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Global Phase Portraits for the Abel Quadratic Polynomial Differential Equations of the Second Kind With Z_2-symmetries

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Abstract. We provide normal forms and the global phase portraits on the Poincaré disk for all Abel quadratic polynomial differential equations of the second kind with $\mathbb{Z}_2$-symmetries.

1 Introduction and Statement of the Main Results

There are more than one thousand papers published on quadratic polynomial differential systems (simply quadratic systems) that are the differential systems of the form $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, where P and Q are real polynomials in the variables x and y, and the maximum of the degrees of P and Q is two. Here the dot denotes the derivative with respect to an independent variable t, usually called time.

The difficulty of studying these differential systems is due to the fact that they depend on twelve parameters. The authors of these published papers studied many subclasses of quadratic systems. A list of them, without trying to be exhaustive, is the following: quadratic systems with a center [42, 46, 48, 60, 67], with no finite real singularities [25, 58], with a unique finite singularity [18, 28, 45, 57, 59, 62, 63], with a focus and one anti-saddle [2], with an integrable saddle [14], with a third order weak focus [5, 40], with all points at infinity as singularities [26, 53], Hamiltonian [3, 4, 30]), bounded [19, 35], Darboux integrable systems [37, 61], homogeneous [64, 65], Lotka–Volterra [55, 56], structurally stable [1, 29], semilinear [41], with invariant lines of total multiplicity greater than or equal to four [49–52, 54], with rational first integrals [15, 38, 39], with polynomial first integrals [24], with a polynomial inverse integrating factor [16]. More recently, the classification of some families of quadratic systems has been made using more modern methods such as the algebraic and geometric invariants; see, for instance, the classification of the quadratic systems with a weak focus of second order [7], the classification of the quadratic systems with a weak focus and

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an invariant straight line [8], and the classification of the geometric configurations of singularities for quadratic systems [6, 9–13].

There are still many open questions regarding quadratic systems. In this paper our objective is to characterize all the global phase portraits in the Poincaré disc of the class of quadratic systems that come from Abel quadratic polynomial differential equations of the second kind modulo, some symmetries.

An Abel differential equation of the second kind is of the form

$$y\frac{dy}{dx} = A(x)y + B(x), \tag{1.1}$$

with $A(x)$ and $B(x)$ non-zero functions, and is equivalent to the differential system $\dot{x} = yc(x)$, $\dot{y} = a(x)y + b(x)$, where $A(x) = a(x)/c(x)$ and $B(x) = b(x)/c(x)$. In this paper we are interested in studying Abel quadratic polynomial differential systems, *i.e.*, the differential systems of the form

$$\begin{aligned} \dot{x} &= yc(x) := y(c_0 + c_1 x), \\ \dot{y} &= a(x)y + b(x) := (a_0 + a_1 x)y + b_0 + b_1 x + b_2 x^2, \end{aligned} \tag{1.2}$$

where $a_0, a_1, b_0, b_1, b_2, c_0, c_1 \in \mathbb{R}$, $c_0^2 + c_1^2 \neq 0$. Otherwise, the system is trivial and such that $\dot{x}$ and $\dot{y}$ do not have a common factor, *i.e.*, either $b_0^2 + b_1^2 + b_2^2 \neq 0$, or $c_0(a_1c_0 - a_0c_1) \neq 0$, or $c_0(b_2c_0^2 - b_1c_0c_1 + b_0c_1^2) \neq 0$, or $c_0 = 0$ and $a_0^2 + b_0^2 \neq 0$. Note that we always have $a_0^2 + a_1^2 \neq 0$, otherwise it would not be an Abel equation of the second kind. Moreover, in order that the system be a quadratic system, we have that $b_2^2 + a_1^2 + c_1^2 \neq 0$. Note that systems (1.2) have seven parameters and, at present, the full classification of their global phase portraits is difficult. So we restrict ourselves to the ones that have a $\mathbb{Z}_2$-symmetry. We recall that there are two types of $\mathbb{Z}_2$-symmetric systems: the *equivariant* and the *reversible*. More precisely, we say that system (1.2) is $\mathbb{Z}_2$-*reversible* if it is invariant under the transformation $(x, y, t) \to (R(x, y), -t)$, where $R(x, y)$ is either $(-x, y)$, or $(x, -y)$, or $(-x, -y)$, and we say that system (1.2) is $\mathbb{Z}_2$-*equivariant* if it is invariant under the transformation $(x, y, t) \to (R(x, y), t)$. For more details on $\mathbb{Z}_2$-symmetric systems, see [31]. These six conditions for reversibility and equivariance can be rewritten as follows.

Let $\mathcal{X}\colon U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field associated with system (1.2). We define the matrix $M = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\mathcal{X}$ is $\mathbb{Z}_2$-equivariant if one of

$$M\mathcal{X}(x, y) = \mathcal{X}(-x, y), \tag{1.3}$$

$$-M\mathcal{X}(x, y) = \mathcal{X}(x, -y), \tag{1.4}$$

$$-\mathcal{X}(x, y) = \mathcal{X}(-x, -y) \tag{1.5}$$

holds, and $\mathcal{X}$ is $\mathbb{Z}_2$-reversible if one of

$$M\mathcal{X}(x, y) = -\mathcal{X}(-x, y), \tag{1.6}$$

$$-M\mathcal{X}(x, y) = -\mathcal{X}(x, -y), \tag{1.7}$$

$$\mathcal{X}(x, y) = \mathcal{X}(-x, -y) \tag{1.8}$$

holds. Different families of planar polynomial vector fields with a $\mathbb{Z}_2$-symmetry have been studied by several authors [33, 36, 66].

Proposition 1.1 *The following statements hold.*

(i) *System* (1.2) *satisfying* (1.3) *becomes*

$$\dot{x} = c_1 x y, \quad \dot{y} = b_0 + a_0 y + b_2 x^2, \tag{1.9}$$

with $a_0 c_1 \neq 0$ *and* $b_0^2 + b_2^2 \neq 0$.

(ii) *System* (1.2) *satisfying* (1.6) *becomes*

$$\dot{x} = c_0 y, \quad \dot{y} = b_1 x + a_1 x y, \tag{1.10}$$

where $a_1 b_1 c_0 \neq 0$.

(iii) *System* (1.2) *satisfying* (1.8) *becomes*

$$\dot{x} = c_1 x y, \quad \dot{y} = b_0 + b_2 x^2 + a_1 x y, \tag{1.11}$$

where $a_1 c_1 b_0 \neq 0$.

(iv) *System* (1.2) *satisfying either* (1.4)*, or* (1.5) *or* (1.7) *is not* $\mathbb{Z}_2$*-symmetric.*

Proof (i) System (1.2) satisfies (1.3) if and only if $a_1 = b_1 = c_0 = 0$, and then it is written as in system (1.9).

(ii) System (1.2) satisfies (1.6) if and only if $a_0 = b_0 = b_2 = c_1 = 0$, and then it is transformed into system (1.10).

(iii) System (1.2) satisfies (1.8) if and only if $a_0 = b_1 = c_0 = 0$ and then it becomes system (1.11).

(iv) System (1.2) satisfies (1.4) if and only if $b_0 = b_1 = b_2 = c_0 = c_1 = 0$, which is not possible. System (1.2) satisfies (1.5) if and only if $a_1 = b_0 = b_2 = c_1 = 0$, which is not possible, because then the differential system would be linear. Finally, system (1.2) satisfies (1.7) if and only if $a_0 = a_1 = c_1 = 0$, which is again not possible, because then the system would not be Abel. ∎

In this work we provide the global phase portraits of systems (1.9), (1.11), and (1.10). We will use the Poincaré compactification of polynomial vector fields, see the Appendix.

We say that two polynomial vector fields in the Poincaré disk are *topologically equivalent* if there exists a homeomorphism from one onto the other which sends orbits to orbits and preserves or reverses the direction of the flow.

Theorem 1.2 *All quadratic polynomial differential systems* (1.9)*, after a linear change of variables and a rescaling of its independent variable* t*, can be written as one of the following three classes, where* $a \in \mathbb{R}$*:*

(I.1) $\dot{x} = xy$, $\dot{y} = a + y$ *with* $a \neq 0$,
(I.2) $\dot{x} = xy$, $\dot{y} = a + y + x^2$,
(I.3) $\dot{x} = xy$, $\dot{y} = a + y - x^2$,

Their phase portraits, which have no limit cycles, are as follows (see Figure 1*),*

- *for system* (I.1)*:* (P1) *if* $a > 0$ *and* (P2) *if* $a < 0$,
- *for system* (I.2)*:* (P3) *if* $a \geq 0$ *and* (P4) *if* $a < 0$,
- *for system* (I.3)*:* (P5) *if* $a > 0$ *and* (P6) *if* $a \leq 0$.

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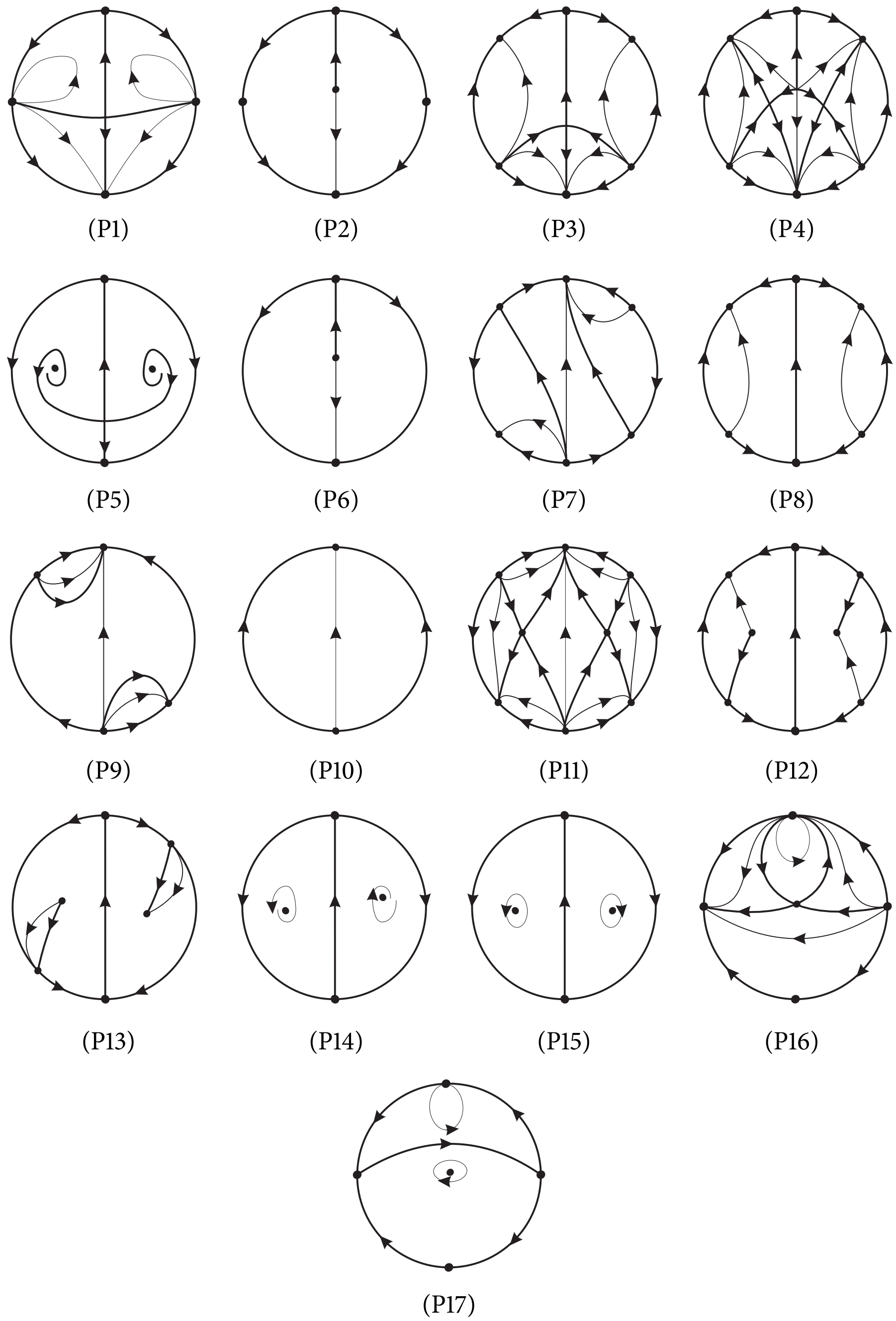

Figure 1: The global phase portraits in the Poincaré disc for the Abel quadratic polynomial differential equations of second kind with $\mathbb{Z}_2$-symmetries.

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The proof of Theorem 1.2 is given in Section 2.

Theorem 1.3 *All quadratic polynomial differential systems* (1.11)*, after a linear change of variables and a rescaling of its independent variable t, can be written as one of the following five classes:*

(II.1) $\dot{x} = xy$, $\dot{y} = \mu + xy$ *with* $\mu \in \{-1, 1\}$;
(II.2) $\dot{x} = xy$, $\dot{y} = 1 + x^2 + axy$;
(II.3) $\dot{x} = xy$, $\dot{y} = -1 - x^2 + axy$;
(II.4) $\dot{x} = xy$, $\dot{y} = -1 + x^2 + axy$;
(II.5) $\dot{x} = xy$, $\dot{y} = 1 - x^2 + axy$;

where $a \in \mathbb{R}$, and their phase portraits, as listed in Figure 1*, are as follows:*

- *for system* (II.1)*:* (P7) *if* $\mu = -1$ *and* (P8) *if* $\mu = 1$,
- *for system* (II.2)*:* (P8),
- *for system* (II.3)*:* (P7) *if* $|a| > 2$, (P9) *if* $|a| = 2$, *and* (P10) *if* $|a| < 2$,
- *for system* (II.4)*:* (P11),
- *for system* (II.5)*:* (P12) *if* $|a| > 2$, (P13) *if* $|a| = 2$, (P14) *if* $|a| < 2$ *and* $a \neq 0$, *and* (P15) *if* $a = 0$,

Moreover, these phase portraits have no limit cycles.

We prove Theorem 1.3 in Section 3.

Theorem 1.4 *All quadratic polynomial differential systems* (1.10)*, after a linear change of variables and a rescaling of its independent variable t, can be written as one of the following systems:*

(III.1) $\dot{x} = y$, $\dot{y} = x(y+1)$;
(III.2) $\dot{x} = y$, $\dot{y} = x(y-1)$.

The global phase portraitsfor systems (III.1) *and* (III.2) *are topologically equivalent to* (P16) *and* (P17) *of Figure* 1*, respectively. Moreover, these phase portraits have no limit cycles.*

The proof of Theorem 1.4 is given in Section 4.

We provide an appendix where we provide some preliminary definitions, notations and theorems about the Poincaré sphere and the Poincaré compactification as well as results on canonical regions that we shall use for proving Theorems 1.4 and 1.3.

For more information on the phase portraits in Figure 1 see Table 2 and the proof of Theorem 1.4.

2 Proof of Theorem 1.2

We consider the linear change of variables and a rescaling of the independent variable (time) of the form $x \to \delta X$, $y \to \beta Y$, $t \to \gamma T$, with $\beta\delta\gamma \neq 0$. Since $a_0 c_1 \neq 0$, we take

$\gamma = \frac{1}{c_1\beta}$ and $\beta = \frac{a_0}{c_1}$. So we have

$$X' = XY, \quad Y' = \frac{b_0 c_1}{a_0^2} + Y + \frac{b_2 c_1 \delta^2}{a_0^2} X^2.$$

If $b_2 = 0$, then $Y' = a + Y$ with $a \neq 0$, and we obtain the normal form (I.1) of the theorem. If $b_2 \neq 0$, then we take

$$\delta = \sqrt{\left|\frac{a_0^2}{b_2 c_1}\right|}.$$

Therefore, $Y' = a + Y + \mu X^2$ with $\mu \in \{-1, 1\}$, and we obtain the normal forms (I.2) and (I.3) of Theorem 1.2.

Systems (I.1)–(I.3) have the invariant straight line $F = x = 0$, and so they have at most one limit cycle [17, 20].

Now we study the finite singular points of systems (I.1)–(I.3). System (I.1) has the finite singular point $p_0 = (0, -a)$, which is a saddle if $a > 0$ and an unstable node if $a < 0$, because the eigenvalues of the linear part of the system at p_0 are $-a, 1$ for $a \in \mathbb{R} \setminus \{0\}$.

System (I.2) has the finite singular points $p_0 = (0, -a)$, $p_1 = (\sqrt{-a}, 0)$, and $p_2 = (-\sqrt{-a}, 0)$; the last two exist as real singularities only if $a < 0$. When $a > 0$, the unique singular point is p_0, which is a saddle, because the eigenvalues of the Jacobian matrix at that point are $-a, 1$. If $a = 0$, the singular points p_0, p_1, p_2 coalesce at the origin and the origin is semi-hyperbolic. Using [23, Theorem 2.19], we get that it is a saddle. If $a < 0$, then the singular point p_0 is an unstable node and p_1, p_2 are hyperbolic saddles because the eigenvalues of the Jacobian matrix at these points have determinant equal to $2a < 0$.

System (I.3) has the finite singular points $p_0 = (0, -a)$, $p_1 = (\sqrt{a}, 0)$, and $p_2 = (-\sqrt{a}, 0)$; the last two exist as real singularities only if $a > 0$. When $a > 0$, the singular point p_0 is a saddle (the eigenvalues of the Jacobian matrix at that point are $-a$ and 1), and the singular points p_1 and p_2 are unstable foci if $a > 1/8$, and unstable nodes if $a \in (0, 1/8]$, because the eigenvalues of the Jacobian matrix at either of these two points are $(1 \pm \sqrt{1 - 8a})/2$. If $a = 0$, the singular points p_0, p_1, p_2 coalesce at the origin and the origin is semi-hyperbolic. Using [23, Theorem 2.19], we get that it is an unstable node. If $a < 0$, then the unique singular point p_0 is an unstable node.

Finally we study the local phase portraits of the infinite singular points.

2.1 System (I.1)

In the local chart U_1 (see the Appendix) system (I.1) becomes

$$\dot{u} = -u^2 + uv + av^2, \quad \dot{v} = -uv.$$

On $v = 0$ we have the singular point $w_0 = (0, 0)$ that is linearly zero. Using the results in [21] we get that if $a > 0$, the local phase portrait at the origin is formed by two elliptic sectors and one parabolic sector separated by infinity and if $a < 0$, it is a saddle-node with the linear part identically zero.

On the local chart U_2 (see again the Appendix) systems (I.1)–(I.3)) become

$$\dot{u} = u(1 - v - av^2), \quad \dot{v} = -v^2(1 + av).$$

The origin $w_3 = (0, 0)$ of U_2 is a singular point. The eigenvalues of the Jacobian matrix at the origin are 1 and 0. So it is semi-hyperbolic. Using [23, Theorem 2.19], we get that it is a saddle-node.

2.2 System (I.2)

In the local chart U_1 we have $\dot{u} = 1 - u^2 + uv + av^2$, $\dot{v} = -uv$. There are two singular points $w_1 = (1, 0)$ and $w_2 = (-1, 0)$. The singular point w_1 is a hyperbolic stable node because the eigenvalues of the Jacobian matrix at w_1 are -2 and -1, and w_2 is an unstable hyperbolic node because the eigenvalues of the Jacobian matrix at w_2 are 2 and 1.

On the local chart U_2 we have $\dot{u} = u(1 - v - u^2 - av^2)$, $\dot{v} = -v(v + u^2 + av^2)$. The origin of U_2, w_3, is a singular point that is semi-hyperbolic because the eigenvalues at the origin are 1 and 0. Using [23, Theorem 2.19], we get that it is a saddle-node.

2.3 System (I.3)

In the local chart U_1 we have $\dot{u} = -1 - u^2 + uv + av^2$, $\dot{v} = -uv$. There are no singular points on the local chart U_1 with $v = 0$. On the local chart U_2 we have

$$\dot{u} = u(1 - v + u^2 - av^2), \quad \dot{v} = v(-v + u^2 - av^2).$$

The origin of U_2, w_3, is a singular point which is semi-hyperbolic because the eigenvalues at the origin w_3 are 1 and 0. Using [23, Theorem 2.19] we get that it is a saddle-node.

In Table 1 we include all the information and phase portraits for the systems (I.1)–(I.3) depending on their parameters. This table uses the following notations: FSP (finite singular points), ISP (infinite singular points), PP (phase portrait), CR (canonical regions), Σ (sepatrices), S (saddle), UN (unstable node), SN (stable node), UF (unstable focus), 2E1P (two elliptic sectors and one parabolic sector separated by the infinity), DS-N (saddle-node with linear part identically zero), and S-N (saddle-node).

Using the information on the local phase portraits described in Table 1 for the finite and infinite singular points, the existence of the invariant straight line $x = 0$ (and consequently the system having at most one limit cycle, see [17, 20]), and the fact that whenever the local phase portrait has two foci it does not have limit cycles due to the symmetry with respect to the y-axis, we conclude that each local phase portrait gives rise to a unique global phase portrait in the Poincaré disc, and we obtain six topologically different global phase portraits as given in Table 1 and provided in Figure 1. Moreover, using the results of the Appendix, for each global phase portrait in Table 1, we give the number of canonical regions (CR) and separatrices (Σ).

Table 1: Study of systems (I)

Family	Conditions	FSP	ISP	PP	CR	Σ
(I.1)	$a > 0$	p_0 =S	w_0 =2E1P, w_3 =S-N	(1)	4	13
(I.1)	$a < 0$	p_0 =UN	w_0 =DS-N, w_3 =S-N	(2)	1	10
(I.2)	$a \geq 0$	p_0 =S	w_1 =SN, w_2 =UN, w_3 =S-N	(3)	4	17
(I.2)	$a < 0$	p_0 =UN, p_1 =S, p_2 =S	w_1 =SN, w_1 =UN, w_3 =S-N	(4)	7	24
(I.3)	$a > 1/8$	p_0 =S, p_1 =UF, p_2 =UF	w_3 =S-N	(5)	2	9
(I.3)	$a \in (0, 1/8]$	p_0 =S, p_1 =UN, p_2 =UN	w_3 =S-N	(5)	2	9
(I.3)	$a \leq 0$	p_0 =UN	w_3 =S-N	(6)	1	6

3 Proof of Theorem 1.3

We consider the linear change of variables and a rescaling of the independent variable (time) of the form $x \to \delta X$, $y \to \beta Y$, $t \to \gamma T$, with $\beta\delta\gamma \neq 0$. Since $c_1 \neq 0$, we take $\gamma = \frac{1}{c_1\beta}$ and so we always have $X' = XY$.

If $b_2 = 0$, then, since $a_1 b_0 \neq 0$, we take

$$\beta = \sqrt{\left|\frac{b_0}{c_1}\right|} \quad \text{and} \quad \delta = \frac{c_1\beta}{a_1}.$$

Then $Y' = \mu + XY$ with $\mu \in \{-1, 1\}$, and we obtain the normal form (II.1) of the theorem.

If $b_2 \neq 0$, since $b_0 \neq 0$, we take

$$\beta = \sqrt{\left|\frac{b_0}{c_1}\right|} \quad \text{and} \quad \delta = \sqrt{\left|\frac{b_0}{b_2}\right|}.$$

Then $Y' = \mu + \nu X^2 + aXY$ with $\mu, \nu \in \{-1, 1\}$ and $a \in \mathbb{R}$. We obtain the normal forms (II.2)–(II.5) of Theorem 1.3.

Now we study the finite singular points of systems (II.1)–(II.5). Systems (II.1)–(II.3) have no finite singular points.

System (II.4) has the finite singular points $p_0 = (1, 0)$ and $p_1 = (-1, 0)$. Both of them are hyperbolic saddles because the eigenvalues are $(a \pm \sqrt{a^2 + 8})/2$ for p_0 and $(-a \pm \sqrt{a^2 + 8})/2$ for p_1, for all $a \in \mathbb{R}$.

System (II.5) has the finite singular points $p_0 = (1, 0)$ and $p_1 = (-1, 0)$. The eigenvalues at p_0 are $(a \pm \sqrt{a^2 - 8})/2$, and the eigenvalues at p_1 are $(-a \pm \sqrt{a^2 - 8})/2$. If $a = 0$, then p_0 and p_1 are centers (the system has the analytic first integral $H = x^2 e^{-(x^2+y^2)}$). If $a \neq 0$ and $a^2 \geq 8$, then p_0 and p_1 are hyperbolic nodes; p_0 is stable if

$a < 0$ and unstable if $a > 0$, while p_1 is stable if $a > 0$ and unstable if $a < 0$. If $a \neq 0$ and $a^2 < 8$, then p_0 and p_1 are foci; p_0 is stable if $a < 0$ and unstable if $a > 0$, while p_1 is stable if $a > 0$ and unstable if $a < 0$.

Finally we study the local phase portraits of the infinite singular points.

3.1 System (II.1)

In the local chart U_1, system (II.1) becomes $\dot{u} = u - u^2 + \mu v^2$, $\dot{v} = -uv$. On $v = 0$ we have the singular points $w_0 = (0,0)$ and $w_1 = (1,0)$. The singular point w_1 is a hyperbolic stable node because the eigenvalues of the Jacobian matrix at w_1 are -1 and 1. The singular point w_0 is semi-hyperbolic because the eigenvalues of the Jacobian matrix at w_0 are 1 and 0. Using [23, Theorem 2.19], we get that it is a saddle if $\mu = -1$ and an unstable node if $\mu = 1$.

On the local chart U_2, systems (I.1)–(I.3) become

$$\dot{u} = u(1 - u - \mu v^2), \quad \dot{v} = -v(u + \mu v^2).$$

The origin, w_2 is a singular point. The eigenvalues of the Jacobian matrix at the origin are $\lambda_1 = 1$ and $\lambda_2 = 0$. So, it is semi-hyperbolic. Using [23, Theorem 2.19], we get that it is a saddle if $\mu = 1$, and an unstable node if $\mu = -1$.

3.2 System (II.2)

In the local chart U_1, we have $\dot{u} = 1 + au - u^2 + v^2$, $\dot{v} = -uv$. There are two singular points $w_0 = (a + \sqrt{4 + a^2})/2, 0)$ and $w_1 = (a - \sqrt{4 + a^2})/2, 0)$. The singular point w_0 is a hyperbolic stable node because the eigenvalues of the Jacobian matrix at w_0 are $-\sqrt{4 + a^2}$ and $-(a + \sqrt{4 + a^2})/2$. The singular point w_1 is a hyperbolic unstable node because the eigenvalues of the Jacobian matrix at w_1 are

$$\sqrt{4 + a^2} \quad \text{and} \quad -(a - \sqrt{4 + a^2})/2.$$

On the local chart U_2, we have $\dot{u} = u(1 - au - u^2 - v^2)$ $\dot{v} = -v(au + u^2 + v^2)$. The origin of U_2 is a singular point which is semi-hyperbolic because the eigenvalues at the origin w_2 are 1 and 0. Using [23, Theorem 2.19], we get that it is a saddle.

3.3 System (II.3)

In the local chart U_1, we have $\dot{u} = -1 + au - u^2 - v^2$, $\dot{v} = -uv$. There are two singular points $w_0 = (a + \sqrt{a^2 - 4})/2, 0)$ and $w_1 = (a - \sqrt{a^2 - 4})/2, 0)$ if and only if $|a| > 2$. If $|a| = 2$, then there is a unique singular point $w = w_0 = w_1 = (a/2, 0)$. When $|a| > 2$, the singular point w_0 is a hyperbolic saddle if $a < 0$ and a hyperbolic stable node if $a > 0$, because the eigenvalues of the Jacobian matrix at w_0 are $-\sqrt{a^2 - 4}$ and $-(a + \sqrt{a^2 - 4})/2$. The singular point w_1 is a hyperbolic saddle if $a > 0$ and a hyperbolic unstable node if $a < 0$, because the eigenvalues of the Jacobian matrix at w_1 are $\sqrt{a^2 - 4}$ and $-(a - \sqrt{a^2 - 4})/2$. When $|a| = 2$, the singular point w is semi-hyperbolic. Using [23, Theorem 2.19], we get that it is a saddle-node.

On the local chart U_2, we have $\dot{u} = u(1 - au + u^2 + v^2)$, $\dot{v} = -v(au - u^2 - v^2)$. The origin of U_2 is a singular point that is semi-hyperbolic because the eigenvalues at the origin w_2 are 1 and 0. Using [23, Theorem 2.19], we get that it is an unstable node.

3.4 System (II.4)

In the local chart U_1, we have $\dot{u} = 1 + au - u^2 - v^2$, $\dot{v} = -uv$. There are two singular points $w_0 = (a + \sqrt{a^2+4})/2, 0)$ and $w_1 = (a - \sqrt{a^2+4})/2, 0)$. The singular point w_0 is a hyperbolic stable node, because the eigenvalues of the Jacobian matrix at w_0 are $-\sqrt{4+a^2}$ and $-(a + \sqrt{4+a^2})/2$. The singular point w_1 is a hyperbolic unstable node, because the eigenvalues of the Jacobian matrix at w_1 are $\sqrt{4+a^2}$ and $-(a - \sqrt{4+a^2})/2$.

On the local chart U_2, we have $\dot{u} = u(1 - au - u^2 + v^2)$, $\dot{v} = -v(au + u^2 - v^2)$. The origin of U_2 is a singular point which is semi-hyperbolic, because the eigenvalues at the origin w_2 are 1 and 0. Using [23, Theorem 2.19], we get that it is an unstable node.

3.5 System (II.5)

In the local chart U_1, we have $\dot{u} = -1 + au - u^2 + v^2$, $\dot{v} = -uv$. There are two singular points $w_0 = (a + \sqrt{a^2-4})/2, 0)$ and $w_1 = (a - \sqrt{a^2-4})/2, 0)$ if and only if $|a| > 2$. If $|a| = 2$, then there is a unique singular point $w = w_0 = w_1 = (a/2, 0)$. When $|a| > 2$, the singular point w_0 is a hyperbolic saddle if $a < 0$ and a hyperbolic stable node if $a > 0$, because the eigenvalues of the Jacobian matrix at w_0 are $-\sqrt{a^2-4}$ and $-(a + \sqrt{a^2-4})/2$. The singular point w_1 is a hyperbolic saddle if $a > 0$ and a hyperbolic unstable node if $a < 0$, because the eigenvalues of the Jacobian matrix at w_1 are $\sqrt{a^2-4}$ and $-(a - \sqrt{a^2-4})/2$. When $|a| = 2$, the singular point w is semi-hyperbolic. Using [23, Theorem 2.19], we get that it is a saddle-node.

On the local chart U_2, we have $\dot{u} = u(1 - au + u^2 - v^2)$, $\dot{v} = -v(au - u^2 + v^2)$. The origin of U_2 is a singular point that is semi-hyperbolic, because the eigenvalues at the origin w_2 are 1 and 0. Using [23, Theorem 2.19], we get that it is a saddle.

In Table 2 we include all the information and phase portraits for the different systems (II.1)–(II.5) depending on their parameters. In this table we have the following notation: FSP (finite singular points), ISP (infinite singular points), PP (phase portrait), CR (canonical regions), Σ (sepatrices, S (saddle), UN (unstable node), SN (stable node), UF (unstable focus), SF (stable focus), C (center) and S-N (saddle-node).

Using the information on the local phase portraits described in Table 2 for the finite and infinite singular points, the existence of the invariant straight line $x = 0$ (and consequently the system having at most one limit cycle, see [17, 20]), and the fact that whenever the local phase portrait has two foci, it does not have limit cycles due to the symmetry with respect to the origin, we conclude that each local phase portrait gives rise to a unique global phase portrait in the Poincaré disc and we obtain nine topologically different global phase portraits as given in Table 2 and provided in Figure 1. Moreover, using the results of the Appendix in Table 2, for each global phase portrait, we give the number of canonical regions (CR) and separatrices (Σ).

Table 2: Study of systems (II)

Family	Conditions	FSP	ISP	PP	CR	Σ
(II.1)	$\mu = -1$	∅	w_0 =S, w_1 =SN, w_2 =UN	(7)	3	14
(II.1)	$\mu = 1$	∅	w_0 =UN, w_1 =SN, w_2 =S	(8)	2	13
(II.2)	$a \in \mathbb{R}$	∅	w_0 =SN, w_1 =UN, w_2 =S	(8)	2	13
(II.3)	$a > 2$	∅	w_0 =SN, w_1 =S, w_2 =UN	(7)	3	14
(II.3)	$a = \pm 2$	∅	w =S-N, w_2 =UN	(9)	3	10
(II.3)	$a \in (-2, 2)$	∅	w_2 =UN	(10)	1	4
(II.3)	$a < -2$	∅	w_0 =S, w_1 =UN, w_2 =UN	(7)	3	14
(II.4)	$a \in \mathbb{R}$	p_0 =S, p_1 =S	w_0 =SN, w_1 =UN, w_2 =UN	(11)	7	22
(II.5)	$a \geq 2\sqrt{2}$	p_0 =UN, p_1 =SN	w_0 =SN, w_1 =S, w_2 =S	(12)	2	17
(II.5)	$a \in (2, 2\sqrt{2})$	p_0 =UF, p_1 =SF	w_0 =SN, w_1-S, w_2 =S	(12)	2	17
(II.5)	$a = 2$	p_0 =UF, p_1 =SF	w =S-N, w_2 =S	(13)	2	13
(II.5)	$a \in (0, 2)$	p_0 =UF, p_1 =SF	w_2 =S	(14)	2	7
(II.5)	$a = 0$	p_0 =C, p_1 =C	w_2 =S	(15)	2	7
(II.5)	$a \in (-2, 0)$	p_0 =SF, p_1 =UF	w_2 =S	(14)	2	7
(II.5)	$a = -2$	p_0 =SF, p_1 =UF	w =S-N, w_2 =S	(13)	2	13
(II.5)	$a \in (-2\sqrt{2}, -2)$	p_0 =SF, p_1 =UF	w_0 =S, w_1 =UN, w_2 =S	(12)	2	17
(II.5)	$a \leq -2\sqrt{2}$	p_0 =SN, p_1 =UN	w_0 =S, w_1 =UN, w_2 =S	(12)	2	17

4 Proof of Theorem 1.4

We first establish the normal form provided in Theorem 1.4. To do that we consider the rescaling of variables $x \to \delta X$, $y \to \beta Y$, $t \to \gamma T$, with $\beta\delta\gamma \neq 0$. Note that $b_1 c_0 \neq 0$. If $b_1 c_0 > 0$, we take $\delta = \frac{a_1}{\sqrt{b_1 c_0}}$, $\beta = \frac{a_1}{b_1}$, $\gamma = \sqrt{b_1 c_0}$, obtaining system (III.1). If $b_1 c_0 < 0$, we take

$$\delta = \frac{a_1}{\sqrt{|b_1 c_0|}}, \quad \beta = -\frac{a_1}{b_1}, \quad \gamma = \sqrt{|b_1 c_0|},$$

obtaining system (III.2).

Now we study the local phase portraits of the finite and infinite singular points of system (III.1).

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The origin is the unique finite singular point which is a hyperbolic saddle because the eigenvalues of the Jacobian matrix at the origin are ±1.

In the local chart U_1, system (III.1) becomes $\dot{u} = u + v - u^2 v$, $\dot{v} = -uv^2$. The unique singular point on $v = 0$ is the origin $w_0 = (0, 0)$ that is semi-hyperbolic, because the eigenvalues of the Jacobian matrix at w_0 are 1 and 0. Using [23, Theorem 2.19], we get that it is a semi-hyperbolic unstable node.

In the local chart U_2 system (III.1) becomes $\dot{u} = v - u^2 - u^2 v$, $\dot{v} = -uv - uv^2$. The origin of the local chart U_2 is nilpotent. Using [23, Theorem 3.5], we obtain that it is locally the union of one elliptic, two parabolic, and one hyperbolic sectors. Applying blow-up techniques, we conclude that the hyperbolic sector is separated from the others by infinity.

Using this information on the local phase portraits of the finite and infinite singular points, that $y = -1$ is an invariant straight line of the system, and that it is symmetric with respect to the y-axis, we conclude that the unique possible global phase portrait is topologically equivalent to Figure 1 (16). Moreover using the Appendix, we conclude that the global phase portrait (16) has 4 canonical regions and 13 separatrices.

Now we study the local phase portraits of the finite and infinite singular points of system (III.2).

The origin is the unique finite singular point that is a center because the eigenvalues of the Jacobian matrix at the origin are $\pm i$ and the system has the analytic first integral $H = (y-1)^2 e^{2y-x^2}$.

In the local chart U_1 system (III.2) becomes $\dot{u} = u - v - u^2 v$, $\dot{v} = -uv^2$. The unique singular point on $v = 0$ is the origin $w_0 = (0, 0)$ that is semi-hyperbolic because the eigenvalues of the Jacobian matrix at w_0 are 1 and 0. Using [23, Theorem 2.19], we get that it is a semi-hyperbolic saddle.

In the local chart U_2 system (III.2) becomes $\dot{u} = v - u^2 + u^2 v$, $\dot{v} = -uv + uv^2$. The origin of the local chart U_2 is nilpotent. Using [23, Theorem 3.5], we get that it is the union of one elliptic sector and one hyperbolic sector. Applying blow-up techniques, we conclude that both sectors are separated by infinity.

Using this information on the local phase portraits of the finite and infinite singular point, that $y = 1$ is an invariant straight line of the system, and that it is symmetric with respect to the y-axis, we conclude that the unique possible global phase portrait is topologically equivalent to Figure 1 (17). Finally, using the Appendix, we conclude that the phase portrait has two canonical regions and ten separatrices.

A Appendix

A.1 Poincaré Compactification

Let $\mathcal{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$ be the quadratic polynomial vector field associated with the polynomial differential system (1.1). The *Poincaré compactified vector field* $p(\mathcal{X})$ *associated with* $\mathcal{X}$ is an analytic vector field on $\mathbb{S}^2$ constructed as follows (see, for instance [27] or [23, Chapter 5]).

The Poincaré sphere is defined as

$$\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$$

and $T_y\mathbb{S}^2$ is the tangent plane to $\mathbb{S}^2$ at the point y. The plane $\mathbb{R}^2$, where we have our polynomial vector field $\mathcal{X}$, is identified with the tangent plane $T_{(0,0,1)}\mathbb{S}^2$. Now we consider the central projection $f\colon T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. It defines two copies of $\mathcal{X}$, one in the northern hemisphere and the other in the southern hemisphere. Denote by $\mathcal{X}'$ the vector field $Df \circ \mathcal{X}$ defined on $\mathbb{S}^2$ except on $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. The equator $\mathbb{S}^1$ is identified with the *infinity* of $\mathbb{R}^2$. In order to extend $\mathcal{X}'$ to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$), it is necessary that $\mathcal{X}$ satisfy suitable conditions. In our case $p(\mathcal{X})$ is the only analytic extension of $y_3\mathcal{X}'$ to $\mathbb{S}^2$. On $\mathbb{S}^2\backslash\mathbb{S}^1$ there are two symmetric copies of $\mathcal{X}$, and knowing the behaviour of $p(\mathcal{X})$ around $\mathbb{S}^1$, we know the behaviour of $\mathcal{X}$ at infinity.

The *Poincaré disc* is just the projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ and is denoted by $\mathbb{D}^2$. We note that the Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$.

As stated in the introduction, we say that two polynomial vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $p(\mathcal{X})$ into orbits of the flow induced by $p(\mathcal{Y})$, preserving or reversing simultaneously the direction of all orbits.

As $\mathbb{S}^2$ is a differentiable manifold, in order to compute $p(\mathcal{X})$, we can consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ for $i = 1, 2, 3$; and the diffeomorphisms $F_i\colon U_i \to \mathbb{R}^2$, $G_i\colon V_i \to \mathbb{R}^2$ for $i = 1, 2, 3$ (here G_i are the inverses of the central projections from the planes tangent at the points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$, and $(0, 0, -1)$, respectively). If we denote by (u, v) the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$ (here (u, v) represents different things according to the local charts under consideration), then some easy computations give the following expressions for $p(\mathcal{X})$:

$$p(\mathcal{X}) = \begin{cases} v^2\Delta(u,v)\Big(Q\big(\frac{1}{v}, \frac{u}{v}\big) - uP\big(\frac{1}{v}, \frac{u}{v}\big), -vP\big(\frac{1}{v}, \frac{u}{v}\big)\Big) & \text{in } U_1, \\ v^2\Delta(u,v)\Big(P\big(\frac{u}{v}, \frac{1}{v}\big) - uQ\big(\frac{u}{v}, \frac{1}{v}\big), -vQ\big(\frac{u}{v}, \frac{1}{v}\big)\Big) & \text{in } U_2, \\ \Delta(u,v)(P(u,v), Q(u,v)) & \text{in } U_3, \end{cases}$$

where $\Delta(u, v) = (u^2 + v^2 + 1)^{-\frac{1}{2}}$. The expression for V_i is the same as that for U_i, except for a multiplicative factor -1. In these coordinates for $i = 1, 2$, $v = 0$ always denotes the points of $\mathbb{S}^1$. In what follows we omit the factor $\Delta(u, v)$ by rescaling the vector field $p(\mathcal{X})$. Thus we obtain a polynomial vector field in each local chart.

A.2 Separatrices and Canonical Regions

We continue to denote by $p(\mathcal{X})$ the Poincaré compactification in the Poincaré disc $\mathbb{D}$ of the polynomial differential system (1.1), and by Φ its analytic flow. We follow the notation in Markus [43] and Neumann [44] and we let (U, Φ) be the flow of a differential system on an invariant set $U \subset \mathbb{D}$ under the flow Φ. As before, two flows (U, Φ) and (V, Ψ) are *topologically equivalent* if and only if there exists a homeomorphism

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$h: U \to V$ that sends orbits of the flow Φ into orbits of the flow Ψ and either preserves or reverses the orientation of all the orbits.

The flow (U, Φ) is *parallel* if it is topologically equivalent to one of the following flows:

- the flow defined in $\mathbb{R}^2$ by the differential system $\dot{x} = 1$, $\dot{y} = 0$, called *the strip flow*,
- the flow defined in $\mathbb{R}^2 \setminus \{(0,0)\}$ by the differential system in polar coordinates $\dot{r} = 0$, $\dot{\theta} = 1$, called *the annular flow*,
- the flow defined in $\mathbb{R}^2 \setminus \{(0,0)\}$ by the differential system in polar coordinates $\dot{r} = r$, $\dot{\theta} = 0$, called *the spiral or radial flow*.

It is well known that the separatrices of the vector field $p(\mathcal{X})$ in the Poincaré disc $\mathbb{D}$ are as follows:

- all orbits of $p(\mathcal{X})$ which are in the boundary $\mathbb{S}^1$ of the Poincaré disc,
- all finite singular points of $p(\mathcal{X})$,
- all limit cycles of $p(\mathcal{X})$,
- all separatrices of the hyperbolic sectors of the finite and infinite singular points of $p(\mathcal{X})$.

Moreover each such vector field $p(\mathcal{X})$ has finitely many separatrices. See for instance [34] for further details.

We denote by $\mathcal{S}$ the union of the separatrices of the flow $(\mathbb{D}, \Phi)$ defined by $p(\mathcal{X})$ in the Poincaré disc $\mathbb{D}$. It is clear that $\mathcal{S}$ is an invariant closed set. If N is a connected component of $\mathbb{D} \setminus \mathcal{S}$, then N is also an invariant set under the flow Φ of $p(\mathcal{X})$, and we call the flow $(N, \Phi|_N)$ a *canonical region* of $(\mathbb{D}, \Phi)$. For a proof of the following proposition. see [34, 44].

Proposition A.1 *If the number of separatrices of the flow* $(\mathbb{D}, \Phi)$ *is finite, then every canonical region of the flow* $(\mathbb{D}, \Phi)$ *is parallel.*

The *separatrix configuration* $\mathcal{S}_c$ of a flow $(\mathbb{D}, \Phi)$ is the union of all the separatrices $\mathcal{S}$ of the flow jointly with an orbit belonging to each canonical region. The separatrix configuration $\mathcal{S}_c$ of the flow $(\mathbb{D}, \Phi)$ is topologically equivalent to the separatrix configuration $\mathcal{S}_c^*$ of the flow $(\mathbb{D}, \Phi^*)$ if there exists an orientation preserving homeomorphism from $\mathbb{D}$ to $\mathbb{D}$ that transforms orbits of $\mathcal{S}_c$ into orbits of $\mathcal{S}_c^*$, and orbits of $\mathcal{S}$ into orbits of $\mathcal{S}^*$.

Theorem A.2 (Markus–Neumann–Peixoto) *Let* $(\mathbb{D}, \Phi)$ *and* $(\mathbb{D}, \Phi^*)$ *be two compactified Poincaré flows with finitely many separatrices coming from two polynomial vector fields* (1.1). *Then they are topologically equivalent if and only if their separatrix configurations are topologically equivalent.*

For a proof of this result we refer the reader to [43, 44, 47].

It follows from the previous theorem that in order to classify the phase portraits in the Poincaré disc of a planar polynomial differential system having finitely many separatrices finite and infinite, it is enough to describe their separatrix configuration.

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