \text{ GeV}) \simeq 0.64_{-0.18}^{+0.36} \pm 0.02 \Longrightarrow \alpha_s(M_Z) \simeq 0.127 \pm 0.009 \pm 0.002 . \tag{28}$$

The error is twice bigger than the one from LEP and $\tau$ decay data, but its value is perfectly consistent with the latter. The theoretical error is mainly due to the uncertainty in $\Delta_\alpha$, while a naïve exponential resummation of the higher order $\alpha_s$ terms leads to the second error much smaller than the previous ones. This value of $\alpha_s$ can be useful for an alternative derivation of this fundamental quantity at low energies and for testing its $q^2$-evolution. *Reciprocally*, using the value of $\alpha_s$ from LEP and $\tau$-decay data as input, one can deduce the prediction of the center of mass (c.o.m) of the $P_J^3$ states given in Table 2.

## 9. $\Upsilon - \eta_b$ mass splitting

For the bottomium, the analysis of the ratios of moments for the $S$ and $P$ waves shows that they are optimized at the same value of $\sigma$, namely [12]:

$$\sigma = (0.35 \pm 0.05) \text{ GeV}^{-2} , \tag{29}$$

which implies for 5 flavours: $\alpha_s(\sigma) \simeq 0.32 \pm 0.06$. Using the conservative values of the two-loop $b$-quark pole mass: $m_b \simeq 4.2-4.7$ GeV, one can deduce:

$$1/x \simeq 28 \pm 7 , \tag{30}$$

where one might (a priori) expect a good convergence of the $1/m$ expansion.

The splitting between the vector $\Upsilon(S_1^3)$ and the pseudoscalar $\eta_b(S_0^1)$ can be done in a similar way than the charmonium one. The double ratio of moments reads numerically:

$$\mathcal{R}_{VP} \simeq \frac{M_V^2}{M_P^2} \simeq \left( 0.9995_{-0.0003}^{+0.0002} \right) \times \left[ 1 + \alpha_s \left( 2.4_{+1.4}^{-0.7} \right) \times 10^{-3} \right.$$
$$\left. + (0.03 \pm 0.01) \text{ GeV}^{-4} \langle \alpha_s G^2 \rangle \right] , \tag{31}$$

where we have used the exact expressions of the QCD corrections. This leads to the mass splitting in Table 2. To this order of perturbation theory, this result is in the range of the potential model estimates [36,37,20], with the exception of the one in [17,22], where in the latter it has been shown that the square of the quark velocity $v^2$ correction can cause a large value of about 100 MeV for the splitting. One should also notice that, to this approximation, the gluon condensate gives still the dominant effect at the $b$-mass scale (0.2% of the leading order) compared to the one 0.08% from the $\alpha_s$-term. However, the $1/m$ series of the QCD $\alpha_s$ correction is badly convergent, showing that the static limit approximation is quantitatively inaccurate in the $b$-channel. Therefore, one expects that the corresponding prediction of $(13^{-7}_{+10})$ MeV is a very crude estimate. In order to control the effect of the unknown higher order terms, it is legitimate to introduce into the sum rule, the coefficient of the Coulombic effect from the QCD potential as given by the $\alpha_s^2$-term in Eq. (12)[5]. Therefore, we deduce the "improved" final estimate in Table 2:

$$M_\Upsilon - M_{\eta_b} \approx \left(63^{-29}_{+51}\right) \text{MeV} , \tag{32}$$

implying the possible observation of the $\eta_b$ from the $\gamma$ radiative decay.

## 10. $\Upsilon - \chi_b$ mass splittings and new estimate of the gluon condensate

As the $S$ and $P$ wave ratios of moments are optimized at the same value of $\sigma$, we can compare directly, with a good accuracy, the different $P$ states with the $\Upsilon_{(S_1^3)}$ one. As the coefficient of the $\alpha_s^2$ corrections, after inserting the expression of $\sigma_{\min}$, are comparable with the one from the Coulombic potential, we expect that the prediction of this splitting is more accurate than in the case of the hyperfine. The different double ratios of moments read numerically for the values in Eqs. (28)–(29):

$$\mathcal{R}_{VS} \simeq \frac{M_V^2}{M_S^2} \simeq \left(0.9696^{+0.0054}_{-0.0083}\right) \times \left[1 - \alpha_s\left(0.071^{-0.006}_{+0.011}\right) - \left(0.50^{+0.18}_{-0.11}\right)\text{GeV}^{-4}\langle\alpha_s G^2\rangle\right] ,$$

$$\mathcal{R}_{VA} \simeq \frac{M_V^2}{M_A^2} \simeq \left(0.9696^{+0.0054}_{-0.0083}\right) \times \left[1 - \alpha_s\left(0.074^{-0.007}_{+0.012}\right) - \left(0.54^{+0.18}_{-0.12}\right)\text{GeV}^{-4}\langle\alpha_s G^2\rangle\right] ,$$

$$\mathcal{R}_{VT} \simeq \frac{M_V^2}{M_T^2} \simeq \left(0.9704^{+0.0051}_{-0.0084}\right) \times \left[1 - \alpha_s\left(0.077^{-0.008}_{+0.006}\right) - \left(0.57^{+0.16}_{-0.13}\right)\text{GeV}^{-4}\langle\alpha_s G^2\rangle\right] ,$$

$$\tag{33}$$

where $V$, $S$, $A$, $T$ refer respectively to the $\gamma$ and to the different $\chi_b$ states $P_0^3$, $P_1^3$, $P_2^3$. Using the value of the gluon condensate obtained previously, these sum rules lead to the mass splittings in Table 2, which is in good agreement with the corresponding data, but definitely higher than the previous predictions of [39], where, among other effects, the values of $\alpha_s$ and of the gluon condensate used there are too low. Reciprocally, one can use the data for re-extracting *independently* the value of the gluon condensate. As usually observed in the

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[5]In this case, the gluon condensate contribution is smaller than the Coulombic one as has been observed in [38].

literature, the prediction is more accurate for the c.o.m., than for the individual mass. The corresponding numerical sum rule is:

$$\frac{M_{\chi_b}^{\text{c.o.m.}} - M_\Upsilon}{M_\Upsilon} \simeq \left(1.53_{-0.42}^{+0.26}\right) \times 10^{-2} + \left(1.20_{-0.2}^{+0.1}\right) \times 10^{-2} + \left(0.28_{-0.06}^{+0.08}\right)\text{GeV}^{-4}\langle\alpha_s G^2\rangle ,$$

(34)

which leads to:

$$\langle\alpha_s G^2\rangle \simeq (6.9 \pm 2.5) \times 10^{-2}\ \text{GeV}^4 .$$

(35)

We expect that this result is more reliable than the one obtained from the $M_\psi - M_{\eta_c}$ as the latter can be more affected by the noncalculated next-next-to-leading perturbative radiative corrections than the former. An average of the two results from the $\psi - \eta_c$ and $\Upsilon - \chi_b$ mass splittings leads to:

$$\langle\alpha_s G^2\rangle \simeq (7.5 \pm 2.5) \times 10^{-2}\ \text{GeV}^4 ,$$

(36)

where we have retained the most precise error.

## 11.  Update average value of $\langle\alpha_s G^2\rangle$

The previous result can be compared with different fits of the heavy and light quark channels given in Table 2, and which range from 4 (SVZ) to 14 in units of $10^{-2}\ \text{GeV}^4$. The most recent estimate from $e^+e^- \to I = 1$ hadrons data using $\tau$-like decay is $(7 \pm 1) \times 10^{-2}\ \text{GeV}^4$, where one should also notice that the different post-SVZ estimates favour higher values of the gluon condensate. If one considers this latter as the update of the light quark channel estimates and the former as an update of the heavy quark one, one can deduce the *update average* from the sum rule:

$$\langle\alpha_s G^2\rangle \simeq (7.1 \pm 0.9) \times 10^{-2}\ \text{GeV}^4 .$$

(37)

More accurate measurements of this quantity than the already available results from $\tau$-decay data [8] are needed for testing the previous phenomenological estimates.

## 12.  Toponium: illustration of the infinite mass limit

Since only in the case of the toponium, the $1/m$-expansion is ideal, we have extended the previous analysis in this channel, though, we are aware that this application can be purely academic because of the eventual non-existence of the corresponding bound states. We use the top mass: $m_t \simeq (173 \pm 14)$ GeV, obtained from the average of the CDF candidates $(174 \pm 16)$ GeV and of the electroweak data $(169 \pm 26)$ GeV as compiled by PDG [25]. We shall work with the ratio of moments in the vector channel for determining the mass of the $S_1^3$ state, and use with a good confidence the leading terms of the expressions given in

Table 1. Using the sum rules in Eqs. (8) and (11) and the value of the minimum $\sigma^{-1/2} \simeq$ 20 GeV from Eq. (9), we deduce the result for the meson-quark mass gap given in Table 2. For the splittings, we use the sum rules in Eqs. (12) and (13), while, for the leptonic width, we use the sum rule in Eq. (14). Our results are summarized in Table 2.

## 13. Conclusions

We have used *new double ratios* of exponential sum rules for directly extracting the mass-splittings of different heavy quarkonia states. Therefore, we have obtained from $M_\psi - M_{\eta_c}$ and $M_{\chi_b} - M_\Upsilon$ a more precise estimate of the value of the gluon condensate given in Eq. (36), which combined with the one from $\tau$-like decay in $e^+e^- \rightarrow I = 1$ hadrons data leads to the *update average* in Eq. (37) from the sum rule. We have also used $M_{\chi_c(P_1^1)} - M_{\chi_c(P_1^3)}$ for an alternative extraction of $\alpha_s$ at low energy (see Eq. (28)), with a value consistent with the one from LEP and $\tau$-decay. Our numerical results are summarized in Table 2, where a comparison with different estimates and experimental data is done.

We have also attempted to connect the sum rules and the potential model approaches, using a $1/m$-expansion. We found, that the Coulombic corrections, which are quite well understood in QCD, agree, in general, in the two approaches, except in the radiative corrections of the hyperfine splitting which requires the knowledge of the next-next-to-leading $\alpha_s$-corrections. Relativistic corrections due to finite value of the quark mass have been included in our analysis. However, the coefficients of the gluon condensate disagree in the two approaches, which may be related to the difficulty encountered by Bell-Bertlmann in finding a bridge between a field theory after SVZ and potential models.

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